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# OPTIMALITY AND ALMOST OPTIMALITY OF MIXTURE STOPPING RULES[1]

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It is shown that for a test of a composite hypothesis on the parameter $\theta$ of an exponential family of distributions, mixture stopping rules are almost optimal with respect to certain criteria of optimality and a unique stopping rule is to be found among them which is optimal with respect to another type of optimality.

**1. Introduction and summary.** Let $J$ denote an open interval of real numbers. Assume that for each $\theta \in J$, $P_\theta$ is a probability measure under which $X_1, X_2, \cdots$ are independent and identically distributed random variables with probability density $h_\theta(x) = \exp\{\theta x - \psi(\theta)\}$ with respect to some $\sigma$-finite measure $\nu$. Let $S_n = \sum_{k=1}^n X_k$ $(n = 0, 1, \cdots; S_0 = 0)$. For a given $\theta_0 \in J$ and $F$ a probability distribution on $J$ define

$$f(x, t) = \int_J \exp\{(y - \theta_0)x - t[\psi(y) - \psi(\theta_0)]\} \, dF(y) \tag{1}$$

and

$$T = \inf\{n \mid f(S_n, n) \geqq \varepsilon\} \quad (\varepsilon > 1) . \tag{2}$$

Any $T$ of this form shall hence be referred to as a mixture stopping rule. It is shown in Robbins (1970) that

$$P_{\theta_0}(T < \infty) \leqq \frac{1}{\varepsilon} \tag{3}$$

and statistical applications of such stopping rules are also discussed there. An approximation for $E_\theta T$ (for $\theta \neq \theta_0$ such that $F$ has a derivative $F'$ with respect to Lebesgue measure in a neighborhood of $\theta$, $F'$ being positive and continuous at $\theta$) is given in Pollak and Siegmund (1975): (as $\varepsilon \to \infty$)

$$E_\theta T = \frac{1}{2I(\theta)} [2 \log \varepsilon + \log \log \varepsilon] + O(1) \tag{4}$$

where $I(\theta) = (\theta - \theta_0)\psi'(\theta) - (\psi(\theta) - \psi(\theta_0))$. (A more explicit form of $O(1)$ can be found in Pollak and Siegmund (1975).)

The purpose of this article is to present optimality properties of mixture stopping rules.

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For statistical applications, it is desirable to choose a stopping time $T$ such that $E_\theta T$ will be as small as possible for a wide range of values of $\theta$. For a given $\theta \neq \theta_0$, by using a mixture stopping rule whose $F$ assigns unit mass to the single point $\theta$, one may obtain $E_\theta T = (\log \varepsilon)/I(\theta) + O(1)$. This is smaller than (4) by a term which is $O(\log\log \varepsilon)$; but it requires prior knowledge of $\theta$ and hence is impossible to implement in general.

A natural consideration under these circumstances would be to employ a minimax approach. (4) would suggest minimizing $\sup_{\theta\neq\theta_0}[E_\theta T - (\log\varepsilon)/I(\theta)]/[(\log\log\varepsilon)/2I(\theta)]$; or equivalently trying to minimize $\sup_{\theta\neq\theta_0} 2I(\theta)E_\theta T$. Unfortunately this is infinite: $\lim_{\theta\to\theta_0}(\theta-\theta_0)^2 E_\theta T = \infty$ by Theorem 1 of Farrell (1964) and $I(\theta) \sim (\theta-\theta_0)^2$ for $\theta$ close to $\theta_0$; also, clearly $\lim_{\theta\to\infty} I(\theta)E_\theta T = \infty$. These considerations lead to attempting to minimize $\sup_{a\leqq\theta\leqq b} 2I(\theta)E_\theta T$ under the restriction that $P_{\theta=\theta_0}(T<\infty) \leqq 1/\varepsilon$, where $[a, b] \subset J$ is an interval of finite length and $\theta_0 \notin [a, b]$.

Theorem 1 states that one cannot hope for anything substantially better than that suggested by (4), i.e., $\inf \sup_{a\leqq\theta\leqq b} 2I(\theta)E_\theta T = 2\log\varepsilon + \log\log\varepsilon + O(1)$. Therefore the class of mixture stopping rules is asymptotically almost optimal—optimal up to a term of order $O(1)$. Theorem 2 presents a Bayesian almost optimal property of mixture stopping rules.

With respect to a different criterion mixture stopping rules form a complete class. Since (4) is exact up to order $O(1)$, multiplying $\varepsilon$ by a constant will not change the right-hand side of (4). Thus if $T$ is defined as the first crossing time of a boundary in the $(n, S_n)$ plane, one would suspect a similar minimax result if one changes $(P_{\theta_0}(T<\infty), E_\theta T)$ to the expected $(\theta_0, \theta)$ number of times the process $(n, S_n)$ remains (above, below) the stopping boundary respectively. Theorem 3 states that the (unique) minimax solution is a mixture stopping rule.

**2. Almost optimality.** Without loss of generality it may be assumed that $0 = \psi(0) = \psi'(0)$.

LEMMA 1. *Let* $0 < a \leqq b < \infty$ *satisfy* $\psi'(a) > \psi(b)/b$, $[a, b] \subset J$. *For any* $\xi > 1$ *and probability measure* $G$ *on* $[a, b]$ *define* $N(\xi; a; b; G) = \inf\{n \mid \int_a^b \exp\{yS_n - n\psi(y)\}\, dG(y) \geqq \xi\}$. *There exist constants* $0 < A, B < \infty$ *independent of* $\xi$, $G$ *such that* $E_\theta N(\xi; a; b; G) \leqq A\log\xi + B$ *for all* $\theta \in [a, b]$ *and* $\xi > 1$.

PROOF. Define $M(\gamma) = \inf\{n \mid \exp\{\gamma S_n - n\psi(\gamma)\} \geqq \xi\}$. It follows from Theorem 1 of Lorden (1970) that there exists $0 < D < \infty$ such that $E_\theta\{S_{M(\gamma)} - [M(\gamma)\psi(\gamma) + \log\xi]/\gamma\} \leqq D$ uniformly in $\theta \in [a, b]$, $\gamma \in [a, b]$, $\xi > 1$ and so (by Wald's lemma) for all $\theta, \gamma \in [a, b]$

$$E_\theta M(\gamma) \leqq [(\log\xi)/\gamma + D]/[\psi'(\theta) - \psi(\gamma)/\gamma] \leqq [(\log\xi)/a + D]/[\psi'(a) - \psi(b)/b]\,. \tag{5}$$

From $\int_a^b \exp\{yS_n - n\psi(y)\}\, dG(y) \geqq \min(\exp\{aS_n - n\psi(a)\}, \exp\{bS_n - n\psi(b)\})$ it follows that $N(\xi; a; b; G) \leqq \max(M(a), M(b)) \leqq M(a) + M(b)$. This and (5) complete the proof of Lemma 1.

LEMMA 2. *Let $\gamma \in (0, 1)$, let $F_0$ be the probability measure wholly concentrated at $\{0\}$, let $G$ be a probability on $[a, b]$, $0 < a \leqq b < \infty$, $[a, b] \subset J$, $\psi'(a) > \psi(b)/b$ and denote $F = \gamma F_0 + (1-\gamma)G$. Consider the optimal stopping problem defined by a prior distribution $F$ on $\theta$ when $X_1, X_2, \cdots$ are* i.i.d.—*$P_\theta$ and each observation costs $c > 0$ if $\theta \neq 0$, zero if $\theta = 0$, with loss $= 1$ for stopping if $\theta = 0$. There exists a constant $0 < M < \infty$ independent of $c$, $F$ such that a Bayes procedure (with probability one) continues sampling whenever the posterior risk of stopping is at least $Mc$.*

PROOF. That a Bayes rule exists can be seen from considerations similar to those of page 108 and Theorem 4.5′ (page 82) of Chow, Robbins and Siegmund (1971).

Let $\infty > Q > A/e$ where $A$ is defined in Lemma 1 and define $T_{Qc}$ to be the first time $n \leqq \infty$ that the posterior risk of stopping is at most $Qc$. It is sufficient to prove for some $Q < M < \infty$ that the (integrated) risk of $T_{Qc}$ is less than $\gamma$ if $\gamma \geqq Mc$. Since the (integrated) risk of any generalized stopping time $T$ is the expected posterior risk of stopping plus $c(1-\gamma)\int_a^b E_\theta T\, dG(\theta)$ it is sufficient to prove for some $0 < M < \infty$ that $(1-\gamma)\int_a^b E_\theta T_{Qc}\, dG(\theta) < \gamma/c - Q$ if $\gamma \geqq Mc$.

Choose $M > Q$ such that $(1 - A/(Qe))M - (B + A/e) > Q$ where $A$, $B$ are the constants defined by Lemma 1. It is enough to look at $c$ for which $Qc < 1$. Notice that

$$\begin{aligned}
T_{Qc} &= \inf\{n \mid Qc \geqq \gamma h_0(x_1)\cdots h_0(x_n)/[\gamma h_0(x_1)\cdots h_0(x_n) \\
&\qquad + (1-\gamma)\textstyle\int_a^b h_\theta(x_1)\cdots h_\theta(x_n)\, dG(\theta)\}\} \qquad (6)\\
&= \inf\left\{n \mid \textstyle\int_a^b \exp\{yS_n - n\psi(y)\}\, dG(y) \geqq \dfrac{\gamma}{1-\gamma}\dfrac{1-Qc}{Qc}\right\} \\
&\leqq \inf\left\{n \mid \textstyle\int_a^b \exp\{yS_n - n\psi(y)\}\, dG(y) \geqq \dfrac{\gamma}{(1-\gamma)Qc}\right\}.
\end{aligned}$$

Noticing that $\sup_{0<y<1} -y(\log y) = 1/e$, apply Lemma 1 to get that if $1 > \gamma \geqq Mc$

$$\begin{aligned}
(1-\gamma)\int_a^b E_\theta T_{Qc}\, dG(\theta) &\leqq (1-\gamma)\left[A\left(\log\frac{\gamma}{Qc} + \log\frac{1}{1-\gamma}\right) + B\right] \\
&\leqq \frac{\gamma}{c}\frac{A}{Q}\frac{Qc}{\gamma}\log\frac{\gamma}{Qc} + B + A(1-\gamma)\log\frac{1}{1-\gamma} \\
&\leqq \frac{\gamma}{c}\frac{A}{Qe} + B + \frac{A}{e} \\
&\leqq \frac{\gamma}{c} - \left(1 - \frac{A}{Qe}\right)M + B + \frac{A}{e} \\
&< \frac{\gamma}{c} - Q.
\end{aligned}$$

This completes the proof of Lemma 2.

LEMMA 3. *Let $0 < a_1 < a < b < b_1 < \infty$ and let $G$ be a probability on $[a_1, b_1] \subset J$*

*with derivative* $g(x) = dG(x)/dx$ *which is positive and continuous on* $[(a_1 + a)/2, (b_1 + b)/2]$. *Let* $T = \inf\{n \mid \int_{a_1}^{b_1} \exp\{yS_n - n\psi(y)\}\, dG(y) \geqq \varepsilon\}$. *Then for all* $\theta \in [a, b]$

$$E_\theta T = \frac{1}{2I(\theta)} [2 \log \varepsilon + \log \log \varepsilon] + O_\theta(1) \tag{7}$$

*where* $\limsup_{\varepsilon\to\infty} \sup_{a \leqq \theta \leqq b} |O_\theta(1)| < \infty$.

PROOF. Notice that Theorem 1 of Pollak and Siegmund (1975) holds uniformly for $\theta \in [a, b]$ so that the right-hand side of (7) is a lower bound for $E_\theta T$. Similar manipulations show the right-hand side of (7) to be an upper bound for $E_\theta T$ uniformly for $\theta \in [a, b]$. See also Lai and Siegmund (1977).

THEOREM 1. *Let* $\theta_0 < a < b < \infty$, $[a, b] \subset J$, $\theta_0 \in J$.

$$\inf_{\{T \mid P_{\theta_0}(T<\infty) \leqq 1/\varepsilon\}} \sup_{a \leqq \theta \leqq b} 2I(\theta) E_\theta T = 2 \log \varepsilon + \log \log \varepsilon + O(1) \tag{8}$$

*where* $\limsup_{\varepsilon\to\infty} |O(1)| < \infty$, *and equality is attained by a mixture stopping rule.*

PROOF. Without loss of generality assume $\theta_0 = 0$ and $\psi'(a) > \psi(b)/b$. That the equality is attained by a mixture stopping rule is the content of Lemma 3. To see that the right side of (8) is a lower bound of the left side of (8), consider the Bayesian problem defined in Lemma 2 when $\gamma = \frac{1}{2}$ and $dG(y)/dy = I(y)/\int_a^b I(y)\, dy$ on $[a, b]$. Let $M$ be the constant derived in Lemma 2 and let $T_{Mc}$ be $T_{Qc}$ for $Q = M$ where $T_{Qc}$ is defined in (6). $T_{Mc}$ is a mixture stopping rule defined by $G$ and $\varepsilon = (1 - Mc)/(Mc)$. By virtue of Lemma 2 there exists a Bayes rule which continues sampling at least as long as $T_{Mc}$. Hence the Bayes risk is at least the sampling cost of $T_{Mc}$, whence for any stopping rule $T$

$$P_{\theta_0}(T < \infty) + c \int_a^b E_\theta T\, dG(\theta) \geqq c \int_a^b E_\theta T_{Mc}\, dG(\theta)\,.$$

Thus if $P_{\theta_0}(T < \infty) \leqq 1/\varepsilon = Mc/(1 - Mc)$

$$\int_a^b E_\theta T\, dG(\theta) \geqq \int_a^b E_\theta T_{Mc}\, dG(\theta) - M/(1 - Mc)\,. \tag{9}$$

There exist $a_1$, $b_1$ such that $0 < a_1 < a < b < b_1 < \infty$ and $\psi'(a_1) > \psi(b_1)/b_1$. Define $\Lambda = \inf\{n \mid \int_{a_1}^{b_1} \exp\{yS_n - n\psi(y)\} I(y)\, dy / \int_a^b I(y)\, dy \geqq \varepsilon\}$. By definition, $T_{Mc} \geqq \Lambda$. $\Lambda$ is a mixture stopping rule defined by $dF(y)/dy = I(y)/\int_{a_1}^{b_1} I(y)\, dy$ on $[a_1, b_1]$ and $\varepsilon' = \varepsilon \int_a^b I(y)\, dy / \int_{a_1}^{b_1} I(y)\, dy$. Thus by Lemma 3

$$E_\theta T_{Mc} \geqq E_\theta \Lambda = \frac{1}{2I(\theta)} [2 \log \varepsilon + \log \log \varepsilon] + O_\theta(1) \tag{10}$$

where $\limsup_{\varepsilon\to\infty} \sup_{a \leqq \theta \leqq b} |O_\theta(1)| < \infty$. Combining (9) and (10) yields

$$\int_a^b E_\theta T\, dG(\theta) \geqq \int_a^b [2 \log \varepsilon + \log \log \varepsilon + O(1)]\, d\theta / (2 \int_a^b I(y)\, dy)$$

whence by definition of $G$

$$\int_a^b [2I(\theta) E_\theta T - (2 \log \varepsilon + \log \log \varepsilon)]\, d\theta \geqq O(1)$$

for all $T$ satisfying $P_{\theta_0}(T < \infty) \leqq 1/\varepsilon$, thus completing the proof of (8).

THEOREM 2. *Let $F$ be a probability on $J$ with $F\{(0,\infty)\} > 0$, $T(\varepsilon) = \inf\{n \mid \int_J \exp\{yS_n - n\psi(y)\}\, dF(y) \geqq \varepsilon\}$ and let $0 < \gamma < 1$. There exists an interval $[a, b] \subset J$ with $0 < a \leqq b < \infty$, $\psi'(a) < \psi(b)/b$ such that $T(\varepsilon)$ is a $\delta$-Bayes solution of the optimal stopping problem described in Lemma 2 (with $dG(y) = dF(y)/F\{[a, b]\}$ for $y \in [a, b]$ and with $c$ defined by $\varepsilon = [\gamma/(1-\gamma)][(1-Mc)/(Mc)]/F\{[a, b]\}$ where $M$ is defined in Lemma 2) where the Bayes solution has a risk of order $(\log \varepsilon)/\varepsilon$ and $\delta = O(1/\varepsilon)$ as $\varepsilon \to \infty$.*

PROOF. There exist $a, b$ satisfying $0 < a \leqq b < \infty$, $[a, b] \subset J$, $\psi'(a) \leqq \psi(b)/b$, $F\{[a, b]\} > 0$. Let $G$, $c$ be defined as above and consider the optimal stopping problem described in Lemma 2. Let $T_{Mc}$ be $T_{Qc}$ for $Q = M$ as defined in (6). Clearly $T_{Mc} \geqq T(\varepsilon)$. Therefore and by virtue of Lemma 2 there exists a Bayes solution which samples at least as many observations as $T(\varepsilon)$ and so the Bayes risk of $T(\varepsilon)$ cannot exceed that of the Bayes solution by more than $\gamma P_0(T(\varepsilon) < \infty) \leqq \gamma/\varepsilon$. Since $c$ is of the order $1/\varepsilon$ and $ET(\varepsilon)$ is of the order $\log \varepsilon$, the order of the risk of sampling of the Bayes solution is $(\log \varepsilon)/\varepsilon$ and the proof is complete.

**3. Exact optimality.** Let $h_\theta{}^{*n}$ be the $n$-fold convolution of $h_\theta$ with itself with respect to $\nu$; let $H_\theta{}^{*n}$ be the measure whose derivative with respect to $\nu$ is $h_\theta{}^{*n}$, and understand $H_\theta{}^{*n}(z) = H_\theta{}^{*n}\{(-\infty, z]\}$. Let $0 \in J$. Denote: $\mathbf{c} = (c_1, c_2, \cdots)$ where $-\infty \leqq c_j \leqq \infty$, $j = 1, 2, \cdots$. Denote

$$\chi(A) = \text{the indicator function of the event } A\,;$$
$$\mathscr{C}_\varepsilon = \{(\mathbf{c}, \boldsymbol{\alpha}) \mid 0 \leqq \alpha_j \leqq 1 \text{ for all } j,\ \textstyle\sum_{j=1}^\infty [1 - (H_0{}^{*j}(c_j) - \alpha_j H_0{}^{*j}\{\{c_j\}\})] = 1/\varepsilon\}, \quad \varepsilon > 1\,;$$
$$\mathscr{G} = \{G \mid G \text{ is a probability on } \{\theta \mid a \leqq \theta \leqq b\}\}\,;$$
$$\mathscr{M}_\varepsilon = \{\mu \mid \mu \text{ is a probability measure on } \{\mathbf{z} \mid \textstyle\sum_{i=1}^\infty (1 - z_i) = 1/\varepsilon\}\}\,;$$
$$\mathscr{E}_\varepsilon = \{W \mid W = \textstyle\sum_{j=1}^\infty [\chi(S_j \leqq c_j{}^w) - \alpha_j{}^w \chi(S_j = c_j{}^w)],\ (\mathbf{c}^w, \boldsymbol{\alpha}^w) \in \mathscr{C}_\varepsilon\}\,;$$
$$\mathscr{R}_\varepsilon = \{T \mid T \text{ is a stopping variable defined by (2)}, \ \varepsilon > 1 \text{ fixed}\}\,;$$
$$\mathscr{R} = \textstyle\bigcup_{\varepsilon > 1} \mathscr{R}_\varepsilon\,;$$
$$\mathscr{Q}_\varepsilon = \{T \mid T \text{ is a stopping variable for } \{S_k\}_{k=1,2,\cdots} \text{ whose stopping boundary is concave, } P_0(T < \infty) = 1/\varepsilon \text{ (where randomization on the boundary is permitted to let this equality hold)}\}\,.$$

$A_T(j)$ is the stopping boundary defining $T \in \mathscr{Q}_\varepsilon$.

THEOREM 3. *Let $r(\theta)$ be any continuous nonnegative function of $\theta$ in $[a, b] \subset J$, $0 < a < b < \infty$, such that $r$ is not identically zero in the interval. Then: $\inf_{W \in \mathscr{E}_\varepsilon} \sup_{a \leqq \theta \leqq b} r(\theta) E_\theta W$ is attained by a unique (up to probability one) $L \in \mathscr{E}_\varepsilon$ defined by coordinates $c_j{}^L$ which are boundary values defining some $T \in \mathscr{R}$.*

SKETCH OF PROOF. Suppose that the closure of the support of $\nu$ is convex (otherwise minor changes must be made in the following). Denote $z_j{}^\theta = H_\theta{}^{*j}(c_j) - \alpha_j H_\theta{}^{*j}\{\{c_j\}\}$. Clearly, there is a 1 : 1 correspondence between $z_j{}^\theta$ and $(\mathbf{c}, \boldsymbol{\alpha})$, and $z_j{}^\theta$ is a continuous and increasing function of $z_j{}^0$. Denote:

$g(\theta, \mathbf{z}^0) = \sum_{j=1}^{\infty} z_j^{\theta}$. By the minimax theorem (cf. Fan (1952)) there exist $\mu_0 \in \mathscr{M}_\varepsilon$, $G_0 \in \mathscr{G}$ such that

$$\begin{aligned} &\inf_{(\mathbf{c},\alpha)\in\mathscr{C}_\varepsilon} \sup_{a\leq\theta\leq b} r(\theta)g(\theta, \mathbf{z}^0) \\ &\quad = \inf_{(\mathbf{c},\alpha)\in\mathscr{C}_\varepsilon} \sup_{G\in\mathscr{G}} \int r(\theta)g(\theta, \mathbf{z}^0)\, dG(\theta) \\ (11) \qquad &\quad \geqq \inf_{\mu\in\mathscr{M}_\varepsilon} \sup_{G\in\mathscr{G}} \int\int r(\theta)g(\theta, \mathbf{z}^0)\, dG(\theta)\, d\mu(\mathbf{z}^0) \\ &\quad = \max_{G\in\mathscr{G}} \min_{\mu\in\mathscr{M}_\varepsilon} \int\int r(\theta)g(\theta, \mathbf{z}^0)\, dG(\theta)\, d\mu(\mathbf{z}^0) \\ &\quad = \min_{\mu\in\mathscr{M}_\varepsilon} \int\int r(\theta)g(\theta, \mathbf{z}^0)\, dG_0(\theta)\, d\mu(\mathbf{z}^0)\,. \end{aligned}$$

Clearly, any $\mu \in \mathscr{M}_\varepsilon$ attaining this minimum must give all of its mass to points $\mathbf{z}^0$ for which $\int r(\theta)g(\theta, \mathbf{z}^0)\, dG_0(\theta)$ reaches its minimum. Since

$$\int r(\theta)g(\theta, \mathbf{z}^0)\, dG_0(\theta) = \int r(\theta) \sum_{j=1}^{\infty} z_j\, dG_0(\theta)$$

and

$$\frac{dz_j^{\theta}}{dz_j^{0}} = \exp\{\theta c_j - j\psi(\theta)\}$$

where $c_j$ is that corresponding to $z_j^0$, one can minimize $\int r(\theta)g(\theta, \mathbf{z}^0)\, dG_0(\theta)$ subject to the constraint $\sum_{i=1}^{\infty}(1 - z_i^0) = 1/\varepsilon$ by the method of Lagrange multipliers (cf. Luenberger (1969) page 186). By differentiating $[\int r(\theta)g(\theta, \mathbf{z}^0)\, dG_0(\theta) + \lambda(\sum_{i=1}^{\infty}(1 - z_i^0) - 1/\varepsilon)]$ with respect to $z_j^0$, one gets that extremum points $\mathbf{z}^0$ must satisfy

$$\int \exp\{\theta c_j - j\psi(\theta)\} r(\theta)\, dG_0(\theta) = \lambda \tag{12}$$

for the corresponding $\mathbf{c}$. Here $c_j$ increases if $\lambda$ is increased. So, because of the constraint, corresponding to $\lambda$ there exists a unique solution $\mathbf{c}^*$ to (12). Because of the constraint, there can be only one $\lambda$ for which a solution to (12) exists. Thus there is a unique $\lambda$ and a unique $\mathbf{c}^*$ satisfying (12), and so $\mathbf{z}^{0*}$ corresponding to $c^*$ must be the unique point of minimum. Therefore there is equality in (11) and the proof is complete.

**4. Remarks.** (a) Lemmas 1 and 2 and Theorem 2 are modeled after Lemmas 2.1 and 2.2 and Theorem 2.1 of Lorden (1967). Obviously, the condition $\psi'(a) < \psi(b)/b$ appearing in Lemma 1 (and in the sequel) can be dispensed with. Theorem 2 can be reformulated to be an analog of Theorem 2.1 of Lorden (1967).

(b) Under certain conditions one can show that $\lambda$ of (12) is of the order of magnitude of $\varepsilon$ as $\varepsilon \to \infty$.

(c) Theorems and lemmas similar to these presented in this article can be formulated for Brownian motion; denote standard Brownian motion by $\omega(t)$ and set $X(t) = \theta t + \omega(t)$, let $I(\theta) = \theta^2/2$, let

$$T_{Qc} = \inf\left\{t \mid \int_a^b \exp\{yS_t - t\psi(y)\}\, dG(y) \geqq \frac{\gamma}{1-\gamma}\frac{1-Qc}{Qc}\right\}$$

replace $T_{Qc}$ of (6) and let $c$ be the cost of sampling per unit time if $\theta \in [a, b]$. An invariance argument leads to an analog of Theorem 3.

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