

## ON WEAK ADMISSIBILITY OF TESTS

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This paper lays the foundation for a general theory of weakly admissible tests, where a test consists of both an experiment (a  $\sigma$ -field of information) and a decision function based on the experiment. A test is weakly admissible unless there is another test with as good a decision function, which uses no more information, and is strictly better in one of these respects. Weakly admissible tests are completely characterized. In the case where the experiment consists of a sequential sampling procedure, a method for constructing such tests is given.

**1. Introduction.** A sequential test is weakly admissible unless there is another test which has no greater error probabilities, never requires more observations and is strictly better in one of these respects. Clearly, weak admissibility is the minimal requirement for a good test. Eisenberg, Ghosh and Simons (1976) show, for arbitrary underlying distributions on the data, that a sequential probability ratio test is always weakly admissible. This follows, of course, from the Wald-Wolfowitz optimality theorem when it is applicable. On the other hand, Neyman-Pearson tests are not always weakly admissible. This paper characterizes weak admissibility. Roughly speaking, a test is weakly admissible if it is a sensible likelihood ratio test which takes no unnecessary observations.

The theory of weakly admissible tests can be developed at a useful level of abstraction that is somewhat removed from the trappings of sequential analysis. Rather than discussing tests  $(N, D)$  where  $N$  is a stopping time and  $D$  is a terminal decision (measurable over the  $\sigma$ -field  $\mathcal{E}_N$  of "events up to time  $N$ "), we shall discuss tests  $(\mathcal{E}, D)$  where  $\mathcal{E}$  is an "experiment" (a  $\sigma$ -field which represents the information available to a decision maker) and  $D$  is a "decision based on the experiment" (an  $\mathcal{E}$ -measurable mapping from the basic space  $\Omega$  into the action space).<sup>2</sup> The decision maker must choose an experiment within a class  $E$  of permissible experiments and then choose an appropriate decision for the experiment chosen. In the case of general sequential tests,  $E$  would be the class of  $\sigma$ -fields  $\mathcal{E}_N$ , where  $N$  is a stopping time.

Properties of decisions are discussed in Section 2. The notion of level  $k$  equivalence of decisions is introduced here. This concept plays an important role in the general theory of hypothesis testing. In particular, it appears in a general version of the Neyman-Pearson lemma, appearing in the section. Many results concerned with the concept are proved throughout the paper.

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<sup>2</sup> A similar use of the term "experiment" has been made by Rényi (1970, page 3).

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A formal definition of weakly admissible tests is given in Section 3. Theorem 6 says that, under general conditions, such tests may be characterized as "curtailed," "proper" likelihood ratio tests. Theorems 7, 8 and 9 (appearing in Section 4) show, under mild assumptions, that a proper likelihood ratio test can always "be curtailed," and the resulting "curtailed version" is a proper likelihood ratio test as well.

Section 5 applies our general theory to several contexts arising in the area of sequential analysis with the bulk of the attention given to tests which permit general (unrestricted) sequential sampling. A method for constructing and identifying the curtailed version of a likelihood ratio test is described.

Generalizations of our theory beyond the context of two competing simple hypotheses could be considered but are not in this paper.

**2. On making a good decision for a given experiment.** Probability measures  $P$  and  $Q$  defined on a measurable space  $(\Omega, \mathcal{F})$  represent two states of nature. The problem is to decide whether the "null state"  $P$  or the "alternative state"  $Q$  is the true state of nature. The decision maker must choose an experiment  $\mathcal{E}$  from a family  $\mathbf{E}$  of possible experiments, and on the basis of the information obtained from this experiment, he must make a decision  $D$  in favor of  $P$  or  $Q$ . Formally, an experiment is a sub- $\sigma$ -field of  $\mathcal{F}$  and a decision, for a given experiment  $\mathcal{E}$ , is an  $\mathcal{E}$ -measurable mapping from the basic space  $\Omega$  into the set  $\{P, Q\}$ . In accordance with conventional terminology,  $\alpha = P(D = Q)$  denotes the size of  $D$  and  $1 - \beta$ , where  $\beta = Q(D = P)$ , denotes the power of  $D$ . Also,  $\alpha$  and  $\beta$  are called error probabilities. Occasionally, they will be written as a vector  $(\alpha, \beta)$ . Each pair  $(\mathcal{E}, D)$ ,  $\mathcal{E} \in \mathbf{E}$ , is called a test. Throughout the paper,  $R$  denotes the probability measure  $\frac{1}{2}(P + Q)$ , which clearly dominates both  $P$  and  $Q$ . The use of  $R$  not only makes many statements more concise but simplifies arguments as well.

Associated with each experiment  $\mathcal{E}$ , there is an extended nonnegative random variable  $\lambda$ , determined up to an  $R$  equivalence, which is  $\mathcal{E}$ -measurable and satisfies the equations

$$(1) \quad Q(E, \lambda \neq \infty) = \int_E \lambda dP, \quad E \in \mathcal{E}.$$

$\lambda$  is called the likelihood ratio for  $\mathcal{E}$ .<sup>3</sup> It always exists and satisfies the reverse relationships

$$P(E, \lambda \neq 0) = \int_E \lambda^{-1} dQ, \quad E \in \mathcal{E},$$

where  $\lambda^{-1} = 0$  when  $\lambda = \infty$ .

For the remainder of this section,  $\mathcal{E}$  will be a fixed experiment,  $\lambda$  will be its likelihood ratio and all decisions will be decisions for  $\mathcal{E}$ .

<sup>3</sup> The reader is referred to Eisenberg, Ghosh and Simons (1976) for details about this notion beyond those described in the present paper. Essentially,  $\lambda$  is the Radon-Nikodym derivative  $dQ/dP$  relative to  $\mathcal{E}$ .

Two decisions,  $D$  and  $D'$ , will be called *equivalent* if

$$(2) \quad R(D = D') = 1,$$

and will be called *level  $k$  equivalent* (for  $\mathcal{E}$ ),  $k \in [0, \infty]$ , if

$$(3) \quad R(D \neq D', \lambda \neq k) = 0$$

and

$$(4) \quad (\alpha, \beta) = (\alpha', \beta'),$$

where these vectors represent the error probabilities of  $D$  and  $D'$  respectively.<sup>4</sup> We will denote level  $k$  equivalence by  $D =_k D'$  when the experiment  $\mathcal{E}$  is understood from the context. When there is a possibility of confusion, we will write  $D =_k D'$  for  $\mathcal{E}$ . It is easily checked that equivalent and level  $k$  equivalent decisions possess the reflexive, symmetric and transitive properties expected of an equivalence.

A decision will be called a (*level  $k$  likelihood ratio decision* (for  $\mathcal{E}$ ) if it is equivalent to a decision which equals  $P$  when  $\lambda < k$  and equals  $Q$  when  $\lambda > k$ . (Its value when  $\lambda = k$  is unspecified in the definition.) If  $D =_k D'$  and  $D$  is a level  $k$  likelihood ratio decision, then so is  $D'$ .

We are now in a position to state a quite general version of the:

**NEYMAN-PEARSON LEMMA.** *If there is a level  $k$  likelihood ratio decision  $D$  which is a most powerful decision of size  $\alpha$ , then every other most powerful decision of size  $\alpha$  is level  $k$  equivalent to  $D$ . In order that there exist such a decision  $D$  for some  $k \in [0, \infty]$ , it is sufficient that  $P$  has no atoms in  $\mathcal{E}$ .*

An assumption that  $P$  has no atoms in  $\mathcal{E}$  is quite mild and is more convenient in the present context than the usual (and equally mild) assumption that randomized decisions are permissible. In point of fact, both assumptions nearly amount to the same thing. Two observations should clarify the situation: (i) Suppose that  $U$  is a random variable which, under  $P$  and  $Q$ , is uniformly distributed on  $[0, 1]$  and is independent of  $\mathcal{E}$ . Then the information represented by the  $\sigma$ -field  $\sigma(U)$  can be used, together with the information in  $\mathcal{E}$ , to make "randomized decisions" relative to  $\mathcal{E}$ . But the likelihood ratio  $\lambda'$  for the experiment  $\mathcal{E}' = \mathcal{E} \vee \sigma(U)$  (the smallest  $\sigma$ -field containing  $\mathcal{E}$  and  $\sigma(U)$ ) will be identical to  $\lambda$ . It follows that a likelihood ratio decision with randomization would just be a likelihood ratio decision relative to  $\mathcal{E}'$ . Hence, there is little reason not to replace  $\mathcal{E}$  by  $\mathcal{E}'$  in the first place; the class of possible decisions is increased, but no unfair statistical advantage is conferred. (ii) If the  $\sigma$ -field  $\mathcal{E}$  is large enough to support a uniformly distributed random variable under  $P$ , then  $P$  is nonatomic on  $\mathcal{E}$  (and conversely).<sup>5</sup>

<sup>4</sup> The level of a decision should not be confused with the *significance* level of a test. The latter terminology will not be used.

<sup>5</sup> It is possible to develop the results of this paper using critical functions instead of decisions. For a given experiment  $\mathcal{E}$ , a critical function would be an  $\mathcal{E}$ -measurable function taking values in  $[0, 1]$  and representing the probability of deciding  $Q$ . Two critical functions  $\phi$  and  $\phi'$  are called level  $k$  equivalent if  $R(\phi \neq \phi', \lambda \neq k) = 0$  and  $\int \phi dR = \int \phi' dR$ .

It cannot be inferred from the Neyman–Pearson lemma that every likelihood ratio decision is a most powerful decision of its size. This is not quite true. Neither would it be correct to conclude that every likelihood ratio decision is a least size decision for its power. The problem is that a level infinity likelihood ratio decision can foolishly choose  $P$  when  $\lambda = \infty$ , and a level zero likelihood ratio decision can foolishly choose  $Q$  when  $\lambda = 0$ . A decision  $D$  will be called *proper (for  $\mathcal{E}$ )* if

$$R(D = Q, \lambda = 0) = R(D = P, \lambda = \infty) = 0.$$

Since  $P(\lambda = \infty) = Q(\lambda = 0) = 0$ , it is easy to show that a given error probability of a proper likelihood ratio decision can be improved upon (reduced) by another decision only at the expense of the other error probability. In particular, every proper likelihood ratio decision is a most powerful decision of its size. Consequently, the Neyman–Pearson lemma yields the following theorem.

**THEOREM 1.** *If  $D$  is a proper likelihood ratio decision of level  $k$  and  $D'$  is a decision with no larger error probabilities, then  $D' =_k D$ .*

The proofs of the Neyman–Pearson lemma and Theorem 1, which rely on equation (1), vary only slightly from standard arguments and, hence, will be omitted.

Level  $k$  equivalence is implicit in standard treatments of hypothesis testing, where a single experiment is being considered. There, it is a simple notion needing little elaboration. In a sequential context, the notion is more complicated.

It should be emphasized that the notions of “level  $k$  equivalence,” “likelihood ratio decision” and “proper decision” are defined only within the context of a specific experiment. For instance,

$$D =_k D' \text{ for } \mathcal{E} \not\Rightarrow D =_k D' \text{ for } \mathcal{E}',$$

even if the decisions  $D$  and  $D'$  are decisions for  $\mathcal{E}'$  (i.e., are  $\mathcal{E}'$ -measurable).

**EXAMPLE 1.** Let  $X_1$  and  $X_2$  be independent Bernoulli variables under  $P$  and  $Q$  with  $P(X_1 = 1) = Q(X_1 = 1) = \frac{1}{2}$ , and  $P(X_2 = 1) = Q(X_2 = 0) = \frac{1}{3}$ . Further, let  $D = Q$  iff  $X_1 = 0$  and  $D' = Q$  iff  $X_1 = 1$ . Then  $D =_1 D'$  for  $\mathcal{E} = \sigma(X_1)$  but  $D \neq_1 D'$  for  $\mathcal{E}' = \sigma(X_1, X_2)$ .  $\square$

In contrast to this example, there are several results which can be very roughly interpreted as saying that a notion holds with respect to  $\mathcal{E}'$  if it holds with respect to  $\mathcal{E}$ , and  $\mathcal{E}' \subseteq \mathcal{E}$ . The next three theorems give three examples where this rough statement can be made precise. Their relevance will be apparent in the next section, where we will be concerned with the possibility and the consequences of replacing a given experiment ( $\sigma$ -field)  $\mathcal{E}$  by a smaller one.

**THEOREM 2.** *Suppose  $D =_k D'$  for  $\mathcal{E}$  and  $\mathcal{E}' \subseteq \mathcal{E}$ . Then  $D =_k D'$  for  $\mathcal{E}'$  providing only that  $D$  and  $D'$  are  $\mathcal{E}'$ -measurable (making them decisions for  $\mathcal{E}'$ ).*

**THEOREM 3.** *Suppose  $D$  is a level  $k$  likelihood ratio decision for  $\mathcal{E}$ . If  $D'$  is a decision for  $\mathcal{E}' \subseteq \mathcal{E}$  and  $D' =_k D$  for  $\mathcal{E}$ , then  $D'$  is a level  $k$  likelihood ratio decision for  $\mathcal{E}'$ .*

**THEOREM 4.** *Suppose  $D$  is a proper decision for  $\mathcal{E}$ . If  $D'$  is a decision for  $\mathcal{E}' \subseteq \mathcal{E}$  and  $D' =_k D$  for  $\mathcal{E}$  and some  $k \in [0, \infty]$ , then  $D'$  is a proper decision for  $\mathcal{E}'$ .*

The proofs of Theorems 2 and 3 depend on the following:

**LEMMA.** *Let  $\mathcal{E}$  and  $\mathcal{E}'$  be two experiments with likelihood ratios  $\lambda$  and  $\lambda'$ , respectively, and  $k \in [0, \infty]$ . If  $\mathcal{E}' \subseteq \mathcal{E}$ , then  $R(E) = 0$  for every  $E \in \mathcal{E}'$  for which  $E \subseteq [\lambda \leq k < \lambda'] \cup [\lambda \geq k > \lambda']$ .*

**REMARK.** The statement of this lemma is made awkward by the fact that the event  $E^* = [\lambda \leq k < \lambda'] \cup [\lambda \geq k > \lambda']$  may not be in  $\mathcal{E}'$  and  $R(E^*) > 0$  is possible. This is apparent from Example 1 if the roles of  $\mathcal{E}$  and  $\mathcal{E}'$  are reversed.

**PROOF.** Without loss of generality, it can be assumed that  $E \subseteq [\lambda \leq k < \lambda']$  or  $E \subseteq [\lambda \geq k > \lambda']$ . (If it is not, consider  $E_1 = E[\lambda' > k]$  and  $E_2 = E[\lambda' < k]$ .) We shall only consider the case  $E \subseteq [\lambda \leq k < \lambda']$  since the other case can be handled similarly. If  $P(E) > 0$ , then  $k < \infty$  and (using (1))

$$\begin{aligned} 0 < \int_E (\lambda' - k) dP &\leq \int_E (\lambda' - \lambda) dP = Q(E, \lambda' \neq \infty) - Q(E, \lambda \neq \infty) \\ &\leq Q(E) - Q(E) = 0, \end{aligned}$$

a contradiction. Thus  $P(E) = 0$ . In turn, if  $R(E) > 0$ , then  $k < \infty$  and (using (1))

$$0 < Q(E) = Q(E, \lambda < \infty) = \int_E \lambda dP = 0,$$

a contradiction. Thus  $R(E) = 0$ .  $\square$

As immediate consequences of the lemma, we have for any event  $A \in \mathcal{E}'$ :  
(i)  $R(A, \lambda > k) = 0 \Rightarrow R(A, \lambda' > k) = 0$ , (ii)  $R(A, \lambda < k) = 0 \Rightarrow R(A, \lambda' < k) = 0$ , and (iii)  $R(A, \lambda \neq k) = 0 \Rightarrow R(A, \lambda' \neq k) = 0$ . By setting  $A = [D \neq D']$  in (iii), one finds that condition (3) for  $\mathcal{E}$  implies condition (3) for  $\mathcal{E}'$ , and Theorem 2 follows. The proof of Theorem 3 requires the setting of  $A = [D' = P]$  in (i) and  $A = [D' = Q]$  in (ii).

**PROOF OF THEOREM 4.** If  $k \neq \infty$ ,  $R(D' = P, \lambda' = \infty) \leq R(D' = P, \lambda = \infty) = R(D = P, \lambda = \infty) = 0$ . If  $k = \infty$ ,  $R(D' = P, \lambda' = \infty) = R(D' = P) - R(D' = P, \lambda' < \infty) \leq R(D' = P) - R(D' = P, \lambda < \infty) = R(D = P) - R(D = P, \lambda < \infty) = R(D = P, \lambda = \infty) = 0$ . Thus  $R(D' = P, \lambda' = \infty) = 0$ . Similarly,  $R(D' = Q, \lambda' = 0) = 0$ , and hence,  $D'$  is a proper decision for  $\mathcal{E}'$ .  $\square$

It is easily seen from Theorem 3 that (under the assumptions of the theorem)

$$(5) \quad R(k \text{ is strictly between } \lambda \text{ and } \lambda') = 0.$$

For the purposes of Section 5, we shall establish a slightly stronger result:

**THEOREM 5.** *Assume  $D$  is a level  $k$  likelihood ratio decision for  $\mathcal{E}$  and there exists*

a decision  $D'$  for  $\mathcal{E}' \subseteq \mathcal{E}$  with  $D' =_k D$  for  $\mathcal{E}$ . Then

$$R(\lambda' \leq k < \lambda) = R(\lambda < k \leq \lambda') = 0.$$

In particular,  $\lambda = k$  on the set  $[\lambda' = k]$  except on an  $R$ -null set.

PROOF. We shall only show that  $R(\lambda < k \leq \lambda') = 0$  since the proof that  $R(\lambda' \leq k < \lambda) = 0$  is similar. By using (3) (since  $D' =_k D$  for  $\mathcal{E}$ ), our task can be reduced to that of showing  $R(E, \lambda < k) = 0$ , where  $E = [D' = P, \lambda' \geq k]$ ; and by using (1), our task can be reduced further to that of showing  $P(E, \lambda < k) = 0$ . Since  $P(\lambda' \geq \infty) = 0$ , we may take  $k$  to be finite, in which case a double application of (1) yields

$$kP(E) \leq \int_E \lambda' dP \leq Q(E) = \int_E \lambda dP + Q(E, \lambda = \infty).$$

But (3) implies  $Q(E, \lambda = \infty) = 0$  and  $P(E, \lambda > k) = 0$ . Hence,

$$\int_{E[\lambda < k]} (\lambda - k) dP = \int_E (\lambda - k) dP = \int_E \lambda dP - kP(E) \geq 0,$$

which implies  $P(E, \lambda < k) = 0$ .  $\square$

**3. Weakly admissible tests.** Some of the terminology introduced in Section 2 to describe decisions will be adapted here, without formal announcement, to the descriptions of tests. For example,  $(\mathcal{E}, D)$  will be called a *likelihood ratio test* if  $D$  is a likelihood ratio decision for  $\mathcal{E} \in \mathbf{E}$ .

Let  $(\mathcal{E}, D)$  be a test with error probabilities  $(\alpha, \beta)$ , and let  $(\mathcal{E}', D')$ , with error probabilities  $(\alpha', \beta')$ , denote a general competitor to  $(\mathcal{E}, D)$ . The test  $(\mathcal{E}, D)$  is called *weakly admissible* (relative to  $\mathbf{E}$ ) unless there exists a test  $(\mathcal{E}', D')$  for which

$$(6) \quad \alpha' \leq \alpha, \quad \beta' \leq \beta, \quad \mathcal{E}' \subseteq \mathcal{E},$$

and which is strictly better in one of these respects, i.e.,  $\alpha' < \alpha$  or  $\beta' < \beta$  or  $\mathcal{E}'$  is a proper subset of  $\mathcal{E}$ . Actually, it is undesirable to distinguish between two experiments whose sets are comparable up to a  $P$  and  $Q$  equivalence. Therefore, in order to avoid problems with the definition of weak admissibility given here, we shall insist that each of the experiments  $\mathcal{E} \in \mathbf{E}$  is complete under the probability measure  $R = \frac{1}{2}(P + Q)$ . That is, for each  $\mathcal{E} \in \mathbf{E}$  and  $E \in \mathcal{E}$ , if  $F \in \mathcal{F}$  and  $R(E \Delta F) = 0$ , then  $F \in \mathcal{E}$ . (Alternatively, one could replace the experiments appearing in (6) by their completions under  $R$ . But this would needlessly complicate the notation and theory which follow.) With this convention, if  $(\mathcal{E}, D)$  is a test and  $D'$  is equivalent to  $D$ , then  $D'$  is  $\mathcal{E}$ -measurable. For this reason, we will no longer distinguish between equivalent decisions; equivalent decisions will be interpreted as equal.

A test  $(\mathcal{E}, D)$  cannot be weakly admissible if there is another test  $(\mathcal{E}', D')$  with  $D'$  equal to  $D$  and  $\mathcal{E}' \in \mathbf{E}$  strictly included in  $\mathcal{E}$ . To be weakly admissible it is therefore necessary that a test be *minimal*. That is, there must be no  $\mathcal{E}'$  in  $\mathbf{E}$  with  $\mathcal{E}'$  strictly included in  $\mathcal{E}$  and such that  $D$  is measurable over  $\mathcal{E}'$ . However, even a minimal and proper likelihood ratio test need not be weakly admissible as the next example demonstrates.

EXAMPLE 2. Let  $X_1$  and  $X_2$  be i.i.d. Bernoulli variables under  $P$  and  $Q$  with means  $\frac{1}{3}$  and  $\frac{2}{3}$ , respectively. Further, let  $\mathbf{E} = \{\mathcal{E}', \mathcal{E}\}$  where  $\mathcal{E}' = \sigma(X_1)$  and  $\mathcal{E} = \sigma(X_1, X_2)$ . Then  $(\mathcal{E}, D)$  is a minimal and proper likelihood ratio test when  $D$  is the decision which chooses  $Q$  iff  $X_2 = 1$ . However,  $(\mathcal{E}, D)$  is not weakly admissible because the test  $(\mathcal{E}', D')$  is better, where  $D'$  is the decision which chooses  $Q$  iff  $X_1 = 1$ .  $\square$

This example suggests that there is a more stringent property than that of minimality which every weakly admissible test must satisfy. A test  $(\mathcal{E}, D)$  is said to be *curtailed* (relative to  $\mathbf{E}$ ) if there is no other test  $(\mathcal{E}', D')$  with  $\mathcal{E}' \in \mathbf{E}$  strictly included in  $\mathcal{E}$  and with  $D' =_k D$  (for  $\mathcal{E}$ ) for some  $k \in [0, \infty]$ . A curtailed test is minimal, but the converse is not true.

THEOREM 6. *Curtailed proper likelihood ratio tests are weakly admissible. Conversely, a weakly admissible test  $(\mathcal{E}, D)$  is a curtailed proper likelihood ratio test if  $P$  is nonatomic on  $\mathcal{E}$ .*

PROOF. If  $(\mathcal{E}, D)$  is a proper likelihood ratio test of level  $k$  and  $(\mathcal{E}', D')$  is any (competing) test which satisfies (6), then by Theorem 1,  $D' =_k D$  for  $\mathcal{E}$ , and hence  $\alpha' = \alpha$  and  $\beta' = \beta$ . If, in addition,  $(\mathcal{E}, D)$  is curtailed, then  $\mathcal{E}'$  cannot be a proper subset of  $\mathcal{E}$ . Thus  $(\mathcal{E}, D)$  is weakly admissible. Conversely, if  $(\mathcal{E}, D)$  is weakly admissible, it obviously is curtailed. Now consider the test  $(\mathcal{E}', D')$ , where  $\mathcal{E}' = \mathcal{E}$  and  $D' = P, D$  or  $Q$  as  $\lambda$  (the likelihood ratio for  $\mathcal{E}$ ) = 0,  $\lambda \in (0, \infty)$  or  $\lambda = \infty$ , respectively. Since  $P(\lambda = \infty) = Q(\lambda = 0) = 0$ , the test  $(\mathcal{E}', D')$  satisfies (6). Since  $(\mathcal{E}, D)$  is weakly admissible,  $\alpha' = \alpha$  and  $\beta' = \beta$ . Then

$$\begin{aligned} R(D = P, \lambda = \infty) &= Q(D = P, \lambda = \infty) = \alpha - Q(D = P, \lambda < \infty) \\ &= \alpha' - Q(D' = P, \lambda < \infty) = Q(D' = P, \lambda = \infty) = 0. \end{aligned}$$

Similarly,  $R(D = Q, \lambda = 0) = 0$ , and hence  $(\mathcal{E}, D)$  is a proper test. Suppose, in addition, that  $P$  is nonatomic on  $\mathcal{E}$ . Then by the Neyman-Pearson lemma (described herein), there is a most powerful likelihood ratio decision  $D'$  for  $\mathcal{E}$  (not to be confused with any previous  $D'$ ), which has the same size as that for  $D$ . That is,  $\alpha' = \alpha$  and  $\beta' \leq \beta$ . Consequently (6) holds if  $\mathcal{E}'$  is set equal to  $\mathcal{E}$ . Since  $(\mathcal{E}, D)$  is weakly admissible,  $\beta'$  must equal  $\beta$ . Thus  $D'$  and  $D$  are both most powerful decisions of size  $\alpha$  (for  $\mathcal{E}$ ). The desired conclusion that  $D$  is a likelihood ratio decision (for  $\mathcal{E}$ ) then follows from the Neyman-Pearson lemma.  $\square$

If randomized tests are allowed, more precisely, if there is a uniformly distributed random variable over every  $\mathcal{E}$  in  $\mathbf{E}$  so that  $P$  is nonatomic on  $\mathbf{E}$ , then Theorem 6 gives a complete characterization of weakly admissible tests.

**4. Curtailed tests and curtailed versions of a test.** While it is a completely trivial matter, in most cases, to check whether a test is a proper likelihood ratio test, it is not always so easy to determine whether a test is curtailed. For

instance, it is obvious that every sequential probability ratio test is a proper likelihood ratio test, but it is not obvious that all such tests are curtailed (with respect to the class of experiments that can be realized through sequential sampling). Nevertheless, this must be the case since Eisenberg, Ghosh and Simons (1976) show that every sequential probability ratio test is weakly admissible. This leads us to raise the following related questions: When is a test curtailed? When can a noncurtailed test be replaced by a curtailed improvement? Will the improvement be unique? If the noncurtailed test is a proper likelihood ratio test, will the improvement be weakly admissible?

A curtailed test (minimal test)  $(\mathcal{E}', D')$  will be called a curtailed version (minimal version) of the test  $(\mathcal{E}, D)$  if  $\mathcal{E}' \subseteq \mathcal{E}$  and  $D' =_k D$  for  $\mathcal{E}$  and some  $k \in [0, \infty]$  (if  $\mathcal{E}' \subseteq \mathcal{E}$  and  $D'$  is equal to  $D$ ). In Example 2, the test  $(\mathcal{E}', D')$  is a curtailed version of  $(\mathcal{E}, D)$ , while each of these two tests is a minimal version of itself.

Because of the importance of the notion of weak admissibility, it is apparent from the previous theorem that a study of curtailed versions should have priority over a study of minimal versions. But since a minimal version is sometimes a curtailed version, we shall first discuss an elementary theorem about minimal versions and then turn our attention to curtailed versions.

**THEOREM 7.** *If  $\mathbf{E}$  is closed under the formation of intersections, then every test  $(\mathcal{E}, D)$  has a unique minimal version.*

**PROOF.** Let  $\mathcal{E}'$  be the intersection of experiments in  $\mathbf{E}$  for which  $D$  is a decision (i.e., for which  $D$  is measurable). Then  $\mathcal{E}' \in \mathbf{E}$ , and  $D$  is  $\mathcal{E}'$ -measurable. Clearly,  $(\mathcal{E}', D)$  is a minimal version of  $(\mathcal{E}, D)$  and no other minimal version is possible.  $\square$

**COROLLARY 7.** *If  $(\mathcal{E}, D)$  is a level  $k$  proper likelihood ratio test and  $R(\lambda = k) = 0$  (where  $\lambda$  is the likelihood ratio for  $\mathcal{E}$ ), then the minimal version promised by Theorem 7 is the unique curtailed version of  $(\mathcal{E}, D)$ .*

**PROOF.** Any candidate  $(\mathcal{E}'', D')$  for the role of being a curtailed version of  $(\mathcal{E}, D)$  must have the same error probabilities as those for  $(\mathcal{E}, D)$ , and, hence, according to Theorem 1, must have  $D' =_k D$  for  $\mathcal{E}$ . This condition, under the assumption  $R(\lambda = k) = 0$ , implies  $D' = D$ . Thus only the unique minimal version  $(\mathcal{E}', D)$ , found in the proof of Theorem 7, can be a curtailed version. The same kind of reasoning shows that  $(\mathcal{E}', D)$  is curtailed, which implies that it is, in fact, a curtailed version of  $(\mathcal{E}, D)$ .  $\square$

The situation when  $R(\lambda = k) > 0$  is more complicated. To prove the existence of a curtailed version of even a proper likelihood ratio test requires the additional (mild) assumption that  $P$  is nonatomic on every experiment in  $\mathbf{E}$ . Even then, there may be more than one curtailed version of the test. Fortunately, there is little reason to be concerned with the existence of curtailed versions for tests which are not proper likelihood ratio tests; so the next theorem is adequate, as far as it goes.



**THEOREM 8.** *Suppose  $(\mathcal{E}, D)$  is a proper likelihood ratio test of level  $k$  and  $R(\lambda = k) > 0$ . If the class  $\mathbf{E}$  is closed under the formation of intersections and  $P$  is nonatomic on (each experiment of)  $\mathbf{E}$ , then  $(\mathcal{E}, D)$  has a curtailed version but (in general) there is no assurance that it is unique.*

**PROOF.** Let  $\mathbf{E}_0$  denote the class of experiments  $\mathcal{E}'$  in  $\mathbf{E}$  for which there is a decision  $D'$  that is level  $k$  equivalent to  $D$  for  $\mathcal{E}$ . A particular pair  $(\mathcal{E}', D')$  is a curtailed version of  $(\mathcal{E}, D)$  if  $\mathcal{E}'$  is a minimal member of  $\mathbf{E}_0$  in the sense of set inclusion. For if  $(\mathcal{E}'', D'')$  is such that  $\mathcal{E}'' \subseteq \mathcal{E}'$  and  $D'' =_k D$  for  $\mathcal{E}'$  and some  $k'$ , then  $D'' =_k D$  for  $\mathcal{E}$ , on account of Theorem 1, and, hence  $\mathcal{E}'' \in \mathbf{E}_0$ . It follows that  $(\mathcal{E}', D')$  is curtailed and, therefore, a curtailed version of  $(\mathcal{E}, D)$ . Now  $\mathbf{E}_0$  possesses at least one minimal member, according to Zorn's lemma, if every "chain" in  $\mathbf{E}_0$  possesses a "lower bound" in  $\mathbf{E}_0$ . That is, if  $\Gamma$  is a linearly order index set and  $\{\mathcal{E}_\gamma, \gamma \in \Gamma\}$  is a nonincreasing collection of experiments in  $\mathbf{E}_0$ , then it must be shown that  $\mathcal{E}_0 = \bigcap_{\gamma \in \Gamma} \mathcal{E}_\gamma$  belongs to  $\mathbf{E}_0$ , i.e., there is an  $\mathcal{E}_0$ -measurable decision  $D_0$  that is level  $k$  equivalent to  $D$ . Briefly, this can be demonstrated as follows: Let  $C_\gamma$  denote the class of  $\mathcal{E}_\gamma$ -measurable "critical functions"  $\phi$  (cf. footnote number 5) which assume their values in  $[0, 1]$ , are zero on  $[\lambda < k]$  and one on  $[\lambda > k]$ , and which have the same size as  $D$ , i.e.,  $\int \phi dP = P(D = Q)$ . Since there is some decision  $D_\gamma$  for  $\mathcal{E}_\gamma$  which is level  $k$  equivalent to  $D$  for  $\mathcal{E}$ ,  $C_\gamma$  contains the indicator function  $\phi = I_{[D_\gamma=Q]}$ , and, hence, is nonempty for every  $\gamma \in \Gamma$ . Let  $C_0 = \bigcap_{\gamma \in \Gamma} C_\gamma$ , an intersection of non-increasing sets.  $C_0$  is nonempty since, as it turns out, each  $C_\gamma$  is an  $L_\infty$ -weak\*-compact set (relative to  $L_1(\Omega, \mathcal{F}, R)$ ), a closed subset of the  $L_\infty$ -weak\*-compact unit ball. Thus  $C_0$  contains a critical function  $\phi_0$ , which, clearly, must be  $\mathcal{E}_0$ -measurable. Since, by assumption,  $P$  is nonatomic on  $\mathcal{E}_0$ ,  $\phi_0$  can be taken to be (replaced by) an indicator function  $I_{[D_0=Q]}$  with  $D_0 =_k D$  for  $\mathcal{E}$ . Thus  $\mathcal{E}_0 \in \mathbf{E}_0$  as required.

The next example shows that the curtailed version  $(\mathcal{E}', D')$  guaranteed by Zorn's lemma does not have to be unique. A less artificial (though more complicated) example appears implicitly in Case 3 of Example 5 below.  $\square$

**EXAMPLE 3.** Let  $X_1$  and  $X_2$  be independent Bernoulli variables with common means  $p = \frac{1}{3}$  under  $P$  and  $p = \frac{2}{3}$  under  $Q$ . Further, let  $U$  be a uniform variable on  $[0, 1]$  which is independent of  $X_1$  and  $X_2$  under  $P$  and  $Q$ . Set  $\mathcal{E} = \sigma(X_1, X_2, U)$ ,  $\mathcal{E}'_1 = \sigma(X_1, U)$ ,  $\mathcal{E}'_2 = \sigma(X_2, U)$ ,  $\mathcal{E}'' = \sigma(U)$  and  $\mathbf{E} = \{\mathcal{E}, \mathcal{E}'_1, \mathcal{E}'_2, \mathcal{E}''\}$ .  $P$  is nonatomic on  $\mathbf{E}$  and  $\mathbf{E}$  is closed under the formation of intersections. Consider the test  $(\mathcal{E}, D)$  which chooses  $P$  if  $X_1 = 0$  and  $Q$  if  $X_1 = 1$ . This is a proper likelihood ratio test of level one. It is easily checked that  $(\mathcal{E}'_1, D'_1)$  and  $(\mathcal{E}'_2, D'_2)$  are each curtailed versions of  $(\mathcal{E}, D)$ , where  $D'_1 \equiv D$  and  $D'_2 = P$  or  $Q$  as  $X_2 = 0$  or 1, respectively.  $\square$

**THEOREM 9.** *A curtailed version of a proper level  $k$  likelihood ratio test is weakly admissible and a proper level  $k$  likelihood ratio test.*

**PROOF.** This follows immediately from Theorems 3, 4 and 6.  $\square$

*A summary.* Our theory has brought us through a series of steps which can be summarized as follows: Start with any test  $(\mathcal{E}, D)$ . By the Neyman-Pearson lemma, there is a likelihood ratio test  $(\mathcal{E}, D')$  (in most cases) which does as well as  $(\mathcal{E}, D)$ . If  $(\mathcal{E}, D')$  is not already a proper test, there is a simple modification  $(\mathcal{E}, D'')$  which is a proper likelihood ratio test and an improvement upon  $(\mathcal{E}, D')$ . If  $(\mathcal{E}, D'')$  is not curtailed and Theorem 8 is applicable (which it usually is in practice), then a further improvement  $(\mathcal{E}, D''')$ , a curtailed version of  $(\mathcal{E}, D'')$ , is possible. Moreover, according to Theorem 9,  $(\mathcal{E}, D''')$  is a proper likelihood ratio test and a weakly admissible test. Thus, through a series of steps one (usually) can obtain a weakly admissible improvement of any test.

We complete this section with an interesting (though very artificial) example which shows that Theorem 8 would be false if the assumption that  $P$  is non-atomic on  $E$  were dropped.

**EXAMPLE 4.** Let  $X_1, X_2, \dots$  be independent uniform variables on  $[0, 1]$  under both  $P$  and  $Q$ . Set  $E = \{\mathcal{E}_n : 1 \leq n \leq \infty\}$ , where  $\mathcal{E}_n, 1 \leq n \leq \infty$ , is the closure under  $R$  of  $\sigma(X_n, X_{n+1}, \dots)$ , and  $\mathcal{E}_\infty = \bigcap_{n=1}^{\infty} \mathcal{E}_n$  is the closure of the tail  $\sigma$ -field of the  $X_i$ 's. Clearly,  $E$  is closed under the formation of intersections, but  $P$  is atomic on  $\mathcal{E}_\infty$  because of Kolmogorov's zero-one law. Also,  $\lambda_n \equiv 1$  is the likelihood ratio of  $\mathcal{E}_n, 1 \leq n \leq \infty$ . Consider the level  $k = 1$  proper likelihood ratio test  $(\mathcal{E}_1, D)$ , where  $D = P$  or  $Q$  as  $X_1 \leq \frac{1}{2}$  or  $> \frac{1}{2}$ , respectively. It has error probabilities  $\alpha = \beta = \frac{1}{2}$ . Clearly there is no curtailed version of this test: the only possible candidates are of the form  $(\mathcal{E}_n, D')$  with  $n < \infty$ , and all such tests can be improved upon by increasing  $n$  and modifying  $D'$  in an obvious way.<sup>6</sup>  $\square$

**5. Applications to sequential tests.** In applications, one is usually concerned with a nested sequence of experiments  $\mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \dots$ , where, typically,  $\mathcal{E}_n$  represents the "information" obtainable from a sample of size  $n$ , i.e.,  $\mathcal{E}_n = \sigma(X_1, \dots, X_n)$ , where  $X_1, \dots, X_n$  denote  $n$  observations. Fixed sample size tests take the basic form  $(\mathcal{E}_n, D)$ , while sequential tests take the basic form  $(\mathcal{E}_N, D)$ , where  $N$  is a stopping time adapted to  $\{\mathcal{E}_n, 1 \leq n \leq \infty\}$ ,  $\mathcal{E}_\infty = \bigvee_{n=1}^{\infty} \mathcal{E}_n$  (the smallest  $\sigma$ -field which includes every  $\mathcal{E}_n, n \geq 1$ ) and  $\mathcal{E}_N$  is the  $\sigma$ -field of "events up to time  $N$ ," i.e.,

$$\mathcal{E}_N = \{E \in \mathcal{F} : E[N = n] \in \mathcal{E}_n, 1 \leq n \leq \infty\}.$$

We shall not be concerned here with how the experiments  $\mathcal{E}_n$  arise, but we shall insist, as we do of experiments in the previous sections, that they are complete under the probability measure  $R$ . It follows that the experiments  $\mathcal{E}_N$  will be complete under  $R$  as well.

There are many different classes  $E$  of possible experiments that have been considered in the past, e.g.,

<sup>6</sup> There clearly does exist an  $\mathcal{E}_\infty$ -measurable critical function  $\phi$  (cf. the previous footnote) which has the right error probabilities, and it would be a curtailed version of  $\mathcal{E}_1$  (in an obvious sense) were we to work with critical functions rather than decisions.

- (i) *fixed sample size experiments*:  $\mathbf{E} = \{\mathcal{E}_n, 1 \leq n < \infty\}$ ,
- (ii) *two-stage experiments*:  $\mathbf{E} = \{\mathcal{E}_N\}$ , where  $N$  is  $\mathcal{C}_m$ -measurable for some  $m \geq 1$  (depending on  $N$ ) and  $N \geq m$ ,
- (iii) *general sequential experiments*:  $\mathbf{E} = \{\mathcal{E}_N\}$ , where  $R(N < \infty) = 1$ ,
- (iv) *truncated sequential experiments*:  $\mathbf{E} = \{\mathcal{E}_N\}$ , where  $N \leq n$  for some  $n \geq 1$  (depending on  $N$  or on the class  $\mathbf{E}$  itself),
- (v) *experiments for power one tests*:  $\mathbf{E} = \{\mathcal{E}_N\}$ , where  $Q(N < \infty) = 1$ .

The tests referred to in (v) have been popularized by Robbins and others. They advocate using the decision  $D$  which chooses  $Q$  when  $N < \infty$  and  $P$  when  $N = \infty$  (i.e., the decision maker acts as if he believes  $P$  is correct until the experiment stops, if ever). The resulting test  $(\mathcal{E}_N, D)$  has power  $Q(D = Q) = Q(N < \infty) = 1$  and size  $\alpha = P(N < \infty)$ , which depends on how  $N$  is chosen.

The assumption in Theorems 7 and 8 that  $\mathbf{E}$  is closed under the formation of intersections holds for all of the classes (i)–(v) except (ii), and even for that class Theorem 8's conclusion, namely that curtailed versions of proper likelihood ratio tests always exist, can be sustained. (The assumption that  $P$  is nonatomic on  $\mathcal{E}$  is still required. Also, the conclusion of Theorem 7 is valid for the class, except for the uniqueness of the minimal version, which seems unlikely.) Thus the problem of existence of curtailed versions is a relatively minor one. The problem of finding them can be more difficult.

Let us turn to this problem within the specific context of general sequential experiments (case (iii)). Theorem 5 points to a partial solution: Let  $(\mathcal{E}_N, D)$  be a fixed general sequential test which (from now on) we suppose to be a proper likelihood ratio test of level  $k$ . Further, let  $\lambda_N$  denote the likelihood ratio for  $\mathcal{E}_N$  and  $\lambda_n$  denote the likelihood ratio for  $\mathcal{E}_n, n \geq 1$ . Finally, let

$$\begin{aligned} N' &= \text{the first } n \geq 1 \text{ such that } R^{\mathcal{E}_n}(\lambda_n \leq k < \lambda_N) \\ &= R^{\mathcal{E}_n}(\lambda_N < k \leq \lambda_n) = 0. \end{aligned}$$

It is readily seen that  $N'$  is a well-defined stopping time and, in particular,  $N' \leq N$  a.s. ( $R$ ). In fact,  $N' \leq M$  a.s. ( $R$ ) for every stopping time  $M$  such that

$$(7) \quad R(\lambda_M \leq k < \lambda_N) = R(\lambda_N < k \leq \lambda_M) = 0.$$

For on  $[M = n]$ ,

$$R^{\mathcal{E}_n}(\lambda_n \leq k < \lambda_N) = R^{\mathcal{E}_n}(\lambda_M \leq k < \lambda_N) = 0 \quad \text{a.s.}$$

and

$$R^{\mathcal{E}_n}(\lambda_N < k \leq \lambda_n) = R^{\mathcal{E}_n}(\lambda_N < k \leq \lambda_M) = 0 \quad \text{a.s.}$$

Now suppose  $(\mathcal{E}_M, D_0)$  is any test for which  $\mathcal{E}_M \subseteq \mathcal{E}_N$  and  $D_0 =_k D$  for  $\mathcal{E}_N$ . By Theorem 5, (7) holds and, hence,  $\mathcal{E}_{N'} \subseteq \mathcal{E}_M$ . In particular, if  $(\mathcal{E}_M, D_0)$  is a curtailed (or the unique minimal) version of  $(\mathcal{E}_N, D)$ , then  $\mathcal{E}_{N'} \subseteq \mathcal{E}_M$ .

**THEOREM 10.** *The following are equivalent:*

- (i) *There exists a curtailed version of  $(\mathcal{E}_N, D)$  of the form  $(\mathcal{E}_{N'}, D)$ .*

- (ii) *There is a decision  $D'$  for  $\mathcal{E}_{N'}$ , which is level  $k$  equivalent to  $D$  for  $\mathcal{E}_N$ .*
- (iii) *There exists no curtailed version of  $(\mathcal{E}_N, D)$  not of the form  $(\mathcal{E}_{N'}, D')$ .*

PROOF. Condition (i) implies (ii) because of the definition of curtailed version. Likewise, condition (ii) implies (iii) because of the definition of a curtailed version, since every curtailed version  $(\mathcal{E}_M, D_0)$  not of the form  $(\mathcal{E}_{N'}, D')$  would have to be such that  $\mathcal{E}_{N'}$  is a *proper* subset of  $\mathcal{E}_M$ . (See the last sentence preceding this theorem.) Finally, condition (iii) implies (i) because, according to Corollary 7 and Theorem 8,  $(\mathcal{E}_N, D)$  must have a curtailed version.  $\square$

THEOREM 11. *If  $R(\lambda_N = k) = 0$ , then  $D$  is  $\mathcal{E}_{N'}$ -measurable and  $(\mathcal{E}_{N'}, D)$  is the unique curtailed version of  $(\mathcal{E}_N, D)$ .*

PROOF. Let  $(\mathcal{E}_M, D)$  denote the unique curtailed version of  $(\mathcal{E}_N, D)$  referred to in Corollary 7. It must be shown that  $\mathcal{E}_{N'} = \mathcal{E}_M$ . The last sentence preceding Theorem 10 points out that  $\mathcal{E}_{N'} \subseteq \mathcal{E}_M$ . It easily follows from the definition of  $N'$  that  $R(\lambda_{N'} \leq k < \lambda_N) = R(\lambda_N < k \leq \lambda_{N'}) = 0$ . Since  $R(\lambda_N = k) = 0$ , it follows that  $R(\lambda_{N'} = k) = 0$ , and consequently, that  $D$  is equivalent to the  $\mathcal{E}_{N'}$ -measurable decision  $D'$  which equals  $P$  on  $[\lambda_{N'} < k]$ , and  $= Q$  on  $[\lambda_{N'} > k]$ . Thus  $(\mathcal{E}_{N'}, D)$  is a test and  $\mathcal{E}_M \subseteq \mathcal{E}_{N'}$ , as required. For otherwise  $(\mathcal{E}_M, D)$  would not be curtailed.  $\square$

There is a less compact way of describing the stopping time  $N'$ , which is somewhat easier to intuit. Since

$$R^{\mathcal{E}_n}(\lambda_n \leq k < \lambda_N) = I_{[\lambda_n \leq k]} R^{\mathcal{E}_n}(\lambda_N > k) \quad \text{and}$$

$$R^{\mathcal{E}_n}(\lambda_N < k \leq \lambda_n) = I_{[\lambda_n \geq k]} R^{\mathcal{E}_n}(\lambda_N < k),$$

(where  $I_A$  denotes the indicator function for an event  $A$ ), due to the smoothing properties of conditional expectations, it follows that  $N'$  is the first integer  $n \geq 1$  such that (a)  $\lambda_n > k$  and  $R^{\mathcal{E}_n}(\lambda_N < k) = 0$ , or (b)  $\lambda_n < k$  and  $R^{\mathcal{E}_n}(\lambda_N > k) = 0$ , or (c)  $\lambda_n = k$  and  $R^{\mathcal{E}_n}(\lambda_N \neq k) = 0$ . For the experimenter, case (a) can be interpreted as saying: If sampling has continued up to time  $n$  and it is found that (i)  $\lambda_n > k$  and (ii)  $\lambda_N$  will not be less than  $k$ , then stop. Cases (b) and (c) have similar interpretations.

In the case  $R(\lambda_N = k) > 0$ , Theorem 10 leaves open the possibility that there is no curtailed version of  $(\mathcal{E}_N, D)$  of the form  $(\mathcal{E}_{N'}, D')$ . Unfortunately, this "possibility" can be realized. The following example illustrates a basic complication that can occur: For a given experiment  $\mathcal{E}_N$ , the class of experiments  $\mathcal{E}_M$  which admit a curtailed version  $(\mathcal{E}_M, D_0)$  of  $(\mathcal{E}_N, D)$  depends on which level  $k$  likelihood ratio decision  $D$  is being considered. This is significant, for it should be remembered that the stopping time  $N'$  is defined with respect to  $k$ , not  $D$ . It also will be apparent that the complication cannot be removed by simply admitting randomized decisions.

EXAMPLE 5. Let  $\mathcal{E}_n = \sigma(X_1, \dots, X_n, U)$  ( $n \geq 1$ ), where the random variables  $X_1, X_2, \dots$  are independent Bernoulli variables with common means equal to

$\frac{1}{3}$  and  $\frac{2}{3}$  under  $P$  and  $Q$ , respectively, and  $U$  is a uniform variable on  $[0, 1]$  which is independent of the  $X_i$ 's under  $P$  and  $Q$  (that is, available for making "randomized decisions," if desired). Further let  $N \equiv 4$ . Then, for  $k = 1$ ,

$$\begin{aligned} N' &= 2 && \text{if } X_1 + X_2 = 0 \text{ or } 2, \\ &= 3 && \text{if } X_1 + X_2 = 1. \end{aligned}$$

Suppose  $D = P$  when  $\lambda_4 < 1$  and  $D = Q$  when  $\lambda_4 > 1$ , so that  $D$  is a likelihood ratio decision for  $\mathcal{E}_N$  of level  $k = 1$ . We shall illustrate three different situations that can arise depending on how  $D$  is defined on  $[\lambda_4 = 1]$ .

CASE I:  $D = Q$  on  $[\lambda_4 = 1, U \leq \frac{1}{2}]$ . Here, there is a unique curtailed version of the form  $(\mathcal{E}_{N'}, D')$ . Either  $\lambda_{N'} < 1$ , in which case  $D' = P$  is mandated, or  $\lambda_{N'} > 1$ , in which case  $D' = Q$  is mandated.

CASE II:  $D = Q$  on  $[\lambda_4 = 1]$ . Here there is a unique curtailed version  $(\mathcal{E}_M, D_0)$ , and it is not of the form  $(\mathcal{E}_{N'}, D')$ .

$$\begin{aligned} M &= 2 && \text{if } X_1 + X_2 = 2, \\ &= 3 && \text{if } X_1 + X_2 = 1 \text{ and } X_3 = 1, \\ &= 3 && \text{if } X_1 + X_2 = 0 \text{ and } X_3 = 0, \\ &= 4 && \text{if } X_1 + X_2 = 0 \text{ and } X_3 = 1, \end{aligned}$$

and  $D_0 = Q$  on  $[\lambda_M \geq 1]$ . Obviously,  $\mathcal{E}_M$  is strictly larger than  $\mathcal{E}_{N'}$ . The decision  $D_0$  is actually the same as  $D$ , and, hence,  $(\mathcal{E}_M, D_0)$  is also the unique minimal version of  $(\mathcal{E}_N, D)$ .

CASE III:  $D = Q$  on  $[\lambda_4 = 1, U \leq \frac{2}{3}]$ . Here, there is no curtailed version  $(\mathcal{E}_M, D_0)$  of the form  $(\mathcal{E}_{N'}, D')$ , but, in contrast to the previous case, there is more than one choice for  $(\mathcal{E}_M, D_0)$ . Two possible choices for  $M$  are as follows:

*First Choice*

$$\begin{aligned} M &= 2 && \text{if } X_1 + X_2 = 0 \text{ or } 2, \\ &= 3 && \text{if } X_1 + X_2 = 1 \text{ and } X_3 = 1, \\ &= 3 && \text{if } X_1 + X_2 = 1, X_3 = 0 \text{ and } U \leq \frac{1}{2}, \\ &= 4 && \text{if } X_1 + X_2 = 1, X_3 = 0 \text{ and } U > \frac{1}{2}. \end{aligned}$$

*Second Choice*

$$\begin{aligned} M &= 2 && \text{if } X_1 + X_2 = 2, \\ &= 3 && \text{if } X_1 + X_2 = 1, \\ &= 3 && \text{if } X_1 + X_2 = 0 \text{ and } X_3 = 0, \\ &= 4 && \text{if } X_1 + X_2 = 0 \text{ and } X_3 = 1. \end{aligned}$$

For both choices,  $D_0 = Q$  on  $[\lambda_M \geq 1]$  is mandated.  $\square$

We can offer no algorithm which finds a curtailed version  $(\mathcal{E}_M, D_0)$  for  $(\mathcal{E}_N, D)$  in every possible situation.

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