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CONDITIONING WITH CONIC SECTIONS IN THE TWO-DIMENSIONAL NORMAL DISTRIBUTION

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Assuming that X has a two-dimensional normal distribution certain conditional distributions of X given that X lies on a hyperbola or a parabola are found. Two of these distributions, related respectively to the parabola and the hyperbola, resemble the von Mises distribution, which can be obtained as a conditional distribution of X given that X lies on a circle. It is, however, proved that the assumptions leading to the conditional distribution in the hyperbolic case are not analogous to those leading to the von Mises distribution.

1. Introduction. Barndorff-Nielsen (1977, 1978) has introduced the d-dimensional hyperbolic distribution and the generalized d-dimensional hyperbolic distributions. For $d = 1$ the hyperbolic distribution has probability density function (pdf)

$$\frac{\kappa}{2\alpha K_1(\kappa)} e^{-\alpha(1+x^2)^{1/2}+\beta x} \qquad x \in R, \tag{1.1}$$

while the pdf of the generalized hyperbolic distribution with index parameter $\lambda = 0$ is

$$\frac{1}{2K_0(\kappa)}(1 + x^2)^{-\frac{1}{2}} e^{-\alpha(1+x^2)^{1/2}+\beta x} \qquad x \in R. \tag{1.2}$$

In both cases $\alpha > 0$, $|\beta| < \alpha$, $\kappa = (\alpha^2 - \beta^2)^{\frac{1}{2}}$ and $K_\nu(\kappa)$ is the modified Bessel function of the third kind and with index ν.

These densities depend on x through x and $(1 + x^2)^{\frac{1}{2}}$ and the transformation $x = \sinh u$ takes (1.2) into

$$\frac{1}{2K_0(\kappa)} e^{-\alpha \cosh u + \beta \sinh u} \qquad u \in R, \tag{1.3}$$

which shows that (1.3) (or (1.2)) may naturally be interpreted as a distribution on a hyperbola. A similar remark holds for (1.1).

Barndorff-Nielsen noted the formal analogy between (1.3) and the von Mises distribution on the unit circle,

$$\frac{1}{2\pi I_0(\kappa)} e^{\kappa \cos\theta \cos v + \kappa \sin\theta \sin v} \qquad v \in (0, 2\pi), \tag{1.4}$$

and pointed out that these distributions have many points of resemblance from a statistical inference point of view.

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It is well known that the von Mises distribution on the unit circle is the conditional distribution of $X/\|X\|$ given $\|X\|$ if X has a two-dimensional normal distribution with arbitrary mean and with the identity matrix as variance (cf. Downs, 1966) and it is clearly of interest to investigate whether a similar result holds for the generalized hyperbolic distribution (1.2). The choice of the curve system determining the conditioning in the hyperbolic case raises a problem. Whereas in the circular case the only natural curve system determining the conditioning is a system of circles with common center several such systems can be defined in the hyperbolic case, but the system given by the hyperbolic coordinates seems to be the immediate analogy to that considered in the circular case. The hyperbolic coordinates (r, u) of $x = (x_1, x_2)$, where $x_1 > |x_2|$, are given by

$$x_1 = r \cosh u \qquad x_2 = r \sinh u. \tag{1.5}$$

Here $u \in R$ and $r > 0$.

In Section 2 it will be shown that the distribution (1.2) can not be obtained as a conditional distribution in a two-dimensional normal distribution when the conditioning is determined by the hyperbolic coordinates. However, Examples 2.1 and 2.2 show that the distributions (1.1) and (1.3) (or (1.2)) are conditional distributions in certain two-dimensional normal distributions.

At the present time it is an open question whether the d-dimensional hyperbolic distribution and the generalized d-dimensional hyperbolic distribution with index parameter $\lambda = (d-1)/2$ can be obtained as a conditional distribution in a $(d+1)$-dimensional normal distribution. At the end of Section 2 a single result for the case $d = 2$ will be mentioned.

Suppose X has an arbitrary two-dimensional normal distribution. Conditional distributions of X given that X lies on a circle are well known (cf. Mardia, 1972). Conditioning with ellipses leads to conditional distributions, which are simple transformations of those obtained in the circular case, and will therefore not be considered. In Section 3 conditioning with the last of the conic sections, the parabola, will be discussed and it will be shown that in this case two different ways of conditioning lead to a conditional distribution which resembles the distributions (1.3) and (1.4).

Throughout this paper $X = (X_1, X_2)$ denotes a random variable having a two-dimensional normal distribution with mean $\xi = (\xi_1, \xi_2)$ and variance

$$\Sigma = \begin{Bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{Bmatrix}.$$

2. Conditional distributions of hyperbolae.

Restricting the sample space of X to the set $C = \{(x_1, x_2) : x_1 > |x_2|\}$ the pdf of X is

$$ce^{-\frac{1}{2}x\Sigma^{-1}x' + x\Sigma^{-1}\xi'}, \qquad x \in C,$$

where

$$c = \left\{P(X_1 > |X_2|)2\pi\sigma_1\sigma_2(1-\rho^2)^{\frac{1}{2}}\right\}^{-1} e^{-\frac{1}{2}\xi\Sigma^{-1}\xi'}.$$

Let (R, U) denote the hyperbolic coordinates of X (see (1.5)).

In the following the conditional distribution of U given R will be found. The pdf of (R, U) is

$$f(r, u) = rc\, e^{-\frac{1}{2}x\Sigma^{-1}x' + x\Sigma^{-1}\xi'}, \qquad r > 0, u \in R, \tag{2.1}$$

where $x = (r \cosh u, r \sinh u)$. The transformation

$$w(u) = e^u$$

is monotone and since

$$\frac{dw}{du} = w,$$

$$\cosh u = \tfrac{1}{2}(w + w^{-1})$$

and

$$\sinh u = \tfrac{1}{2}(w - w^{-1})$$

it follows that the pdf of r may be written

$$\begin{aligned} f(r) &= rc\, e^{-r^2/(4(1-\rho^2))(1/\sigma_1^2 - 1/\sigma_2^2)} \int_0^\infty e^{-\frac{1}{2}(\gamma w^2 + \varphi w^{-2}) + \frac{1}{2}(\alpha w^{-1} + \beta w)} \frac{1}{w}\, dw \\ &= rc\, e^{-r^2/(4(1-\rho^2))(1/\sigma_1^2 - 1/\sigma_2^2)} I(\alpha, \beta, \gamma, \varphi), \end{aligned} \tag{2.2}$$

where

$$\alpha = \frac{r}{1-\rho^2}\left(\frac{\xi_1}{\sigma_1^2} - \frac{\xi_2}{\sigma_2^2} - \frac{\rho(\xi_2 - \xi_1)}{\sigma_1\sigma_2}\right) \qquad \beta = \frac{r}{1-\rho^2}\left(\frac{\xi_1}{\sigma_1^2} + \frac{\xi_2}{\sigma_2^2} - \frac{\rho(\xi_2 + \xi_1)}{\sigma_1\sigma_2}\right)$$

$$\gamma = \frac{r^2}{4(1-\rho^2)}\left(\frac{1}{\sigma_1^2} - \frac{2\rho}{\sigma_1\sigma_2} + \frac{1}{\sigma_2^2}\right)$$

and

$$\varphi = \frac{r^2}{4(1-\rho^2)}\left(\frac{1}{\sigma_1^2} + \frac{2\rho}{\sigma_1\sigma_2} + \frac{1}{\sigma_2^2}\right).$$

The integral $I(\alpha, \beta, \gamma, \varphi)$ may be rewritten as

$$I(\alpha, \beta, \gamma, \varphi) = \tfrac{1}{2}\int_0^\infty e^{-\frac{1}{2}(\varphi w^{-1} + \gamma w) + \frac{1}{2}(\alpha w^{-1/2} + \beta w^{1/2})} \frac{1}{w}\, dw. \tag{2.3}$$

Let I_ν denote the modified Bessel function of the first kind and with index ν. Using the Taylor expansion of $\exp(\frac{1}{2}(\alpha w^{-\frac{1}{2}} + \beta w^{\frac{1}{2}}))$, interchanging the order of integration and summation and applying the formulae

$$\int_0^\infty w^{-\nu-1} e^{-\frac{1}{2}(\varphi w^{-1} + \gamma w)}\, dw = 2\left(\frac{\gamma}{\varphi}\right)^{\nu/2} K_\nu\left(\gamma^{\frac{1}{2}}\varphi^{\frac{1}{2}}\right)$$

and

$$I_\nu(z) = \Sigma_{m=0}^{\infty} \frac{(z/2)^{2m+\nu}}{m!\Gamma(m+\nu+1)}$$

(cf. Gradshteyn and Ryzhik, 1965) one obtains

$$(2.4) \qquad I(\alpha, \beta, \gamma, \varphi) = \Sigma_{\nu=-\infty}^{\infty} \left[\frac{\alpha^{\frac{1}{2}}}{\beta^{\frac{1}{2}}}\right]^{\nu} \left(\frac{\gamma}{\varphi}\right)^{\nu/4} K_{\nu/2}\left(\gamma^{\frac{1}{2}}\varphi^{\frac{1}{2}}\right) I_{|\nu|}\left(\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}\right),$$

where $w^{\frac{1}{2}}$ is to be understood as $i|w|^{\frac{1}{2}}$ for w negative.

If $\alpha = 0$ and/or $\beta = 0$ the value of the integral is to be interpreted as the limiting value. (For $\nu \neq -1, -2, \cdots$ and for $z \to 0$ one has the asymptotic formula $I_\nu(z) \sim (\frac{1}{2}z)^\nu / \Gamma(\nu + 1)$ (cf. Abramowitz and Stegun, 1965, page 375).)

The probability density function of the conditional distribution of U given R is determined by (2.1), (2.2) and (2.3) or (2.4).

In order to investigate whether a result similar to that of Downs' holds for the generalized one-dimensional hyperbolic distribution (1.2) when the conditioning is determined by the hyperbolic coordinates, assume that $\sigma_1^2 = \sigma_2^2 = 1$ and $\rho = 0$. Letting

$$a = \xi_1 - \xi_2$$

and

$$b = \xi_1 + \xi_2,$$

it follows that the pdf of the conditional distribution of U given $R = r$ is

$$f(u|r) = \frac{\exp\left(-\frac{1}{2}r^2 \cosh 2u + r\xi_1 \cosh u + r\xi_2 \sinh u\right)}{\Sigma_{\nu=-\infty}^{\infty}\left(\frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}\right)^{\nu} K_{\nu/2}\left(\frac{r^2}{2}\right) I_{|\nu|}\left(ra^{\frac{1}{2}}b^{\frac{1}{2}}\right)}.$$

Thus, the conditional distribution of X_2/R given $R = r$ has pdf

$$\frac{\exp\left(-r^2\left(x^2 + \frac{1}{2}\right) + r\xi_1(1 + x^2)^{\frac{1}{2}} + r\xi_2 x\right)}{(1 + x^2)^{\frac{1}{2}}\Sigma_{\nu=-\infty}^{\infty}\left(\frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}\right)^{\nu} K_{\nu/2}\left(\frac{r^2}{2}\right) I_{|\nu|}\left(ra^{\frac{1}{2}}b^{\frac{1}{2}}\right)},$$

which is not the pdf corresponding to the generalized one-dimensional hyperbolic distribution (1.2) and consequently the considered analogy to Downs' result does not hold. However, Examples 2.1 and 2.2 show that the one-dimensional hyperbolic distribution (1.1) and the generalized one-dimensional hyperbolic distribution (1.2) can be obtained as conditional distributions in a two-dimensional normal distribution using curve systems different from that given by the hyperbolic coordinates to determine the conditioning. In both examples the mean of the normal distribution is assumed to be 0 while the correlation coefficient ρ varies freely in $(-1, 1)$.

EXAMPLE 2.1. Suppose $\xi_1 = \xi_2 = 0$ and assume for the sake of simplicity that $\sigma_1^2 = \sigma_2^2 = 1$. Let (R, U) denote the hyperbolic coordinates of X, i.e.,

$$X_1 = R \cosh U$$
$$X_2 = R \sinh U.$$

It is easily seen (directly, or from the calculations above) that the conditional distribution of U given $R = r$ has pdf

$$\frac{1}{K_0\left[\dfrac{r^2}{2(1-\rho^2)^{\frac{1}{2}}}\right]} \exp(-r^2/(2(1-\rho^2))(\cosh 2u - \rho \sinh 2u)).$$

If $S = \sinh 2U = 2X_1X_2/R^2$ it follows that

$$f(s|r) = \frac{1}{2K_0\left[\dfrac{r^2}{2(1-\rho^2)^{\frac{1}{2}}}\right]} (1 + S^2)^{-\frac{1}{2}} \exp\left(-r^2/(2(1-\rho^2))\left((1+s^2)^{\frac{1}{2}} - \rho s\right)\right),$$

which means that the conditional distribution of S given $R = r$ is the generalized hyperbolic distribution (1.2) with parameters

$$\alpha = \frac{r^2}{2(1-\rho^2)}$$

and

$$\beta = \frac{\rho r^2}{2(1-\rho^2)}.$$

Let $x = (x_1, x_2)$, where $x_1 > |x_2|$. In the hyperbolic coordinate system the position of x on the hyperbola $x_1^2 - x_2^2 = r^2$ is determined by $u = \operatorname{argtanh}(x_2/x_1)$. If x is the point corresponding to (r, s) then x is the intersection of the hyperbola $x_1^2 - x_2^2 = r^2$ and the line $x_2 = ((1 + s^2)^{\frac{1}{2}} - 1)x_1/s$ (see Figure 1), i.e., the position of x on the hyperbola $x_1^2 - x_2^2 = r^2$ in the '(r, s)-coordinate system' is given by $s = 2x_1x_2/(x_1^2 - x_2^2)$.

EXAMPLE 2.2. Assume that the sample space of X is restricted to the set $\{(x_1, x_2) \in R^2 : x_1 > 0\}$. Let $\xi_1 = \xi_2 = 0$ and suppose again for simplicity that $\sigma_1^2 = \sigma_2^2 = 1$.

Consider the transformation given by

$$x_1 = e^{-t} \cosh u$$
$$x_2 = e^{t} \sinh u$$

where $t \in R$ and $u \in R$. It is easy to show that the pdf of (T, U) is

$$\frac{\cosh 2u}{\pi(1-\rho^2)^{\frac{1}{2}}} \exp(1/(2(1-\rho^2))\{\sinh 2t - \cosh 2u \cosh 2t + \rho \sinh 2u\}). \tag{2.5}$$

Using (1.1) it follows that the pdf of T is

$$\frac{K_1(\kappa(t))}{2\pi\kappa(t)(1-\rho^2)^{\frac{3}{2}}} \cosh 2t \, e^{\frac{1}{2(1-\rho^2)} \sinh 2t}, \tag{2.6}$$

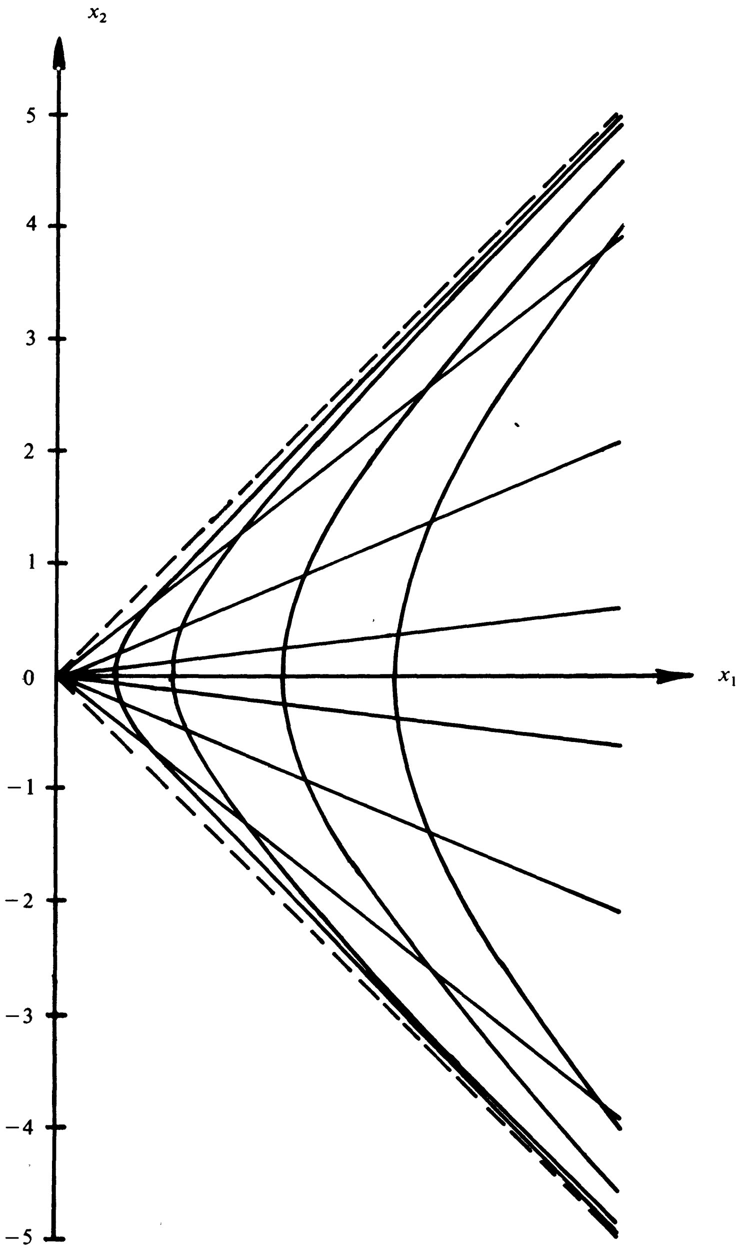

FIG. 1. *The hyperbolae* $x_1^2 - x_2^2 = r^2$ *for* $r = 1/2, 1, 2, 3$ *and the lines* $x_2 = ((1 + s^2)^{\frac{1}{2}} - 1)x_1/s$ *for* $s = 4, 1, 1/4, -1/4, -1, -4.$

where

$$\kappa(t) = \frac{1}{2(1-\rho^2)}(\cosh^2 2t - \rho^2)^{\frac{1}{2}}.$$

Let $s = \sinh 2u = 2x_1x_2$ and consider the transformation of x into (t, s). If $x = (x_1, x_2)$, where $x_1 > 0$ and $x_2 \neq 0$, corresponds to (t, s) then x is the intersection of the hyperbolae given by $e^{2t}x_1^2 - e^{-2t}x_2^2 = 1$ and $2x_1x_2 = s$ (see Figure 2).

From (2.5) and (2.6) one obtains that the conditional distribution of S given $T = t$ has pdf

$$\frac{\kappa(t)(1-\rho^2)}{\cosh 2tK_1(\kappa(t))}\exp\Big(-1/\left(2(1-\rho^2)\right)\big\{(1+s^2)^{\frac{1}{2}}\cosh 2t - \rho s\big\}\Big).$$

Consequently, letting

$$\alpha(t) = \frac{\cosh 2t}{2(1-\rho^2)}$$

and

$$\beta = \frac{\rho}{2(1-\rho^2)},$$

the conditional distribution of S given $T = t$ is the hyperbolic distribution (1.1) with parameters $\alpha(t)$ and β.

It is remarkable also that the generalized one-dimensional hyperbolic distribution with index parameter $\lambda = 0$ can be obtained as a conditional distribution under these assumptions. To see this, note that formulae (1.3) and (2.5) imply that the pdf of U is

$$\frac{K_0(\tilde{\kappa}(u))}{\pi(1-\rho^2)^{\frac{1}{2}}}\cosh 2u\, e^{\frac{\rho}{2(1-\rho^2)}\sinh 2u} \qquad u \neq 0.$$

Here

$$\tilde{\kappa}(u) = \frac{1}{2(1-\rho^2)}(\cosh^2 2u - 1)^{\frac{1}{2}}.$$

Let $Y = \sinh 2T$. It follows that the conditional distribution of Y given $U = u$, where $u \neq 0$, has pdf

$$\frac{1}{2K_0(\tilde{\kappa}(u))}(1+y^2)^{-\frac{1}{2}}\exp\Big(-1/\left(2(1-\rho^2)\right)\big\{(1+y^2)^{\frac{1}{2}}\cosh 2u - y\big\}\Big),$$

which is the pdf corresponding to the generalized hyperbolic distribution (1.2) with parameters $\tilde{\alpha}(u)$ and $\tilde{\beta}$, where

$$\tilde{\alpha}(u) = \frac{\cosh 2u}{2(1-\rho^2)}$$

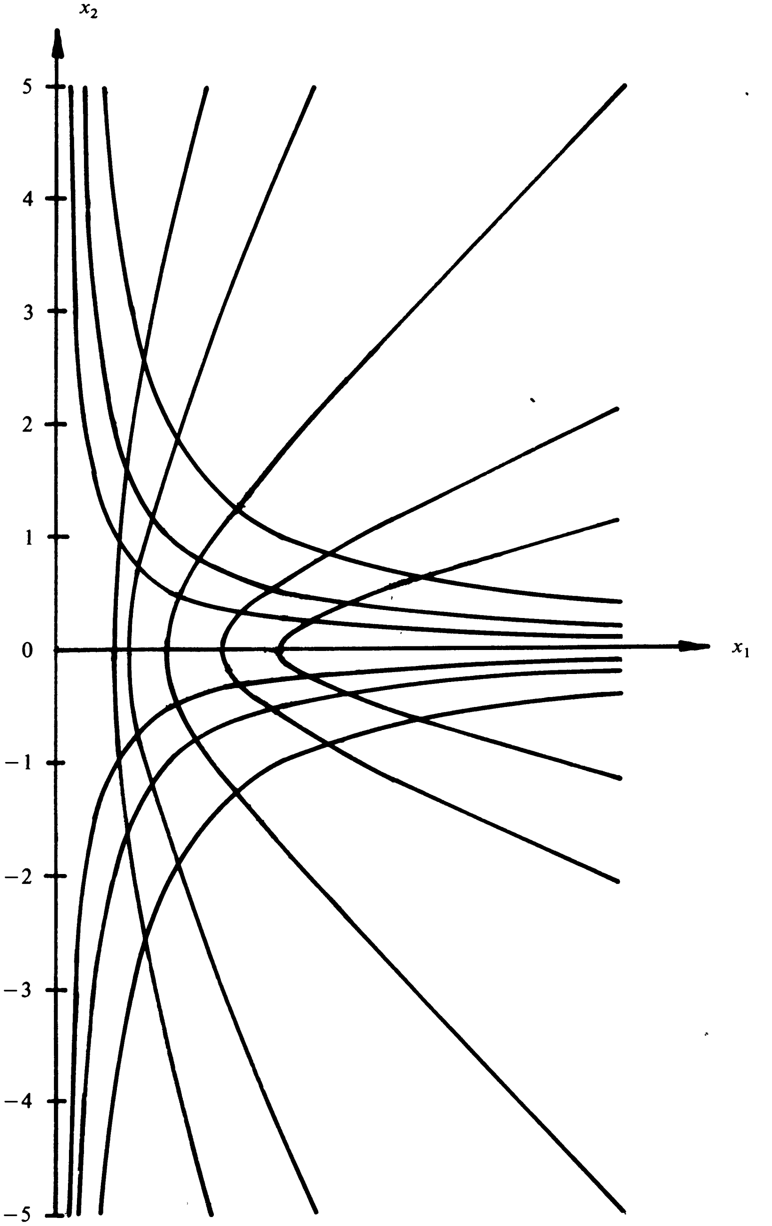

FIG. 2. *The hyperbolae* $x_1x_2 = s/2$ *for* $s/2 = 2, 1, 1/2, -1/2, -1, -2$ *and* $e^{2t}x_1^2 - e^{-2t}x_2^2 = 1$ *for* $e^t = 2, 3/2, 1, 2/3, 1/2$.

and

$$\tilde{\beta} = \frac{1}{2(1-\rho^2)}.$$

An attempt to obtain the d-dimensional hyperbolic distribution and the generalized d-dimensional hyperbolic distribution with index parameter $\lambda = (d-1)/2$ as conditional distributions in a $(d+1)$-dimensional normal distribution has been made. The only result worth mentioning at the present time is the following. Suppose (X_1, X_2, X_3) has a three-dimensional normal distribution with mean 0 and let the variance $\tilde{\Sigma}$ be given by

$$\tilde{\Sigma}^{-1} = \begin{Bmatrix} 1 & -\gamma_1 & -\gamma_2 \\ -\gamma_1 & 1 & 0 \\ -\gamma_2 & 0 & 1 \end{Bmatrix},$$

where $1 - \gamma_1^2 - \gamma_2^2 > 0$. Consider the transformation determined by

$$x_1 = 4t + \cosh u$$
$$x_2 = \sinh u \cos v$$
$$x_3 = \sinh u \sin v,$$

where $u > 0$, $v \in (0, 2\pi)$ and $t \in R$, and let $(z_1, z_2) = \sinh 2u \cos v$, $\sinh 2u \sin v)$. Setting $\alpha = 1/2$, $\beta_i = \gamma_i/2$, $i = 1, 2$, and $\kappa = (\alpha^2 - \beta_1^2 - \beta_2^2)^{\frac{1}{2}}$ the pdf of the conditional distribution of (Z_1, Z_2) given $T = 0$ is

$$\frac{\kappa e^{\kappa}}{2\pi}(1 + z_1^2 + z_2^2)^{-\frac{1}{2}}\exp\Big(-\alpha(1 + z_1^2 + z_2^2)^{\frac{1}{2}} + \beta_1 z_1 + \beta_2 z_2\Big) \qquad (z_1, z_2) \in R^2,$$

which is the pdf of the generalized two-dimensional hyperbolic distribution with index parameter $\lambda = 1/2$ (cf. Barndorff-Nielsen, 1978).

3. Conditioning with parabolae. In this section the conditional distribution of X given that X lies on a parabola will be found in two cases using different curve systems to determine the conditioning. In both cases the resulting conditional distribution has pdf of the form

$$(3.1) \qquad \frac{2\gamma^{\frac{1}{2}}}{\varphi^{\frac{1}{2}} K_{1/4}\left(\frac{\varphi^2}{8\gamma}\right)} \exp\left(-\frac{\varphi^2}{8\gamma} - \gamma v^4 - \varphi v^2\right) \qquad v \in R.$$

The domain of variation for the parameter (γ, φ) is the set $\{(\gamma, \varphi) \in R^2 : \gamma \geqslant 0, \varphi \geqslant 0\} \setminus \{(0, 0)\}$.

Just as for the densities (1.3) and (1.4) the norming constant for the density (3.1) is expressed by means of a modified Bessel function. Furthermore, it may be noted that the three families of distributions (1.3), (1.4) and (3.1) are exponential families with densities of the particular form

$$a(\theta)e^{\theta \cdot T(v)}$$

where the general form is

$$a(\theta)b(v)e^{\theta \cdot T(v)}.$$

Fisher (1922) has proved that the probability density functions for which the estimation by the method of moments is efficient are of the form

$$a(\mathbf{c})\exp(-c_4v^4 - c_3v^3 - c_2v^2 - c_1v - c_0) \qquad v \in R,$$

where $c_4 > 0$ and $a(\mathbf{c})$ is a norming constant. The distribution (3.1) is a special case of these distributions.

It follows from the asymptotic formula

$$K_\nu(z) \sim (\pi/2)^{\frac{1}{2}} z^{-\frac{1}{2}} e^{-z} \qquad as \quad z\uparrow\infty \qquad \nu \in R$$

(cf. Abramowitz and Stegun, 1965, page 378) that the distribution (3.1) tends to the normal distribution with mean 0 and variance $1/(2\varphi)$ for $\gamma \to 0$.

The first conditioning to be considered is determined by the transformation

$$x_1 = v$$

$$x_2 = \alpha v^2 + u$$

where $v \in R$, $u \in R$, and where $\alpha \neq 0$ is fixed.

The distribution of (U, V) has pdf

(3.2)

$$\frac{1}{2\pi\sigma_1\sigma_2(1-\rho^2)^{\frac{1}{2}}}\exp\left(\frac{-1}{2(1-\rho^2)}\left\{\frac{(v-\xi_1)^2}{\sigma_1^2} - \frac{2\rho(v-\xi_1)(u+\alpha v^2-\xi_2)}{\sigma_1\sigma_2} + \frac{(u+\alpha v^2-\xi_2)}{\sigma_2^2}\right\}\right).$$

Letting

$$c_4 = \frac{\alpha^2}{\sigma_2^2},$$

$$c_3 = \frac{-2\rho\alpha}{\sigma_1\sigma_2},$$

$$c_2 = \frac{1}{\sigma_1^2} + \frac{2\rho\alpha\xi_1}{\sigma_1\sigma_2} + \frac{2\alpha(u-\xi_2)}{\sigma_2^2},$$

$$c_1 = \frac{-2\xi_1}{\sigma_1^2} - \frac{2\rho(u-\xi_2)}{\sigma_1\sigma_2},$$

and

$$c_0 = \frac{\xi_1^2}{\sigma_1^2} + \frac{2\rho\xi_1(u-\xi_2)}{\sigma_1\sigma_2} + \frac{(u-\xi_2)^2}{\sigma_2^2},$$

the pdf of U is

$$f(u) = \frac{1}{2\pi\sigma_1\sigma_2(1-\rho^2)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \exp\left(\frac{-1}{2(1-\rho^2)}\{c_4v^4 + c_3v^3 + c_2v^2 + c_1v + c_0\}\right) dv.$$

This expression simplifies considerably if $c_1 = c_3 = 0$ and $c_2 > 0$. If $\xi_1 = \rho = 0$ and $\alpha(u - \xi_2) > -\sigma_2^2/(2\sigma_1^2)$ these conditions are fulfilled and using that

$$\int_0^\infty t^{-\frac{1}{2}} e^{-at^2 - st} dt = \frac{s^{\frac{1}{2}}}{2a^{\frac{1}{2}}} e^{\frac{s^2}{8a}} K_{1/4}\left(\frac{s^2}{8a}\right),$$

if $a > 0$ and $s > 0$ (cf. Robert and Kaufman, 1966), the pdf of U is in this case found to be

$$f(u) = \frac{c_2^{\frac{1}{2}} e^{-\frac{1}{2}c_0}}{4\pi\sigma_1\sigma_2 c_4^{\frac{1}{2}}} e^{\frac{c_2^2}{16c_4}} K_{1/4}\left(\frac{c_2^2}{16c_4}\right). \tag{3.3}$$

Setting

$$\gamma = \frac{c_4}{2} = \frac{\alpha^2}{2\sigma_2^2}$$

and

$$\varphi = \frac{c_2}{2} = \frac{1}{2\sigma_1^2} + \frac{\alpha(u - \xi_2)}{\sigma_2^2}$$

it follows from (3.2) and (3.3) that if $\xi_1 = \rho = 0$ and $\alpha(u - \xi_2) > -\sigma_2^2/(2\sigma_1^2)$ then (3.1) is the pdf of the conditional distribution of V given $U = u$.

The second conditioning to be considered is determined by the 'parabolic coordinates', i.e., by the transformation

$$x_1 = v$$
$$x_2 = rv^2,$$

where $r \in R$ and $v \in R$.

Arguing as above it is easily seen that if $\rho = \xi_1 = 0$ and $\sigma_2^2/(2\sigma_1^2) > r\xi_2$, where $r \neq 0$, then the conditional distribution of V given $R = r$ has pdf (3.1) with

$$\gamma = \frac{r^2}{2\sigma_2^2}$$

and

$$\varphi = \frac{1}{2\sigma_1^2} - \frac{r\xi_2}{\sigma_2^2}.$$

In particular, if $\rho = \xi_1 = \xi_2 = 0$ then (3.1) with $\gamma = r^2/2$ and $\varphi = 1/2$ is the pdf of the conditional distribution of V given $R = r$, whatever the value of $r \neq 0$.

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