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POWER-ONE TESTS BASED ON SAMPLE SUMS[1]

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This paper studies the properties of open-ended power-one tests of H_0: $\theta \leqq \theta_0$ versus H_1: $\theta > \theta_0$ or of H: $\theta = \theta_0$ versus K: $\theta \neq \theta_0$ based on sample sums stopped at moving boundaries. The behavior of the expected sample size is analyzed and certain asymptotic results as $\theta \to \theta_0$ are obtained in the case of a location parameter and also in the case of an exponential family of distributions.

1. Introduction. Suppose $X_1, X_2, \cdots$ are i.i.d. random variables with distribution function F_θ, $\theta \in \Theta$, where Θ is an open subset of the real line. Let $\theta_0 \in \Theta$. We want to test the hypothesis H_0: $\theta \leqq \theta_0$ versus H_1: $\theta > \theta_0$. Assume that the family of distributions F_θ is stochastically increasing, and that for all $\theta > \theta_0$, $E_\theta X_1$ exists and is $> \mu$ for some real number μ. Given $0 < \alpha < 1$, let $b(n)$ be a sequence of real numbers such that

$$\text{(1.1)} \qquad \limsup_{n\to\infty} b(n)/n \leqq \mu \quad \text{and} \quad P_{\theta_0}[S_n \geqq b(n) \text{ for some } n \geqq m] \leqq \alpha\,,$$

where $S_n = X_1 + \cdots + X_n$. Consider the following test: Stop sampling at stage

$$\text{(1.2)} \qquad T = \inf\{n \geqq m: S_n \geqq b(n)\}$$

and reject H_0. (We do not reject H_0 as long as we continue sampling.) For $\theta > \theta_0$, since $E_\theta X_1 > \mu$ and $\limsup_{n\to\infty} b(n)/n \leqq \mu$, the strong law of large numbers implies that $P_\theta[\text{Reject } H_0] = P_\theta[T < \infty] = 1$, and in fact $E_\theta T < \infty$. Since the family of distribution functions F_θ is stochastically increasing, it follows from (1.1) that for $\theta \leqq \theta_0$, $P_\theta[\text{Reject } H_0] = P_\theta[T < \infty] \leqq \alpha$, i.e., the Type I error probability is $\leqq \alpha$. Hence we have a level-α power-one test of H_0 versus H_1. The expected sample size $E_\theta T$ is infinite for $\theta \leqq \theta_0$ and is a finite nonincreasing function of θ for $\theta > \theta_0$. This monotone property of $E_\theta T$ follows easily from the fact that the family of distribution functions F_θ is stochastically increasing (cf. [8]). In Sections 2 and 3, we shall study the asymptotic behavior of $E_\theta T$ as $\theta \downarrow \theta_0$.

Similar ideas as above can be used to obtain level-α power-one tests of the two-sided hypothesis H: $\theta = \theta_0$ versus K: $\theta \neq \theta_0$. Here we assume that $E_{\theta_0} X_1 = \mu$ and that for $\theta \neq \theta_0$, $E_\theta X_1$ exists and is not equal to μ. Given $0 < \alpha < 1$, let

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$b_1(n) > b_2(n)$ be two sequences of real numbers satisfying the following condition:

$$\text{(1.3)} \qquad \lim_{n\to\infty} b_i(n)/n = \mu \quad \text{for} \quad i = 1, 2 \quad \text{and}$$
$$P_{\theta_0}[b_2(n) < S_n < b_1(n) \ \text{ for all } \ n \geqq m] \geqq 1 - \alpha \, .$$

We stop sampling at stage

$$\text{(1.4)} \qquad N = \inf\{n \geqq m : S_n \notin (b_2(n), b_1(n))\}$$

and reject H when we stop. For $\theta \neq \theta_0$, since $E_\theta X_1 \neq \mu$ and $\lim_{n\to\infty} b_i(n)/n = \mu$ ($i = 1, 2,$), the strong law of large numbers implies that $P_\theta[N < \infty] = 1$, and in fact $E_\theta N < \infty$. By (1.3),

$$P_{\theta_0}[\text{Reject } \ H] = P_{\theta_0}[N < \infty] \leqq \alpha \, ,$$

and so the test has level α. The asymptotic behavior of $E_\theta N$ as $\theta \to \theta_0$ will be treated in Sections 2 and 3.

In [1], [2], [3], [4], [9], [10], [13], [14], [15], [16], by making use of sharp martingale inequalities, sequences $b(n)$ satisfying (1.1) and $b_1(n)$, $b_2(n)$ satisfying (1.3) have been constructed for certain families of distributions. In the statistical literature, open-ended tests based on sample sums as described above for the unknown parameters of various parametric families have been considered by Darling and Robbins [1], [2], [3], [4], Fabian [5], Farrell [6], Robbins [14], Robbins and Siegmund [15], [17], [18], Pollak and Siegmund [13] and Lai and Siegmund [11]. The references [4], [11] and [13] contain some Monte Carlo studies of the expected sample sizes and the error probabilities of these tests.

In [18, page 429], Robbins and Siegmund have discussed the importance of finding the asymptotic behavior of the expected sample size of a power-one test at alternatives close to θ_0 for two-sided tests of $H\colon \theta = \theta_0$ versus $K\colon \theta \neq \theta_0$ or for one-sided tests of $H_0\colon \theta \leqq \theta_0$ versus $H_1\colon \theta > \theta_0$. This kind of asymptotic behavior was first investigated by Farrell [6] who, in the case of a normalized exponential family of distributions, found the asymptotic mean (as $\theta \to \theta_0 = 0 = \mu$) of the two-sided stopping rule N as defined in (1.4) with $b_1(n) = -b_2(n) = b(n)$ and

$$\text{(1.5)} \qquad b(n) = \sigma\{2n[\log_2(n + e) + c \log_3(n + e^e)]\}^{\frac{1}{2}} \, ,$$

where $\sigma^2 = E_0 X_1^2$, $c > \frac{3}{2}$ and $\log_2$ denotes log log, etc. For the special case where $X_1, X_2, \cdots$ are i.i.d. normal random variables, Farrell's result has been extended to the one-sided stopping rule T of (1.2) by Robbins and Siegmund [17] who, besides considering the iterated-logarithm boundary of (1.5), also consider the logarithmic case $b(n) \sim (n \log n)^{\frac{1}{2}}$ such that $b(n)$ is concave and increasing.

The above results of Farrell, Robbins and Siegmund will be extended to a general class of boundaries $b(n)$ in Section 3 where we study the asymptotic behavior (as $\theta \to \theta_0$) of the expected sample size of open-ended power-one tests of H_0 versus H_1 or of H versus K in the case of a general exponential family of distributions. These results are similar to the corresponding results in the case where θ is a location parameter. The location-parameter problem, which has a simpler formulation, will be treated in Section 2 and we now briefly sketch the main results thereof. Since θ is a location parameter, we can write $X_i = Z_i + \theta$ so that the distribution of Z_i does not depend on θ. Assuming that $EZ_1 = 0$ and letting $\tilde{S}_n = Z_1 + \cdots Z_n$, the stopping rule of the power-one test of $H_0: \theta \leqq 0$ versus $H_1: \theta > 0$ as given by (1.2) can be written as

$$T(\theta) = \inf\{n \geqq m: \tilde{S}_n + n\theta \geqq b(n)\}\,. \tag{1.6}$$

Suppose that $b(t)$ is a concave, increasing, positive continuous function on $[m, \infty)$ such that $\lim_{t\to\infty} b(t)/t = 0$ and $P[\tilde{S}_n < b(n) \text{ for all } n \geqq m] > 0$. Then the equation $\theta t = b(t)$ has a unique root $t = g(\theta) > m$ for all sufficiently small positive θ, and under certain weak regularity conditions, we shall show in Section 2 that

$$\lim_{\theta\downarrow 0} ET(\theta)/g(\theta) = P[\tilde{S}_n < b(n) \quad \text{for all} \quad n \geqq m]\,. \tag{1.7}$$

Since power-one tests of $H_0: \theta \leqq 0$ versus $H_1: \theta > 0$ are of particular interest when detection of a small positive value of θ is important, it is desirable to know the expected number of observations required for such detection when θ is small, and (1.7) gives us a general asymptotic formula for evaluating this expected sample size.

The result (1.7) also suggests a new class of problems in the field of extended renewal theory and first passage times. For example, it would be interesting to compare (1.7) with corresponding results for lower-class boundaries. The particular lower-class boundary $b(n) = 0$ has been studied by Lan [12] who has shown that if $X_1, X_2, \cdots$ are i.i.d. with $EX_1 = 0$ and $EX_1^2 = \sigma^2 > 0$ and if $T^*(\theta) = \inf\{n \geqq m: S_n + n\theta \geqq 0\}$, then

$$\lim_{\theta\downarrow 0} \theta ET^*(\theta) = 2^{-\frac{1}{2}}\sigma \exp\{\textstyle\sum_{n=1}^{\infty} n^{-1}(P[S_n < 0] - \frac{1}{2})\} < \infty\,. \tag{1.8}$$

In the case of an exponential family of distributions, as will be shown in Section 3, the analogue of (1.7) becomes

$$\lim_{\theta\downarrow 0} E_\theta T/g(\mu_\theta) = P_0[T = \infty]\,, \tag{1.9}$$

where $\mu_\theta = E_\theta X_1$ and the family is normalized so that $\mu_0 = 0$. For the iterated-logarithm boundary defined by (1.5),

$$g(\mu_\theta) \sim \mu_\theta^{-2}(2\sigma^2 \log_2 \mu_\theta^{-1}) \qquad \text{as} \quad \theta \downarrow 0\,. \tag{1.10}$$

The above first-order asymptotic approximation does not involve the constant c of (1.5). In view of (1.9), (1.10) and the law of the iterated logarithm, one may suspect that power-one tests using iterated-logarithm boundaries as defined in (1.5) have asymptotically minimal (up to the first-order approximation) $E_\theta T$ as $\theta \downarrow 0$ among all tests of the form (1.2) having the same significance level. In fact, it has been shown by Farrell [6] that for any stopping rule τ such that $P_0[\tau = \infty] > 0$,

$$\limsup_{\theta\downarrow 0} \mu_\theta^2 E_\theta \tau/(2\sigma^2 \log_2 \mu_\theta^{-1}) \geqq P_0[\tau = \infty]\,. \tag{1.11}$$

While τ in (1.11) need not be of the form (1.2), it suffices, however, to restrict ourselves to the class of stopping rules of the form (1.2). In fact, from considerations of sufficiency and monotonicity for the exponential family, it can be shown that given any stopping rule τ such that $P_0[\tau = \infty] > 0$ and $P_\theta[\tau < \infty] = 1$ for $\theta > 0$, there exists a sequence $\beta(n)$ of constants such that the stopping rule $T = \inf\{n \geqq 1 : S_n \geqq \beta(n)\}$ satisfies $P_0[T = \infty] = P_0[\tau = \infty]$ and $P_\theta[T > n] \leqq P_\theta[\tau > n]$ for all $n = 1, 2, \cdots$ and $\theta > 0$. In particular $E_\theta T \leqq E_\theta \tau$ for all $\theta > 0$. (See [18, pages 424–425] and [6, pages 48–55]).

2. Power-one tests for a location parameter. The main results of this section are contained in Theorems 1, 2 and Corollary 1 and are illustrated in Example 1. Let Ψ be a known distribution function such that $\int_{-\infty}^{\infty} |x| \, d\Psi(x) < \infty$. Let $F_\theta(x) = \Psi(x - \theta)$, $\theta \in \Theta$, i.e., θ is a location parameter. The family F_θ then is stochastically increasing. Suppose we want to test $H_0 : \theta \leqq \theta_0$ versus $H_1 : \theta > \theta_0$. It suffices just to consider the case $\theta_0 = 0$ and $\int_{-\infty}^{\infty} x \, d\Psi(x) = 0$. In this case, we use the stopping rule $T = \inf\{n \geqq m : \sum_1^n X_i \geqq b(n)\}$, where $b(n)$ is a sequence of positive numbers such that $\lim_{n\to\infty} b(n)/n = 0$ and $P_0[T = \infty] > 0$. Let $Z_1, Z_2, \cdots$ be i.i.d. with distribution function Ψ and let $T(\theta) = \inf\{n \geqq m : \sum_1^n Z_i + n\theta \geqq b(n)\}$. Obviously the distribution of $T(\theta)$ is the same as the distribution of T under P_θ. As indicated in Section 1, $E_\theta T = \infty$ if $\theta \leqq 0$ and $E_\theta T$ is a finite nonincreasing function of θ for $\theta > 0$. From the definition of $T(\theta)$, it is easy to see that $E_\theta T$ is right continuous in θ for $\theta > 0$, and if $\Psi(x)$ is continuous, then $E_\theta T$ is continuous for $\theta > 0$. Considering the random variables $T(\theta)$, we observe that $T(0) = \infty$ implies that $\lim_{\theta \downarrow 0} T(\theta) = \infty$, and so $E_\theta T \geqq ET(\theta) I_{[T(0)=\infty]} \to \infty$ as $\theta \downarrow 0$ since $P_0[T = \infty] > 0$. The following theorem studies the asymptotic behavior of $E_\theta T$ as $\theta \downarrow 0$.

THEOREM 1. *Suppose $Z_1, Z_2, \cdots$ are* i.i.d. *random variables such that $EZ_1 = 0$ and $E(Z_1^+)^\nu < \infty$ for some $\nu > 1$. Let $S_n = Z_1 + \cdots + Z_n$. Let $b(t)$ be a positive continuous function on $[m, \infty)$ with $m \geqq 1$ satisfying the following conditions*:

(2.1) $$b(t) \text{ is concave, increasing and } \lim_{t\to\infty} b(t)/t = 0\,;$$

(2.2) $$\text{For } \varepsilon_0 \leqq \varepsilon < 1 \text{ (where } \varepsilon_0 \text{ is some positive number } < 1),$$
$$\beta(\varepsilon) = \lim_{t\to\infty} b(\varepsilon t)/b(t) \text{ exists and is } > \varepsilon, \text{ and } \lim_{\varepsilon\uparrow 1} \beta(\varepsilon) = 1\,;$$

(2.3) $$P[S_n < b(n) \text{ for all } n \geqq m] > 0\,;$$

(2.4) $$\lim_{t\to\infty} t^{-1/\nu} b(t) = \infty\,;$$

$$\text{For any } 0 < \rho < 1\,,$$
(2.5) $$\lim_{n\to\infty} P[S_i < b(i) - i\rho b(n)/n \text{ for all } n \geqq i \geqq m]$$
$$= P[S_i < b(i) \text{ for all } i \geqq m]\,.$$

For each $\theta > 0$ such that $\theta m < b(m)$, let $t = g(\theta)$ be the root of $\theta t = b(t)$ with $t > m$, and let $T(\theta) = \inf\{n \geqq m : S_n + n\theta \geqq b(n)\}$. Then

(2.6) $$\lim_{\theta\downarrow 0} ET(\theta)/g(\theta) = P[S_n < b(n) \text{ for all } n \geqq m]\,.$$

REMARKS. (i) If $b(t)$ is any continuous concave function on $[m, \infty)$ such that $\lim_{t\to\infty} b(t) = \infty$, then for $0 < \varepsilon < 1$, $(b(\varepsilon t) - b(m))/(\varepsilon t - m) \geqq (b(t) - b(m))/(t - m)$ and so $\liminf_{t\to\infty} b(\varepsilon t)/b(t) \geqq \varepsilon$. Condition (2.2) in Theorem 1 requires that $\lim_{t\to\infty} b(\varepsilon t)/b(t)$ actually exists and is $> \varepsilon$. If there exists $\alpha \in (0, 1)$ such that $b(t) = t^\alpha U(t) + O(t^\alpha)$, where $U(t)$ is a slowly varying function such that $\lim_{t\to\infty} U(t) = \infty$, or if $b(t) = ct^\alpha + o(t^\alpha)$, where $c > 0$, then (2.2) holds.

(ii) Suppose $Z_1, Z_2, \cdots$ are i.i.d. with $EZ_1 = 0$, $0 < EZ_1^2 < \infty$. Let $b(t)$ be a positive continuous function on $[m, \infty)$ $(m \geqq 1)$ satisfying conditions (2.1), (2.2), (2.3) of Theorem 1. Suppose further that $b(t)$ satisfies

$$\lim_{t\to\infty} b(t)/(t \log_2 t)^{\frac{1}{2}} = \infty\,. \tag{2.7}$$

Then condition (2.5) holds. To see this, we note that for $0 < \rho < 1$ and $i_0\,(> m)$ sufficiently large, if $n \geqq i \geqq i_0$, then

$$\begin{aligned} b(i) - b(m) &\geqq \{(i - m)/(n - m)\}(b(n) - b(m)) \\ &\geqq \tfrac{1}{2}(1 + \rho)ib(n)/n\,, \end{aligned}$$

and so $i\rho b(n)/n \leqq 2\rho b(i)/(1 + \rho)$ for $n \geqq i \geqq i_0$. Therefore in view of (2.7), the law of the iterated logarithm implies that given $\varepsilon > 0$, we can choose $n_0 \geqq i_0$ such that for $n > n_0$,

$$\begin{aligned} &P[S_i \geqq b(i) - i\rho b(n)/n \quad \text{for some} \quad n \geqq i \geqq n_0] \\ &\qquad \leqq P[S_i \geqq (1 - \rho)b(i)/(1 + \rho) \quad \text{for some} \quad i \geqq n_0] < \varepsilon\,. \end{aligned}$$

Since $\lim_{n\to\infty} b(n)/n = 0$, it is now clear that (2.5) holds. Noting that $EZ_1^2 < \infty$, we can set $\nu = 2$ in Theorem 1 so that (2.7) also implies (2.4).

(iii) The growth condition (2.7) in (ii) rules out the most delicate (such as given by (1.5)) among the upper-class boundaries. To show that condition (2.5) still holds for these boundaries, we shall use a more delicate argument by considering the Wiener process and using the Skorohod embedding. This argument, which will be given in detail after the proof of Theorem 1, yields the following corollary of Theorem 1.

COROLLARY 1. *Suppose $Z_1, Z_2, \cdots$ are* i.i.d. *nondegenerate random variables with $EZ_1 = 0$ and $E\{Z_1 \log_2 (|Z_1| + e)\}^2 < \infty$. Let $b(t)$ be a positive continuous function on $[m, \infty)$ belonging to the upper class so that* (2.3) *holds. Assume that $b(t)$ satisfies the regularity conditions* (2.1), (2.2) *and*

$$t^{-\frac{1}{2}}b(t) \quad \textit{is ultimately nondecreasing.} \tag{2.8}$$

Define $T(\theta)$ and $g(\theta)$ as in Theorem 1. *Then* (2.6) *holds.*

We now proceed to prove Theorem 1. First we establish the following lemma:

LEMMA 1. *Let $b(t)$ be a positive continuous function on $[m, \infty)$ $(m \geqq 1)$ such that $\lim_{t\to\infty} b(t) = \infty$ and conditions* (2.1), (2.2) *of Theorem* 1 *are satisfied. Let the function $g(\theta)$ be as defined in Theorem* 1. *Then*

(a) *$g(\theta)$ is a strictly decreasing function and* $\lim_{\theta\downarrow 0} g(\theta) = \infty$.

(b) *Let* $f\colon (0, a) \to [m, \infty)$ *where a is any positive number. Then*

$$\liminf_{\theta\downarrow 0} b(f(\theta))/(\theta f(\theta)) \geqq 1 \Rightarrow \limsup_{\theta\downarrow 0} f(\theta)/g(\theta) \leqq 1\,,$$
$$\limsup_{\theta\downarrow 0} b(f(\theta))/(\theta f(\theta)) \leqq 1 \Rightarrow \liminf_{\theta\downarrow 0} f(\theta)/g(\theta) \geqq 1\,,$$
$$b(f(\theta)) \sim \theta f(\theta) \quad as \quad \theta \downarrow 0 \Rightarrow f(\theta) \sim g(\theta) \quad as \quad \theta \downarrow 0\,.$$

(c) *Take any* $\theta^* > 0$ *such that* $\theta^* m < b(m)$. *Then there exists* $\psi\colon (0, \theta^*) \to [m, \infty)$ *such that* $\lim_{\theta\downarrow 0}\psi(\theta) = \infty$, $\lim_{\theta\downarrow 0}\psi(\theta)/g(\theta) = 0$ *and* $\lim_{\theta\downarrow 0} b(\psi(\theta))/b(g(\theta)) = 0$.

(d) *Let* $b^*(t)$ *be any positive, continuous, strictly increasing and concave function on* $[m^*, \infty)$ *with* $m^* \geqq m$ *such that* $b^*(t) \sim b(t)$ *as* $t \to \infty$. *For all* $\theta > 0$ *such that* $\theta m^* < b^*(m^*)$, *let* $t = g^*(\theta)$ *be the root of* $\theta t = b^*(t)$ *with* $t > m^*$. *Then* $g^*(\theta) \sim g(\theta)$ *as* $\theta \downarrow 0$.

(e) *There exists a function* $\gamma\colon [\rho_1, 1) \to (0, 1)$ *where* ρ_1 *is some positive number* < 1 *such that* $\limsup_{\theta\downarrow 0}\gamma(\rho)g(\rho\theta)/g(\theta) \leqq 1$ *for all* $\rho \in [\rho_1, 1)$ *and* $\lim_{\rho\uparrow 1}\gamma(\rho) = 1$.

PROOF. (a) is obvious. We shall only prove the second implication of (b), since the first implication can be proved similarly and the third implication follows from the first two implications. Suppose that $\limsup_{\theta\downarrow 0} b(f(\theta))/(\theta f(\theta)) \leqq 1$. Then $\lim_{\theta\downarrow 0} f(\theta) = \infty$ and so $\lim_{\theta\downarrow 0} b(f(\theta)) = \infty = \lim_{\theta\downarrow 0}\theta f(\theta)$. Assume that $\liminf_{\theta\downarrow 0} f(\theta)/g(\theta) < 1$. Then there exist a positive sequence $\theta_n \downarrow 0$ and $1 > \varepsilon > \varepsilon_0$ such that $m < f(\theta_n) < \varepsilon g(\theta_n)$. Using (2.2) and the concavity of $b(t)$, we obtain that

$$\begin{aligned} 1 &< \lim_{n\to\infty}\{b(\varepsilon g(\theta_n)) - b(m)\}/\{\varepsilon b(g(\theta_n)) - m\theta_n\} \\ &= \lim_{n\to\infty}\{b(\varepsilon g(\theta_n)) - b(m)\}/\{\theta_n(\varepsilon g(\theta_n) - m)\} \\ &\leqq \liminf_{n\to\infty}\{b(f(\theta_n)) - b(m)\}/\{\theta_n(f(\theta_n) - m)\}\,, \end{aligned}$$

contradicting the assumption that $1 \geqq \limsup_{\theta\downarrow 0} b(f(\theta))/(\theta f(\theta))$.

To prove (c), define $\pi(x) = \inf\{y \geqq m\colon b(y) \geqq (b(x))^{\frac{1}{2}}\}$ for $x > m$. Then $\lim_{x\to\infty}\pi(x) = \infty$ and $b(\pi(x)) = (b(x))^{\frac{1}{2}}$ for all large x. By the concavity of b, $\lim_{x\to\infty}\pi(x)/x = 0$. Let $\psi(\theta) = \pi(g(\theta))$. Then ψ satisfies the desired conclusions. To prove (d), we observe that as $\theta \downarrow 0$, $g^*(\theta) \to \infty$ and $b(g^*(\theta)) \sim b^*(g^*(\theta)) = \theta g^*(\theta)$, and so $g^*(\theta) \sim g(\theta)$ by (b).

We now prove (e). Let $\rho_0 = \varepsilon_0/\beta(\varepsilon_0) < 1$. For $\rho \in [\rho_0, 1)$, let $\bar\gamma(\rho) = \sup\{\varepsilon\colon \varepsilon_0 \leqq \varepsilon < 1$ and $\rho\beta(\varepsilon) \geqq \varepsilon\}$. Since $\lim_{\varepsilon\uparrow 1}\varepsilon/\beta(\varepsilon) = 1 > \rho$, $\bar\gamma(\rho) < 1$. It is easy to see that $\lim_{\rho\uparrow 1}\bar\gamma(\rho) = 1$. Therefore we can choose $\rho_1 \in [\rho_0, 1)$ such that for all $\rho \in [\rho_1, 1)$, $2\bar\gamma(\rho) - 1 > \varepsilon_0$. For each $\rho \in [\rho_1, 1)$, we can choose $\gamma(\rho) \in (2\bar\gamma(\rho) - 1, \bar\gamma(\rho)]$ such that $\rho\beta(\gamma(\rho)) \geqq \gamma(\rho)$. Obviously $\lim_{\rho\uparrow 1}\gamma(\rho) = 1$. By (2.2), as $\theta \downarrow 0$,

$$b(\gamma(\rho)g(\rho\theta)) \sim \beta(\gamma(\rho))b(g(\rho\theta)) = \rho\theta\beta(\gamma(\rho))g(\rho\theta) \geqq \theta\gamma(\rho)g(\rho\theta)\,,$$

and so $\limsup_{\theta\downarrow 0}\gamma(\rho)g(\rho\theta)/g(\theta) \leqq 1$ by (b). □

PROOF OF THEOREM 1. Set $P^* = P[S_i < b(i)$ for all $i \geqq m]$. We shall first show that

$$\liminf_{\theta\downarrow 0} ET(\theta)/g(\theta) \geqq P^*\,. \tag{2.9}$$

For $n > m$, define $c_n = (b(n) - b(m))/(n - m)$. Then c_n is nonincreasing and $\lim_{n\to\infty} c_n = 0$. Let ρ_1 and γ be given by Lemma 1(e). Take any $\rho \in [\rho_1, 1)$ and any sequence θ_n such that $\rho c_{n+1} \leqq \theta_n \leqq \rho c_n$. Obviously

$$\begin{aligned} P[T(\theta_n) > n] &= P[S_i + i\theta_n < b(i) \text{ for all } n \geqq i \geqq m] \\ &\leqq P[S_i < b(i) \text{ for all } n \geqq i \geqq m] = P^* + o(1) \quad \text{as} \quad n \to \infty . \end{aligned}$$

The key to our proof of (2.9) lies in the fact that the reverse inequality also holds as $n \to \infty$ so that

$$\lim_{n\to\infty} P[T(\theta_n) > n] = P^* . \tag{2.10}$$

To see this, we take $\rho' \in (\rho, 1)$ and note that for all large n,

$$\begin{aligned} P[T(\theta_n) > n] &\geqq P[S_i < b(i) - i\rho' b(n)/n \text{ for all } n \geqq i \geqq m] \\ &\to P^* \quad \text{as} \quad n \to \infty \quad \text{by condition (2.5).} \end{aligned}$$

Since $(n + 1 - m)c_{n+1} = b(n + 1) - b(m)$, Lemma 1(d) implies that $g(c_{n+1}) \sim n$. Using this and (2.10), we obtain that

$$ET(\theta_n) \geqq nP[T(\theta_n) > n] \sim g(c_{n+1})P^* \geqq (1 + o(1))\gamma(\rho)g(\theta_n)P^* . \tag{2.11}$$

To see the last inequality above, we note that $c_{n+1} \leqq \theta_n/\rho$ implies that $g(c_{n+1}) \geqq g(\theta_n/\rho)$ and apply Lemma 1(e). Therefore we have proved that for any $\rho \in [\rho_1, 1)$ and any sequence θ_n such that $\rho c_{n+1} \leqq \theta_n \leqq \rho c_n$, (2.11) holds. This implies that $\liminf_{\theta\downarrow 0} ET(\theta)/g(\theta) \geqq \gamma(\rho)P^*$. Since $\lim_{\rho\uparrow 1}\gamma(\rho) = 1$, we have established (2.9).

For $\theta > 0$, $S_{T(\theta)} + \theta T(\theta) \leqq b(T(\theta)) + X_{T(\theta)}^+ + \theta + |S_m| + m\theta$. Therefore applying Wald's lemma, we have

$$\theta ET(\theta) \leqq Eb(T(\theta)) + E^{1/\nu}(X_{T(\theta)}^+)^\nu + \theta + m(E|X_1| + \theta) . \tag{2.12}$$

Now $E(X_{T(\theta)}^+)^\nu \leqq E\sum_{i=1}^{T(\theta)} (X_i^+)^\nu = (ET(\theta))E(X_1^+)^\nu$. From condition (2.4), it follows that $\lim_{\theta\downarrow 0} b(g(\theta))/(g(\theta))^{1/\nu} = \infty$, and so $\lim_{\theta\downarrow 0}\theta(g(\theta))^{1-1/\nu} = \infty$, implying that $\lim_{\theta\downarrow 0}\theta(ET(\theta))^{1-1/\nu} = \infty$ in view of (2.9). Hence $\lim_{\theta\downarrow 0}\theta ET(\theta) = \infty$ and $E^{1/\nu}(X_{T(\theta)}^+)^\nu = O(E^{1/\nu}T(\theta)) = o(\theta ET(\theta))$. Putting this in (2.12), we obtain

$$\theta ET(\theta)(1 + o(1)) \leqq Eb(T(\theta)) . \tag{2.13}$$

Let ψ be the function constructed in Lemma 1(c). We note that

$$\begin{aligned} Eb(T(\theta)) &\leqq E[b(T(\theta)) \mid T(\theta) > \psi(\theta)]P[T(\theta) > \psi(\theta)] \\ &\qquad + b(\psi(\theta))P[T(\theta) \leqq \psi(\theta)] \\ &\leqq b(E[T(\theta) \mid T(\theta) > \psi(\theta)])P[T(\theta) > \psi(\theta)] + o(b(g(\theta))) . \end{aligned} \tag{2.14}$$

The last relation above follows from Jensen's inequality (since b is concave) and Lemma 1(c). Now $b(g(\theta)) = \theta g(\theta) = O(\theta ET(\theta))$ by (2.9). Hence letting $f(\theta) = E[T(\theta) \mid T(\theta) > \psi(\theta)]$, we obtain that

$$\begin{aligned} 1 &\leqq \liminf_{\theta\downarrow 0} b(f(\theta))P[T(\theta) > \psi(\theta)]/(\theta ET(\theta)) , \quad \text{by (2.13) and (2.14)} , \\ &\leqq \liminf_{\theta\downarrow 0} b(f(\theta))/(\theta f(\theta)) , \quad \text{since} \quad ET(\theta) \geqq f(\theta)P[T(\theta) > \psi(\theta)] . \end{aligned} \tag{2.15}$$

By Lemma 1(b), (2.15) implies that

$$\limsup_{\theta\downarrow 0} f(\theta)/g(\theta) \leqq 1 . \tag{2.16}$$

Since $\lim_{\theta\downarrow 0}\psi(\theta)/g(\theta) = 0$ by Lemma 1(c), we obtain using (2.16) that

$$\begin{aligned} ET(\theta) &\leqq \psi(\theta) + f(\theta)P[T(\theta) > \psi(\theta)] \\ &\leqq o(g(\theta)) + (1 + o(1))g(\theta)P[T(0) > \psi(\theta)] . \end{aligned} \tag{2.17}$$

Noting that $\lim_{\theta\downarrow 0} P[T(0) > \psi(\theta)] = P[T(0) = \infty] = P^*$, the desired conclusion (2.6) follows from (2.9) and (2.17). □

Proof of Corollary 1. Without loss of generality, we shall assume that $EZ_1^2 = 1$. Since b belongs to the upper class and (2.8) holds, $\lim_{t\to\infty} t^{-\frac{1}{2}}b(t) = \infty$ by the Feller-Kolmogorov integral test (cf. [7, page 132]), i.e., (2.4) holds if we set $\nu = 2$ in Theorem 1. Therefore it remains to show that condition (2.5) is also satisfied. For $0 < \rho < 1$,

$$\begin{aligned} &P[S_i \geqq b(i) - i\rho b(n)/n \ \text{ for some } \ n \geqq i \geqq m] \\ &\qquad \geqq P[S_i \geqq b(i) \ \text{ for some } \ n \geqq i \geqq m] \\ &\qquad \to P[S_i \geqq b(i) \ \text{ for some } \ i \geqq m] \quad \text{ as } \quad n \to \infty . \end{aligned} \tag{2.18}$$

For each n, let $k(n)$ be a positive integer $\geqq m$ such that

$$\lim_{n\to\infty} k(n) = \infty \qquad \text{and} \qquad \lim_{n\to\infty} k(n)b(n)/n = 0 . \tag{2.19}$$

Given $\varepsilon > 0$, since $b(t) - 1$ obviously also belongs to the upper class, we can choose n_0 such that $P[S_i \geqq b(i) - 1 \text{ for } i \geqq n_0] \leqq \varepsilon$ and $\rho b(n)k(n)/n \leqq 1$ for $n \geqq n_0$. Then for $n \geqq n_0$,

$$\begin{aligned} &P[S_i \geqq b(i) - \rho i b(n)/n \ \text{ for some } \ m \leqq i \leqq k(n)] \\ &\qquad \leqq P[S_i \geqq b(i) - \rho n_0 b(n)/n \ \text{ for some } \ m \leqq i \leqq n_0] + \varepsilon \\ &\qquad = P[S_i \geqq b(i) \ \text{ for some } \ m \leqq i \leqq n_0] + o(1) + \varepsilon \quad (\text{as } \ n \to \infty) \\ &\qquad \leqq P[S_i \geqq b(i) \ \text{ for some } \ i \geqq m] + \varepsilon + o(1) . \end{aligned} \tag{2.20}$$

In view of (2.18) and (2.20), it suffices to show that

$$\lim_{n\to\infty} P[S_i \geqq b(i) - i\rho b(n)/n \ \text{ for some } \ k(n) \leqq i \leqq n] = 0 . \tag{2.21}$$

Since $EZ_1 = 0$, $EZ_1^2 = 1$ and $E\{Z_1 \log_2(|Z_1| + e)\}^2 < \infty$, by the Skorohod embedding theorem (cf. [7, page 128]), we have (by redefining the random variables on a new probability space if necessary)

$$\lim_{n\to\infty} |S_n - S_n^*|/\{n^{\frac{1}{2}}(\log_2 n)^{-\frac{1}{2}}\} = 0 \quad \text{a.s.,} \tag{2.22}$$

where $S_n^* = Y_1 + \cdots + Y_n$ and $Y_1, Y_2, \cdots$ are i.i.d. standard normal random variables. Take any $\lambda > 0$ and let $\beta(t) = b(t) - \lambda t^{\frac{1}{2}}(\log_2 t)^{-\frac{1}{2}}$. A standard argument involving the Feller-Kolmogorov test shows that $\beta(t)$ also belongs to the upper class. Therefore by Lemma 2 below, letting $\rho < \rho' < 1$,

$$\begin{aligned} \lim_{n\to\infty} P[S_i^* + i\rho'(b(n) - b(m))/(n - m) \geqq \beta(i) \\ \text{for some } \ k(n) \leqq i \leqq n] = 0 . \end{aligned} \tag{2.23}$$

From (2.22) and (2.23), the desired conclusion (2.21) follows. □

LEMMA 2. *Suppose that under P_θ, $X_1, X_2, \cdots$ are* i.i.d. $N(\theta, 1)$ *random variables. Set* $S_n = X_1 + \cdots + X_n$. *Let* $\beta(n)$ *be a positive sequence such that* $\lim_{n\to\infty} \beta(n)/n = 0$, $\lim_{n\to\infty} n^{-\frac{1}{2}}\beta(n) = \infty$ *and* $P_0[S_n < \beta(n)$ *for all large* $n] = 1$. *Let* $c(n)$ *be a positive nonincreasing sequence such that* $\lim_{n\to\infty} c(n) = 0$ *and* $\lim_{n\to\infty} nc(n)/\beta(n) = 1$. *Suppose* $k(n) \leqq n$ *is a sequence of positive integers such that* $\lim_{n\to\infty} k(n) = \infty$. *Then for any* $\rho \in (0, 1)$,

$$\lim_{n\to\infty} P_{\rho c(n)}[S_i \geqq \beta(i) \text{ for some } k(n) \leqq i \leqq n] = 0 . \tag{2.24}$$

PROOF. We shall use the ideas of Farrell's proof of Lemma 4 in [6]. Let $m(n)$ be a sequence of nonnegative integers such that

$$\lim_{n\to\infty} m(n) = \infty , \qquad \lim_{n\to\infty} c(m(n))/c(n) = \infty \qquad \text{and} \qquad \lim_{n\to\infty} c(n)(m(n))^{\frac{1}{2}} = \infty . \tag{2.25}$$

Such a sequence can be constructed as follows. Set $d_n = n^{\frac{1}{2}}c(n)$. Then $d_n \to \infty$ since $nc(n) \sim \beta(n)$. Let α_n be a decreasing sequence of positive numbers such that $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} \alpha_n d_n = \infty$. Define $m(n) = \sup\{m : c(m)/c(n) \geqq \alpha_m d_m\}$ $(\sup \varnothing = 0)$. Then $m(n)$ satisfies (2.5).

Let $T_n = \inf\{i \geqq k(n) : S_i \geqq \beta(i)\}$. We now make use of a standard likelihood-ratio argument in sequential analysis. If $k(n) \leqq m(n)$, we have

$$P_\theta[T_n \leqq m(n)] = \sum_{i=k(n)}^{m(n)} \int_{[T_n=i]} e^{\theta S_i - i\theta^2/2}\, dP_0 . \tag{2.26}$$

Hence

$$\frac{d^2}{d\theta^2} P_\theta[T_n \leqq m(n)] = \sum_{i=k(n)}^{m(n)} \int_{[T_n=i]} \{(S_i - i\theta)^2 - i\} e^{\theta S_i - i\theta^2/2}\, dP_0 . \tag{2.27}$$

Choose n_1 such that for $i \geqq n_1$, $\beta(i) > \rho i c(i) + i^{\frac{1}{2}}$. Therefore if $i > n_1$, $\theta \leqq \rho c(i)$ and $x \geqq \beta(i)$, then $(x - i\theta)^2 - i = (x - i\theta - i^{\frac{1}{2}})(x - i\theta + i^{\frac{1}{2}}) > 0$. Since $S_i \geqq \beta(i)$ on the event $[T_n = i]$ and $c(i) \geqq c(j)$ if $i \leqq j$, it then follows from (2.27) that we can choose n_2 such that for $n \geqq n_2$, $P_\theta[T_n \leqq m(n)]$ is a convex function of θ in the interval $[0, \rho c(m(n))]$. Therefore by (2.25), for all large n, $c(n) < c(m(n))$ and

$$\begin{aligned} &P_{\rho c(n)}[T_n \leqq m(n)] \\ &\quad \leqq (c(n)/c(m(n))P_{\rho c(m(n))}[T_n \leqq m(n)] \\ &\qquad + \{1 - c(n)/c(m(n))\}P_0[S_i \geqq \beta(i) \text{ for some } k(n) \leqq i \leqq m(n)] \\ &\quad \to 0 \quad \text{as} \quad n \to \infty . \end{aligned} \tag{2.28}$$

Clearly $m(n) < n$ for large n. Again by using likelihood ratios, we obtain

$$\begin{aligned} &P_{\rho c(n)}[m(n) \leqq T_n \leqq n] \\ &\quad = \sum_{i=m(n)}^{n} \int_{[T_n=i]} \exp\left\{-(1-\rho)c(n)S_i + \frac{i}{2}(1-\rho^2)c^2(n)\right\} dP_{c(n)} \\ &\quad \leqq \max_{m(n)\leqq i\leqq n} \exp\left\{-(1-\rho)c(n)\left(\beta(i) - \frac{i}{2}(1+\rho)c(n)\right)\right\} , \end{aligned} \tag{2.29}$$

since $S_i \geqq \beta(i)$ on the event $[T_n = i]$. Let $1 > \varepsilon > (1+\rho)/2$. Then for all large i, $\beta(i)/i > \varepsilon c(i) \geqq \varepsilon c(n)$ if $n \geqq i$. Hence letting $\delta = (1-\rho)\{\varepsilon - \frac{1}{2}(1+\rho)\}$, we have for all large n and $m(n) \leqq i \leqq n$,

$$\exp\left\{-(1-\rho)c(n)\left(\beta(i) - \frac{i}{2}(1+\rho)c(n)\right)\right\} \leqq \exp\{-\delta c^2(n)m(n)\} \to 0 \tag{2.30}$$

in view of (2.25). From (2.28), (2.29) and (2.30), the desired conclusion (2.24) follows. □

EXAMPLE 1. Let $Z_1, Z_2, \cdots$ be i.i.d. with $EZ_1 = 0$, $0 < EZ_1^2 < \infty$ and let $S_n = Z_1 + \cdots + Z_n$. Let $b(t)$ be a concave, increasing, positive continuous function on $[m, \infty)$ such that $P[S_n < b(n) \text{ for all } n \geqq m] = P^* > 0$. Define $T(\theta)$ as in Theorem 1.

(a) If $b(t) \sim (t \log t)^{\frac{1}{2}}$, then by Theorem 1 (see Remarks (i) and (ii)),

$$\lim_{\theta \downarrow 0} \theta^2 ET(\theta)/(2|\log \theta|) = P^* .$$

(b) Let $\frac{1}{2} < \alpha < 1$. If $b(t) \sim t^\alpha$, then again by Theorem 1,

$$\lim_{\theta \downarrow 0} \theta^{1/(1-\alpha)} ET(\theta) = P^* .$$

The analogue of Theorem 1 for stopping rules of two-sided power-one tests also holds. More specifically, we have the following theorem.

THEOREM 2. *Suppose $Z_1, Z_2, \cdots$ are* i.i.d. *random variables such that $EZ_1 = 0$ and $E|Z_1|^\nu < \infty$ for some $\nu > 1$. Let $S_n = Z_1 + \cdots + Z_n$. For $i = 1, 2$, let $b_i(t)$ be a positive continuous function on $[m, \infty)$ with $m \geqq 1$ satisfying the following conditions*:

$$b_i(t) \quad \textit{is concave, increasing and} \quad \lim_{t\to\infty} b_i(t)/t = 0 ; \tag{2.31}$$

(2.32) *For* $\varepsilon_0 \leqq \varepsilon < 1$ (*where* ε_0 *is some positive number* < 1),

$$\beta_i(\varepsilon) = \lim_{t\to\infty} b_i(\varepsilon t)/b_i(t) \quad \textit{exists and is} \quad > \varepsilon , \quad \textit{and} \quad \lim_{\varepsilon \uparrow 1} \beta_i(\varepsilon) = 1 ;$$

$$P[-b_2(n) < S_n < b_1(n) \quad \textit{for all} \quad n \geqq m] > 0 ; \tag{2.33}$$

$$\lim_{t\to\infty} t^{-1/\nu} b_i(t) = \infty ; \tag{2.34}$$

For any $0 < \rho < 1$,

$$\begin{aligned} &\lim_{n\to\infty} P[-b_2(k) < S_k < b_1(k) - k\rho b_1(n)/n \quad \textit{for all} \quad n \geqq k \geqq m] \\ &\quad = \lim_{n\to\infty} P[-b_2(k) + k\rho b_2(n)/n < S_k < b_1(k) \quad \textit{for all} \quad n \geqq k \geqq m] \\ &\quad = P[-b_2(k) < S_k < b_1(k) \quad \textit{for all} \quad k \geqq m] . \end{aligned} \tag{2.35}$$

$$P[S_n = b_1(n)] = P[S_n = -b_2(n)] = 0 \qquad \textit{for all} \quad n \geqq m . \tag{2.36}$$

For each $\theta > 0$ such that $\theta m < b_1(m)$, let $t = g_1(\theta)$ be the root of $\theta t = b_1(t)$ with $t > m$, while for each $\theta < 0$ such that $\theta m > -b_2(m)$, let $t = g_2(\theta)$ be the root of $\theta t = -b_2(t)$ with $t > m$. For any real θ, let

$$N(\theta) = \inf\{n \geqq m : S_n + n\theta \geqq b_1(n) \quad \textit{or} \quad S_n + n\theta \leqq -b_2(n)\} .$$

Then $EN(\theta) < \infty$ *if* $\theta \neq 0$, $EN(0) = \infty$, $\lim_{|\theta|\to\infty} EN(\theta) = m$, *and*

$$\lim_{\theta\to 0+} EN(\theta)/g_1(\theta) = \lim_{\theta\to 0-} EN(\theta)/g_2(\theta) = P[-b_2(n) < S_n < b_1(n) \text{ for all } n \geqq m] .$$

PROOF. We shall only restrict ourselves to the asymptotic behavior as $\theta \to 0+$ since the argument for $\theta \to 0-$ is similar. Set $p = P[-b_2(i) < S_i < b_1(i)$ for all $i \geqq m]$. For $n > m$, define $c_n = (b_1(n) - b_1(m))/(n - m)$. Let ρ_1 and γ be given by Lemma 1(e) with $g = g_1$. Take any $\rho \in [\rho_1, 1)$ and any sequence θ_n such that $\rho c_{n+1} \leqq \theta_n \leqq \rho c_n$. We shall first show that

$$\lim_{n\to\infty} P[N(\theta_n) > n] = p . \tag{2.37}$$

By condition (2.35), taking $\rho' \in (\rho, 1)$, we obtain that for all large n,

$$P[N(\theta_n) > n] \geqq P[-b_2(i) < S_i < b_1(i) - i\rho' b_1(n)/n \text{ for all } n \geqq i \geqq m] \to p \quad \text{as} \quad n \to \infty .$$

On the other hand, in view of (2.33), given $\varepsilon > 0$, we can choose $n_0 > m$ such that $P[S_i \leqq -b_2(i)$ for some $i \geqq n_0] < \varepsilon$, and so for $n \geqq n_0$,

$$\begin{aligned} P[N(\theta_n) > n] &\leqq P[-b_2(i) - i\rho c_n < S_i < b_1(i) \text{ for all } n \geqq i \geqq m] \\ &\leqq P[-b_2(i) < S_i < b_1(i) \text{ for all } n \geqq i \geqq m] \\ &\quad + P[-b_2(i) - i\rho c_n < S_i \leqq -b_2(i) \text{ for some } n_0 \geqq i \geqq m] + \varepsilon \\ &\to p + \varepsilon \quad \text{as} \quad n \to \infty \quad \text{by condition (2.36).} \end{aligned}$$

Since ε is arbitrary, we have established (2.37). Making use of (2.37), we can then prove as in Theorem 1 that

$$\liminf_{\theta\downarrow 0} EN(\theta)/g_1(\theta) \geqq p . \tag{2.38}$$

Let $\psi(\theta)$ be as constructed in Lemma 1(c) with $b = b_1$ and $g = g_1$. Set $f(\theta) = E[N(\theta) \mid N(\theta) > \psi(\theta)]$. Noting that for $\theta > 0$, $S_{N(\theta)} + \theta N(\theta) \leqq b_1(N(\theta)) + X^+_{N(\theta)} + \theta + |S_m| + m\theta$, we can apply Wald's lemma and Jensen's inequality as in the proof of Theorem 1 to show that

$$\limsup_{\theta\downarrow 0} f(\theta)/g_1(\theta) \leqq 1 . \tag{2.39}$$

Define $N_+(\theta) = \inf\{n \geqq m : S_n + n\theta \geqq b_1(n)\}$, $N_-(\theta) = \inf\{n \geqq m : S_n + n\theta \leqq -b_2(n)\}$. Then by (2.39) and Lemma 1(c), we have for $\theta > 0$,

$$\begin{aligned} EN(\theta) &\leqq \psi(\theta) + f(\theta)P[N(\theta) > \psi(\theta)] \\ &\leqq o(g_1(\theta)) + (1 + o(1))g_1(\theta)P[N_+(0) > \psi(\theta), N_-(\theta) > \psi(\theta)] . \end{aligned} \tag{2.40}$$

Observe that

$$\begin{aligned} 0 &\leqq P[N_+(0) > \psi(\theta), N_-(\theta) > \psi(\theta)] - P[N_+(0) > \psi(\theta), N_-(0) > \psi(\theta)] \\ &\leqq P[S_i > -b_2(i) - i\theta \text{ for all } m \leqq i \leqq \psi(\theta) \\ &\qquad \text{and } S_i \leqq -b_2(i) \text{ for some } m \leqq i \leqq \psi(\theta)] \\ &\to 0 \quad \text{as} \quad \theta \downarrow 0 \quad \text{by (2.33) and (2.36).} \end{aligned}$$

Since $\lim_{\theta \downarrow 0} P[N_+(0) > \psi(\theta), N_-(0) > \psi(\theta)] = p$, it follows from (2.40) that

$$\limsup_{\theta \downarrow 0} EN(\theta)/g_1(\theta) \leqq p . \tag{2.41}$$ □

Suppose $Z_1, Z_2, \cdots$ are i.i.d. with $EZ_1 = 0$ and $0 < EZ_1^2 < \infty$. For $i = 1, 2$, let $b_i(t)$ be a positive continuous function on $[m, \infty)$ satisfying conditions (2.31), (2.32), (2.33) and (2.36) of Theorem 2. Suppose further that

$$\lim_{t \to \infty} b_i(t)/(t \log_2 t)^{\frac{1}{2}} = \infty \qquad \text{for} \quad i = 1, 2 . \tag{2.42}$$

Then condition (2.34) holds with $\nu = 2$ and condition (2.35) is also satisfied. (See Remark (ii) to Theorem 1.) Alternatively, instead of assuming (2.42), we can obtain the same conclusion by assuming that

$$E\{Z_1 \log_2 (|Z_1| + e)\}^2 < \infty \quad \text{and} \quad t^{-\frac{1}{2}} b_i(t) \quad \text{is ultimately nondecreasing} \quad \text{for} \quad i = 1, 2 . \tag{2.43}$$

(See the proof of Corollary 1.) Hence Theorem 2 gives the analogue of Corollary 1 for $N(\theta)$.

3. The case of an exponential family. Suppose $X_1, X_2, \cdots$ are i.i.d. random variables having a common density $\exp(\theta x - h(\theta))$ with respect to some nondegenerate measure π on the real line, where $\theta \in \Theta$ is an unknown natural parameter of the exponential family and Θ is an open interval of the real line. Let $\mu_\theta = E_\theta X_1 = h'(\theta)$. For a given $\theta_0 \in \Theta$, we want to test the hypothesis H: $\theta = \theta_0$ versus K: $\theta \neq \theta_0$. By considering $X_i^* = X_i - h'(\theta_0)$ and $\theta^* = \theta - \theta_0$ if necessary, we may assume without loss of generality that $\theta_0 = 0$ and $\mu_0 = 0$. The following theorem studies the asymptotic behavior of the expected sample size as $\theta \to 0$ for power-one tests of H versus K based on sample sums and suitable stopping boundaries $b_1(n)$ and $-b_2(n)$.

THEOREM 3. *Suppose that under P_θ, $X_1, X_2, \cdots$ are* i.i.d. *with a common density $\exp(\theta x - h(\theta))$ with respect to a fixed nondegenerate measure π on the real line such that $\mu_0 = 0$, where $\mu_\theta = E_\theta X_1$, $\theta \in \Theta$, Θ being an open interval (possibly infinite) containing* 0. *Let $S_n = X_1 + \cdots + X_n$. For $i = 1, 2$, let $b_i(t)$ be a positive continuous function of $[m, \infty)$ with $m \geqq 1$ satisfying conditions* (2.31), (2.32) *of Theorem* 2 *and the following two conditions*:

$$\lim_{t \to \infty} t^{-\frac{1}{2}} b_i(t) = \infty \qquad \textit{for} \quad i = 1, 2 ; \tag{3.1}$$

$$P_0[-b_2(n) < S_n < b_1(n) \ \textit{for all} \ n \geqq m] > 0 . \tag{3.2}$$

Let $N = \inf\{n \geqq m : S_n \geqq b_1(n)$ or $S_n \leqq -b_2(n)\}$. Define $g_1(\theta)$ for $\theta > 0$ *and $g_2(\theta)$ for $\theta < 0$ as in Theorem* 2. *Then $E_\theta N < \infty$ if $\theta \neq 0$ and*

$$\lim_{\theta \to 0+} E_\theta N/g_1(\mu_\theta) = \lim_{\theta \to 0-} E_\theta N/g_2(\mu_\theta) = P_0[N = \infty] . \tag{3.3}$$

PROOF. As shown in [6, page 40], $P_\theta[N < \infty, S_N \geqq b_1(N)]$ is a nondecreasing and left continuous function of θ, while $P_\theta[N < \infty, S_N \leqq -b_2(N)]$ is a nonincreasing and right continuous function of θ. This fact will be used in the proof. We shall only consider the case $\theta \to 0+$ in (3.3) as the proof for $\theta \to 0-$ is exactly analogous. Let $\sigma^2 = E_0 X_1^2$. As is well known, $\mu_\theta \sim \sigma^2\theta$ as $\theta \to 0$.

For $n > m$, define $c_n = (b_1(n) - b_1(m))/\{\sigma^2(n - m)\}$. Let ρ_1 and γ be given by Lemma 1(e) with $g = g_1$. Take any $\rho \in [\rho_1, 1)$ and any sequence θ_n such that $\rho c_{n+1} \leqq \theta_n \leqq \rho c_n$. Like Farrell's proof of Lemma 4 of [6] (see also our proof of Lemma 2 above), it can be shown that

$$\lim_{n\to\infty} P_{\theta_n}[N < n, S_N \geqq b_1(N)] = P_0[N < \infty, S_N \geqq b_1(N)] . \tag{3.4}$$

(This result corresponds to relation (41) of [6].) By the right continuity of $P_\theta[N < \infty, S_N \leqq -b_2(N)]$ as a function of θ, we obtain that

$$\lim_{n\to\infty} P_{\theta_n}[N < \infty, S_N \leqq -b_2(N)] = P_0[N < \infty, S_N \leqq -b_2(N)] . \tag{3.5}$$

From (3.4) and (3.5), it follows that

$$\begin{aligned}
&\lim_{n\to\infty} P_{\theta_n}[N \geqq n, S_N \geqq b_1(N)] \\
&\quad = 1 - P_0[N < \infty, S_N \geqq b_1(N)] - P_0[N < \infty, S_N \leqq -b_2(N)] \\
&\quad = P_0[N = \infty] .
\end{aligned} \tag{3.6}$$

Since $(n + 1 - m)\sigma^2 c_{n+1} = b_1(n + 1) - b_1(m)$, Lemma 1(d) implies that $g_1(\sigma^2 c_{n+1}) \sim n$. Using this and (3.6), we obtain that

$$\begin{aligned}
E_{\theta_n} N &\geqq nP_{\theta_n}[N \geqq n, S_N \geqq b_1(N)] \sim g_1(\sigma^2 c_{n+1})P_0[N = \infty] \\
&\geqq g_1(\sigma^2\theta_n/\rho)P_0[N = \infty] \geqq (1 + o(1))\gamma(\rho)g_1(\sigma^2\theta_n)P_0[N = \infty] .
\end{aligned} \tag{3.7}$$

Since $\lim_{\theta\to 0} \mu_\theta/\theta = \sigma^2$, the monotonicity of g_1 and Lemma 1(e) imply that

$$g_1(\sigma^2\theta) \sim g_1(\mu_\theta) \quad \text{as} \quad \theta \to 0 . \tag{3.8}$$

From (3.7) and (3.8), it then follows as in the proof of Theorem 1 that

$$\liminf_{\theta\downarrow 0} E_\theta N/g_1(\mu_\theta) \geqq P_0[N = \infty] . \tag{3.9}$$

We note that $E_\theta|X_N| \leqq E_\theta^{\frac12}(\sum_1^N X_i^2) \sim \sigma E_\theta^{\frac12} N$ as $\theta \downarrow 0$ by Wald's lemma. Since $\lim_{\theta\downarrow 0} (g_1(\theta))^{-\frac12} b_1(g_1(\theta)) = \infty$ by (3.1), $\lim_{\theta\downarrow 0} \theta^2 g_1(\theta) = \infty$. Therefore $\lim_{\theta\downarrow 0} \mu_\theta E_\theta^{\frac12} N = \infty$ in view of (3.9). Hence $\lim_{\theta\downarrow 0} \mu_\theta E_\theta N = \infty$ and $E_\theta|X_N| = o(\mu_\theta E_\theta N)$. Therefore using Wald's lemma, we obtain that

$$\begin{aligned}
\mu_\theta E_\theta N = E_\theta S_N &\leqq E_\theta b_1(N) + E_\theta|X_N| + mE_\theta|X_1| \\
&= E_\theta b_1(N) + o(\mu_\theta E_\theta N)
\end{aligned} \tag{3.10}$$

as $\theta \downarrow 0$. For $\theta > 0$, let $\psi^*(\theta)$ be the greatest integer $\leqq |\log \theta|$. Since μ_θ is a strictly increasing function of θ, $E_\theta[N \mid N > \psi^*(\theta)]$ can be written as a function of μ_θ, say $E_\theta[N \mid N > \psi^*(\theta)] = f(\mu_\theta)$. By Jensen's inequality,

$$E_\theta b_1(N) \leqq b_1(f(\mu_\theta))P_\theta[N > \psi^*(\theta)] + b_1(\psi^*(\theta)) . \tag{3.11}$$

Since $b_1(\psi^*(\theta)) = o(\psi^*(\theta))$ and $\mu_\theta g_1(\mu_\theta) = b_1(g_1(\mu_\theta)) \geqq b_1(\mu_\theta^{-2}) > \theta^{-1} > \psi^*(\theta)$ for all small positive θ, we obtain that $b_1(\psi^*(\theta)) = o(\mu_\theta E_\theta N)$ as $\theta \downarrow 0$ in view of (3.9). Therefore from (3.10) and (3.11), it follows as in the proof of Theorem 1 that $\liminf_{\theta\downarrow 0} b_1(f(\mu_\theta))/(\mu_\theta f(\mu_\theta)) \geqq 1$. By Lemma 1(b), this implies that

$$\limsup_{\theta\downarrow 0} f(\mu_\theta)/g_1(\mu_\theta) \leqq 1 . \tag{3.12}$$

Therefore as $\theta \downarrow 0$,

$$E_\theta N \leqq \phi^*(\theta) + f(\mu_\theta) P_\theta[N > \phi^*(\theta)] \\ \leqq o(g_1(\mu_\theta)) + (1 + o(1)) g_1(\mu_\theta) P_\theta[N \geqq \phi^*(\theta)] . \tag{3.13}$$

In view of (3.9) and (3.13), it remains to prove that

$$\lim_{\theta \downarrow 0} P_\theta[N \geqq \phi^*(\theta)] = P_0[N = \infty] . \tag{3.14}$$

We note that for $0 < \theta < 1$, $\theta \leqq \exp(-\phi^*(\theta)) = o(c_{\phi^*(\theta)})$. Therefore given any $\rho \in (0, 1)$, $\theta < \rho c_{\phi^*(\theta)}$ for all small positive θ. Since for every fixed k, $P_\theta[N < k, S_N \geqq b_1(N)]$ is nondecreasing in θ, it then follows that as $\theta \downarrow 0$,

$$P_{\rho c_{\phi^*(\theta)}}[N < \phi^*(\theta), S_N \geqq b_1(N)] \geqq P_\theta[N < \phi^*(\theta), S_N \geqq b_1(N)] \\ \geqq P_0[N < \phi^*(\theta), S_N \geqq b_1(N)] \\ \to P_0[N < \infty, S_N \geqq b_1(N)] . \tag{3.15}$$

Now $\lim_{\theta \downarrow 0} P_{\rho c_{\phi^*(\theta)}}[N < \phi^*(\theta), S_N \geqq b_1(N)] = P_0[N < \infty, S_N \geqq b_1(N)]$ by (3.4), and so (3.15) implies that

$$\lim_{\theta \downarrow 0} P_\theta[N < \phi^*(\theta), S_N \geqq b_1(N)] = P_0[N < \infty, S_N \geqq b_1(N)] . \tag{3.16}$$

Given any $k \geqq m$, we have for all small positive θ,

$$P_0[N < \infty, S_N \leqq -b_2(N)] \geqq P_\theta[N < \infty, S_N \leqq -b_2(N)] \\ \geqq P_\theta[N < \phi^*(\theta), S_N \leqq -b_2(N)] \\ \geqq P_\theta[N \leqq k, S_N \leqq -b_2(N)] \\ \to P_0[N \leqq k, S_N \leqq -b_2(N)] \quad \text{as} \quad \theta \downarrow 0 . \tag{3.17}$$

Since (3.17) holds for all $k \geqq m$, it follows that

$$\lim_{\theta \downarrow 0} P_\theta[N < \phi^*(\theta), S_N \leqq -b_2(N)] = P_0[N < \infty, S_N \leqq -b_2(N)] . \tag{3.18}$$

From (3.16) and (3.18), (3.14) follows immediately. □

EXAMPLE 2. Let $X_1, X_2, \cdots$ be the same as in Theorem 3 and let $\sigma^2 = E_0 X_1^2$, $b_1(t) = b_2(t) = \sigma\{2t[\log_2(t + e) + c \log_3(t + e^e)]\}^{\frac{1}{2}}$, where $c > \frac{3}{2}$. Choose $m \geqq 1$ such that b_1, b_2 are concave on $[m, \infty)$ and $P_0[-b_2(n) < S_n < b_1(n)$ for all $n \geqq m] > 0$. Then for $i = 1, 2$, $0 < \varepsilon < 1$, $\lim_{t\to\infty} b_i(\varepsilon t)/b_i(t) = \varepsilon^{\frac{1}{2}}$. Solving $b_1(t) = \theta t$ for t in terms of θ, we obtain that $t = g_1(\theta) \sim \theta^{-2}(2\sigma^2 \log_2 \theta^{-1})$ as $\theta \to 0+$. Likewise $g_2(\theta) \sim \theta^{-2}(2\sigma^2 \log_2 |\theta|^{-1})$ as $\theta \to 0-$. Hence defining N as in Theorem 3, we obtain by the theorem that

$$\lim_{\theta\to 0} \mu_\theta^2 E_\theta N/(2\sigma^2 \log|\log|\mu_\theta||) = P_0[N = \infty] , \tag{3.19}$$

which was obtained by Farrell in [6]

Using a similar argument as the proof of Theorem 3, we can prove the following analogue of Theorem 3 for power-one tests of the one-sided hypothesis $H_0 : \theta > 0$ based on sample sums and upper-class boundaries.

THEOREM 4. *Let* $X_1, X_2, \cdots, S_n, \Theta, P_\theta, \mu_\theta$ *be as in Theorem* 3 ($\mu_0 = 0$). *Let* $b(t)$ *be a positive continuous function on* $[m, \infty)$ *with* $m \geqq 1$ *such that* $\lim_{t\to\infty} t^{-\frac{1}{2}} b(t) = \infty$ *and* $P_0[S_n < b(n)$ *for all* $n \geqq m] > 0$. *Assume that* $b(t)$ *also satisfies the regularity conditions* (2.1) *and* (2.2) *of Theorem* 1 *and define* $g(\theta)$ *as in Theorem* 1. *Let* $T = \inf\{n \geqq m : S_n \geqq b(n)\}$. *Then*

$$\lim_{\theta \downarrow 0} E_\theta T / g(\mu_\theta) = P_0[T = \infty] . \tag{3.20}$$

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