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ASYMPTOTIC DISTRIBUTION OF AN ESTIMATOR OF THE BOUNDARY PARAMETER OF AN UNSTABLE PROCESS[1]

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The limit distribution of the least squares estimator $\hat{\alpha}$ of the parameter α of the first order stochastic difference equation, in the boundary case $|\alpha| = 1$, is presented. With this, the asymptotic distributional problem for any real α in the first order case is completely settled.

1. Introduction. Let $\{X_t, t \geqq 1\}$ be a time series generated by the stochastic difference equation:

$$X_t = \alpha X_{t-1} + \varepsilon_t\,, \qquad t \geqq 1,\ \alpha \in \mathbb{R}\,, \tag{1}$$

where $X_t = 0$ for $t \leqq 0$, and $\{\varepsilon_t, t \geqq 1\}$ are independent identically distributed random variables, on a probability space, with mean zero and variance $\sigma^2 > 0$. Here α is an unknown parameter, and the least squares estimator $\hat{\alpha}_N$ of α is given by

$$\hat{\alpha}_N = \textstyle\sum_{t=1}^N X_t X_{t-1} / \sum_{t=1}^N X_{t-1}^2\,. \tag{2}$$

Then it is known that $p \lim_N \hat{\alpha}_N = \alpha$ (consistency). The next important problem is to obtain the asymptotic distribution of $\hat{\alpha}_N$. Thus if $s(n) = |\alpha|^n/(\alpha^2 - 1)$ for $|\alpha| > 1$, $= n/2^{\frac{1}{2}}$ for $|\alpha| = 1$, and $[n(1-\alpha^2)]^{-\frac{1}{2}}$ for $|\alpha| < 1$, then $s(n)(\hat{\alpha}_n - \alpha)$ has a limit distribution if $|\alpha| < 1$ which is normal with mean zero and variance 1 (i.e., $N(0, 1)$), and if $|\alpha| > 1$ then again this limit exists but *it depends on the distribution of* ε_t's. In particular if ε_t's are $N(0, 1)$, then the last limit distribution is Cauchy. These results were proved under essentially the present generality by Anderson ([1], Theorems 2.5, 2.7, 4.3) and under the normality assumption ($|\alpha| > 1$) by White [9]. The earlier important study (for $|\alpha| < 1$) of Mann–Wald in 1943, and the consistency problem by Rubin in 1950 (for $\alpha \in \mathbb{R}$, cf. references in [9]), should also be recalled. Both the consistency and limit distribution for $|\alpha| > 1$, in exactly the present generality, are contained in [7] (Theorems I and II).

The result in the boundary case $|\alpha| = 1$ needs a quite different method (as seen in the next section) compared to the other cases. A preliminary study of this was made in [9]. The purpose of this note is to present a solution of this boundary parameter problem which has been open from at least 1958.

2. The result. The desired result may be given as follows.

THEOREM. *Let* $\{X_t, t \geqq 1\}$ *be a process generated by* (1) *where the* $\{\varepsilon_t, t \geqq 1\}$

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are independent, identically and symmetrically distributed with means zero and variance one. Let $|\alpha| = 1$, and $\hat{\alpha}_N$ be given by (2). Then

$$\lim_{N\to\infty} P[s(N)(\hat{\alpha}_N - \alpha) < x] = \int_{-\infty}^{x} h(u)\, du\,, \tag{3}$$

where the density function $h(\cdot)$ of the limit distribution is given by:

$$h(x) = (8\pi^2)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \frac{\rho(x,t)}{[r(x,t)]^{\frac{3}{2}}} \cos\left(\delta(x,t) - \tfrac{3}{2}\theta(x,t)\right) \times [\chi_{R_+^2}(x,t) + \chi_{R_+^2}(-x,-t)] \frac{dt}{(tx)^{\frac{1}{2}}}\,. \tag{4}$$

Here ρ, r, δ, θ are defined by the following expressions:

$$r(x,t)^2 = \sinh^2(2tx)^{\frac{1}{2}} + \cos^2(2tx)^{\frac{1}{2}} + \frac{t}{2x}\left(\sinh^2(2tx)^{\frac{1}{2}} + \sin^2(2tx)^{\frac{1}{2}}\right) + \frac{\alpha}{2}\left(\frac{t}{x}\right)^{\frac{1}{2}}\left(\sin(8tx)^{\frac{1}{2}} - \sinh(8tx)^{\frac{1}{2}}\right), \tag{5}$$

$$\theta(x,t) = \arctan\left[\frac{1 - (\alpha/2)(t/x)^{\frac{1}{2}}(\coth(2tx)^{\frac{1}{2}} + \cot(2tx)^{\frac{1}{2}})}{1 - (\alpha/2)(t/x)^{\frac{1}{2}}(\tanh(2tx)^{\frac{1}{2}} - \tan(2tx)^{\frac{1}{2}})} \times \tan(2tx)^{\frac{1}{2}}\tanh(2tx)^{\frac{1}{2}}\right], \tag{6}$$

$$\rho(x,t)^2 = 2\left(1 - \frac{\alpha}{(8x^2)^{\frac{1}{2}}}\right)^2 \left(\sinh^2(2tx)^{\frac{1}{2}} + \sin^2(2tx)^{\frac{1}{2}}\right) + \frac{t}{x}\left(\sinh^2(2tx)^{\frac{1}{2}} + \cos^2(2tx)^{\frac{1}{2}}\right) - \alpha(t/x)^{\frac{1}{2}}\left(1 - \alpha/(8x^2)^{\frac{1}{2}}\right)\left(\sin(8tx)^{\frac{1}{2}} + \sinh(8tx)^{\frac{1}{2}}\right), \tag{7}$$

$$\delta(x,t) = \arctan\left(\frac{C\cos(2)^{\frac{1}{2}}\alpha t - D\sin(2)^{\frac{1}{2}}\alpha t}{C\sin(2)^{\frac{1}{2}}\alpha t + D\cos(2)^{\frac{1}{2}}\alpha t}\right), \tag{8}$$

where

$$C = \left(1 - \frac{\alpha}{(8x^2)^{\frac{1}{2}}}\right)\left(\sinh(2tx)^{\frac{1}{2}}\cos(2tx)^{\frac{1}{2}} - \cosh(2tx)^{\frac{1}{2}}\sin(2tx)^{\frac{1}{2}}\right) + \alpha(t/x)^{\frac{1}{2}}\sinh(2tx)^{\frac{1}{2}}\sin(2tx)^{\frac{1}{2}}\,, \tag{9}$$

$$D = \left(1 - \frac{\alpha}{(8x^2)^{\frac{1}{2}}}\right)\left(\sinh(2tx)^{\frac{1}{2}}\cos(2tx)^{\frac{1}{2}} + \cosh(2tx)^{\frac{1}{2}}\sin(2tx)^{\frac{1}{2}}\right) - \alpha(t/x)^{\frac{1}{2}}\cosh(2tx)^{\frac{1}{2}}\cos(2tx)^{\frac{1}{2}}\,.$$

REMARK. This formidable expression is presented in the hope that it could be used for a numerical evaluation in some problems. More particularly, it may be of interest in considering the kth order equation ($k > 1$) with some of the roots of its characteristic equation lying on the (boundary of the) unit circle. After presenting the proof, simple modifications of the argument will be pointed out whereby the assumption of symmetry for the common distribution of ε_t's may be suppressed in the above theorem.

If $|\alpha| < 1$, the process $\{Y_t, t \geqq 1\}$ of (1) is called *stable*, $|\alpha| = 1$ it is *unstable* and $|\alpha| > 1$ it is *explosive*. An excellent account of the stable problem can be found in ([2], Section 5.5) while a study of the explosive case may be seen in [9], [1], [7], and [8], among others.

3. Proof of the result. From (1) and (2) one obtains

$$\hat{\alpha}_N - \alpha = \sum_{t=1}^N \varepsilon_t X_{t-1} / \sum_{t=1}^N X_{t-1}^2 = U_N/V_N \,, \tag{10}$$

where U_N and V_N stand for the numerator and denominator sums respectively. Since (1) also implies $X_t = \sum_{t=0}^{t-1} \alpha^i \varepsilon_{t-i} = \sum_{i=0}^{t-1} u_{t-i}$, with $u_{t-i} = \alpha^i \varepsilon_{t-i}$, one may conclude that the u_i are independent and identically distributed with mean zero and variance one by using the symmetry of the distribution of ε_t's. This and a theorem of Erdös and Kac [5] yield that

$$\lim_{N\to\infty} P\left[\frac{2}{N^2} V_N < x\right] = F(x) \,, \qquad F(0) = 0 \,, \tag{11}$$

where the characteristic function ξ of F is given by $\xi(u) = (\sec (4iu)^{\frac{1}{2}})^{\frac{1}{2}}$. Similarly $U_N = \sum_{t=1}^N (X_t - \alpha X_{t-1})X_{t-1}$ and since ε_t and $\alpha^i \varepsilon_t$ are identically distributed (by the symmetry assumption) one can deduce from the same considerations of [5] leading to the *invariance principle*, that

$$\lim_{N\to\infty} P\left[\frac{2^{\frac{1}{2}}}{N} U_N < y\right] = G(y) \tag{12}$$

exists. In fact, using Donsker's result ([4], Theorem 4.4), one deduces that the vector $((2^{\frac{1}{2}}/N)U_N, (2/N^2)V_N)$ converges in distribution, as $N \to \infty$, to a function which is independent of the distribution of ε_t's (or of the u_t's) and the limit distribution is that of a suitable functional of the Brownian motion process. In the present case, this functional is as follows. Let H_1, H_2 be the functionals defined by (the first one is the (Itô) stochastic integral, the second pointwise Lebesgue):

$$H_1(B) = \textstyle\int_0^1 B(t)\, dB(t) \,, \qquad H_2(B) = \int_0^1 B^2(t)\, dt \tag{13}$$

where $\{B(t), t \in [0, 1]\}$ is the Brownian motion, and let a_1, a_2 be real numbers. Then, on observing that U_N, V_N are both Borel functions of $X_t = \sum_{i=0}^{t-1} u_{t-i}$, the partial sum of the desired sequence of independent identically distributed random variables (for [4]), it follows that

$$\lim_{N\to\infty} P\left[a_1 \frac{2^{\frac{1}{2}}}{N} U_N + \frac{2a_2}{N^2} V_N < a_1 x + a_2 y\right] = G_\alpha(a_1 x + a_2 y) \,, \tag{14}$$

where $G_\alpha(\cdot)$ is the distribution of $a_1 H_1(B) + (a_1 \alpha - a_1 + a_2)H_2(B)$. The point here is that the left side limit of (14) exists for each vector (a_1, a_2) which then implies the earlier statement. Next to calculate G_α, following the method of [5], one chooses a convenient distribution of ε_t's (hence of u_t's) and determines the characteristic function of the limit distribution G_α.

The obvious choice of the distribution of ε_t's is that they be $N(0, 1)$ so u_t's also have the same distribution. Thus if $\varphi_N(\cdot, \cdot)$ is the characteristic function defined by

$$\varphi_N(t, u) = E\left(\exp\left[\frac{2^{\frac{1}{2}} it}{N} U_N + \frac{2iu}{N^2} V_N\right]\right) ,$$

then it is easy to calculate φ_N. In fact this was done in [9] and one finds that $\varphi_N \to \varphi$ (as this must obtain by the preceding paragraph and the easy half of the continuity theorem for characteristic functions). Moreover,

$$\varphi(t, u) = e^{2^{\frac{1}{2}}\alpha it}\left(\cos 2(iu)^{\frac{1}{2}} - \frac{\alpha it}{(2iu)^{\frac{1}{2}}}\sin 2(iu)^{\frac{1}{2}}\right)^{-\frac{1}{2}}. \tag{15}$$

Observe that $\varphi(o, u)$ is precisely $\xi(u)$ of (11), as calculated in [5]. If (U, V) is the limit random vector whose characteristic function is (15) then it is clear that $V > 0$ with probability 1. (Observe that $(2^{\frac{1}{2}}/N)U_N \to U$ in distribution, and U has the shifted gamma distribution.) Further, one finds, after an elementary but somewhat tedious computation, that $|\varphi(t, u)| \to 0$ as $|t| \to \infty$ and $|u| \to \infty$. Hence (by the Riemann–Lebesgue lemma) one concludes that the joint distribution of (U, V) is of continuous type. Since φ is differentiable (several times) this limit distribution has at least two moments finite.

From the preceding facts, one can apply Cramér's theorem ([3], page 317, Exercise 6, which is based on his 1937 theorem noted there), the random variable U/V has the density given by

$$h(x) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\partial\varphi}{\partial u}(t, -tx)\,dt, \tag{16}$$

if the integral converges uniformly in x. To see that this $h(\cdot)$ is the desired density, let $W_N^x = (U_N/s(N)) - x(V_N/s^2(N))$ and $W^x = U - xV$. Then

$$E(e^{i\tau W_N^x}) = \varphi_N(\tau, -x\tau) \to \varphi(\tau, -x\tau) = E(e^{i\tau W^x}), \qquad \text{as} \quad N \to \infty. \tag{17}$$

Moreover,

$$\begin{aligned} \lim_{N\to\infty} P[s(N)(\hat{\alpha}_N - \alpha) < x] &= \lim_{N\to\infty} P[W_N^x < 0] \\ &= P[W^x < 0] = P\left[\frac{U}{V} < x\right]. \end{aligned} \tag{18}$$

Then Cramér's theorem implies that $h(x) = (d/dx)P[W^x < 0]$, and so it remains to establish (16), which is the essence of the proof.

Thus differentiating $\varphi(\cdot, \cdot)$ and simplifying one gets

$$\begin{aligned} &\frac{\partial\varphi}{\partial u}(t, -tx) \\ &= \frac{e^{2^{\frac{1}{2}}\alpha it}}{(8tx)^{\frac{1}{2}}}\frac{[(2i)^{\frac{1}{2}}\sinh 2(itx)^{\frac{1}{2}}(1 - \alpha/(8x^2)^{\frac{1}{2}}) - \alpha i(t/x)^{\frac{1}{2}}\cosh 2(itx)^{\frac{1}{2}}]}{[\cosh 2(itx)^{\frac{1}{2}} - \alpha(it/2x)^{\frac{1}{2}}\sinh 2(itx)^{\frac{1}{2}}]^{\frac{3}{2}}}. \end{aligned} \tag{19}$$

Next observe that $i^{\frac{1}{2}} = (1 + i)/2^{\frac{1}{2}}$ and then substituting (19) in (16) with $\beta = (2tx)^{\frac{1}{2}}$,

$$\begin{aligned} h(x) &= \frac{1}{4\pi i(2x)^{\frac{1}{2}}}\int_{-\infty}^{\infty}\frac{e^{2^{\frac{1}{2}}\alpha it}}{t^{\frac{1}{2}}} \\ &\quad\times\frac{[(1+i)(1-\alpha/(8x^2)^{\frac{1}{2}})\sinh(1+i)\beta - \alpha i(t/x)^{\frac{1}{2}}\cosh(1+i)\beta]}{[\cosh 2(1+i)\beta - (\alpha/2)(t/x)^{\frac{1}{2}}(1+i)\sinh(1+i)\beta]^{\frac{3}{2}}}. \end{aligned} \tag{20}$$

Using the identities $\sinh(\beta + i\beta) = i\cosh\beta\sin\beta + \sinh\beta\cos\beta$, and $\cosh(\beta + i\beta) = \cosh\beta\cos\beta + i\sinh\beta\sin\beta$, and simplifying (20), one finds, after a straightforward but long computation, the following:

$$h(x) = \frac{1}{2^{\frac{5}{2}}\pi}\int_{-\infty}^{\infty} \frac{F - iE}{(A + iB)^{\frac{3}{2}}} \frac{dt}{(tx)^{\frac{1}{2}}}, \tag{21}$$

where A, B, E, F are given by:

$$A = \cosh\beta\cos\beta - \frac{\alpha}{2}\left(\frac{t}{x}\right)^{\frac{1}{2}}(\sinh\beta\cos\beta - \cosh\beta\sin\beta),$$

$$B = \sin\beta\sinh\beta - \frac{\alpha}{2}\left(\frac{t}{x}\right)^{\frac{1}{2}}(\sin\beta\cosh\beta + \cos\beta\sinh\beta),$$

$$E = C\cos 2^{\frac{1}{2}}\alpha t - D\sin 2^{\frac{1}{2}}\alpha t,$$

$$F = C\sin 2^{\frac{1}{2}}\alpha t + D\cos 2^{\frac{1}{2}}\alpha t,$$

with C and D as in (9).

It is now necessary to consider (i) $x \geqq 0$ and (ii) $x < 0$ separately on the t-intervals $(-\infty, 0)$ and $[0, \infty)$. This is done as follows:

CASE (i). $x \geqq 0$. If $t \geqq 0$, then $\beta \geqq 0$ so that A, B, C, D, E, F are real. Let $F + iE = \rho e^{i\delta}$, $A + iB = re^{i\theta}$, so that $\rho^2 = C^2 + D^2 = E^2 + F^2$, and $r^2 = A^2 + B^2$. Also $\delta = \arctan(E/F)$, $\theta = \arctan(B/A)$ and then ρ, r, θ, δ are seen to reduce to (5)—(8), when $\alpha^2 = 1$ is used. If $t = -\tau < 0$ $(\tau > 0)$, then it is noted that $(A + iB)(-\tau) = \overline{(A + iB)}(\tau)$, $C(-\tau) = -iC(\tau)$, $D(-\tau) = iD(\tau)$ so that $(F - iE)(-\tau) = iF(\tau) - E(\tau)$. With these reductions (since $x \geqq 0$) (21) reduces to the following:

$$\begin{aligned} h(x) &= \frac{1}{2^{\frac{5}{2}}\pi}\left[\int_0^\infty \frac{F - iE}{(A + iB)^{\frac{3}{2}}} \frac{dt}{(tx)^{\frac{1}{2}}} + \int_0^\infty \frac{\overline{F - iE}}{\overline{(A + iB)^{\frac{3}{2}}}} \frac{d\tau}{(\tau x)^{\frac{1}{2}}}\right] \\ &= \frac{1}{(8\pi^2)^{\frac{1}{2}}}\int_0^\infty \frac{\rho}{r^{\frac{3}{2}}}\cos\left(\delta - \frac{3\theta}{2}\right)\frac{dt}{(tx)^{\frac{1}{2}}}. \end{aligned} \tag{22}$$

CASE (ii). $x < 0$. If $t < 0$ then $\beta \geqq 0$, and A, B, C, D, E, F are again real. If $t \geqq 0$, $\tau = -t > 0$ so that $xt = -x\tau$ and the same relations obtain as in the preceding case (i.e., the second integral is the complex conjugate of the first). Thus (22) again holds.

Combining the two parts, one has

$$\begin{aligned} h(x) &= \frac{1}{8^{\frac{1}{2}}\pi}\int_0^\infty \frac{\rho(x, t)}{r(x, t)^{\frac{3}{2}}}\cos\left(\delta(x, t) - \tfrac{3}{2}\theta(x, t)\right)\frac{dt}{(tx)^{\frac{1}{2}}}, & x \geqq 0, \\ &= \frac{1}{8^{\frac{1}{2}}\pi}\int_{-\infty}^0 \frac{\rho(x, t)}{r(x, t)^{\frac{3}{2}}}\cos\left(\delta(x, t) - \tfrac{3}{2}\theta(x, t)\right)\frac{dt}{(tx)^{\frac{1}{2}}}, & x < 0. \end{aligned} \tag{23}$$

Note that ρ, r, δ, θ are *not* symmetric in x or t. Also x is not a singularity of the integrands in either parts of (23), and the integrals exist uniformly in x. Then by the important Cramér's theorem noted above, this $h(\cdot)$ must be the desired density, and (23) is just (4) in a different form. This completes the proof.

REMARKS. 1. In the above proof, the fact that ε_t's are symmetrically distributed is used in concluding that u_t's are identically distributed which was needed for Donsker's theorem, employed in (13)—(14). However, this hypothesis on u_t's was used in [4] and [5] in deducing the fact that the u_t (or the partial sums X_t's) obey the central limit theorem. For the latter, it is sufficient (as well as necessary) that the u_t obey the classical Lindeberg condition. This was used by Prokhorov ([6], Theorem 3.1) in extending Donsker's theorem for nonidentically distributed summands. Now if ε_t's are independent and identically distributed with means zero and variance one, then u_t's satisfy the Lindeberg condition, as a simple computation shows. Thus the above theorem *is true if the symmetry assumption is dropped from the distribution of* ε_t's. Also if the Lindeberg condition is violated (and the latter is true of u_t's if $|\alpha| > 1$) then by Prokhorov's result, noted above, the invariance principle fails. This observation helps in appreciating Anderson's conclusions [1].

2. The preceding analysis and an extension of Cramér's theorem to higher dimensions may lead to the solutions of the asymptotic distributional problems of $\hat{\alpha}_{Ni}$'s of [1] and [7] when some of the roots are on the unit circle. This is presently unexplored.

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