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DISCOUNTED AND RAPID SUBFAIR RED-AND-BLACK

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A gambler seeks to maximize the expected utility earned upon reaching a goal in a game where he is allowed at each stage to stake any amount of his current fortune. He wins each bet with probability w. In the discounted case the utility at the goal is β^n where β, the discount factor, is in (0, 1) and n is the number of plays used to reach the goal. In the rapid case the utility at the goal is 1 and the gambler seeks to minimize his expected playing time given he reaches the goal. Here all optimal strategies are characterized when $w \leqq \frac{1}{2}$ for the discounted case and when $w < \frac{1}{2}$ for the rapid case. It is shown that when $w < \frac{1}{2}$ the set of rapidly optimal strategies coincides with the set of optimal strategies for the discounted case.

1. Introduction. In red-and-black gambling problems the gambler can stake any amount s of his current fortune f, $0 \leqq s \leqq f$. If he stakes s his fortune becomes $f + s$ with probability w and $f - s$ with probability $\bar{w} = 1 - w$ where $0 \leqq w \leqq 1$. The gambler is allowed to gamble repeatedly with a basic objective of reaching fortune 1. The problem is to find a strategy which makes the probability of reaching the goal as large as possible.

In the more precise notation and terminology of Dubins and Savage (1965), a red-and-black problem is defined by the set of fortunes $F = [0, \infty)$; utility function $u(f) = 0$ or 1 according as $0 \leqq f < 1$ or $f \geqq 1$; and for each $f \in F$, the set $\Gamma_w(f) = \{w\delta(f + s) + \bar{w}\delta(f - s) : 0 \leqq s \leqq f\}$ of available gambles at f. The symbol $\delta(f)$ denotes the measure which assigns mass 1 to $\{f\}$. To distinguish this problem from the modifications introduced later it will henceforth be referred to as basic red-and-black.

A strategy σ available at f in Γ_w is a sequence $\sigma_0, \sigma_1, \cdots$ where $\sigma_0 \in \Gamma_w(f)$ and, for each positive n and each finite sequence $(f_1, \cdots, f_n)$ of elements of F, $\sigma_n(f_1, \cdots, f_n) \in \Gamma_w(f_n)$. Define an incomplete stop rule t by $t(f_1, f_2, \cdots) = \inf\{n : f_n = 0 \text{ or } f_n \geqq 1\}$. The utility of a strategy σ is given by $u(\sigma) = E_\sigma[u(f_t)] = P_\sigma[f_t \geqq 1]$. Define $U_w(f) = \sup u(\sigma)$ where the supremum is taken over all σ available at f and call a strategy σ optimal if $u(\sigma) = U_w(f)$. For a fortune f, let $R_w(f)$ be the set of all optimal strategies that are available at f in Γ_w.

The first modification to be discussed here is the introduction of a *discount factor* $0 < \beta < 1$. It reflects the concept that the value of money decreases with time. A direct way of incorporating the discount factor into basic red-and-black is to change the definition of the utility of a strategy to $u(\sigma) = E_\sigma[\beta^t u(f_t)]$. Let

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$D_{w,\beta}(f)$ be the set of all optimal strategies available at f for this problem. In Section 2 an equivalent, but more useful, model for the discounted problem is given and then the *bold* strategy is shown to be optimal when $0 < w \leqq \frac{1}{2}$ and $0 < \beta < 1$. In Section 3 it is shown that $D_{w,\beta}(f)$ does not depend on w and β for $0 < w < \frac{1}{2}$ and $0 < \beta < 1$ and its members are characterized. Section 4 shows that for $w = \frac{1}{2}$, $D_{w,\beta}(f)$ does not depend on β for $0 < \beta < 1$ and this set proves to strictly contain $D_{w,\beta}(f)$ for $0 < w < \frac{1}{2}$.

The second modification provides a more direct way of encouraging the gambler to quickly resolve the game. Let $T(\sigma) = E_\sigma[t \mid f_t \geqq 1]$ be the rapid utility of strategy σ and $T_w(f) = \inf T(\sigma)$ with the infimum taken over all $\sigma \in R_w(f)$. A strategy σ available at f is called *rapidly optimal* if $\sigma \in R_w(f)$ and $T(\sigma) = T_w(f)$. In Section 5 some properties of this problem are investigated. In Section 6 it is shown that for $0 < w < \frac{1}{2}$ the set of rapidly optimal strategies is $D_{w,\beta}(f)$ for any $0 < \beta < 1$.

2. Discounted red-and-black. The discounted red-and-black problem is formally defined by $\Gamma_{w,\beta}(f) = \{w\beta\delta(f+s) + \bar{w}\beta\delta(f-s) + \bar{\beta}\delta(0) : 0 \leqq s \leqq f\}$ for all $f \geqq 0$. All other defining items are identical to those for basic red-and-black. To see that this is the same problem as the one given in Section 1, note that for any strategy the gambler's expected utility, given that it took exactly n plays to reach the goal, will be β^n times the same expected utility for basic red-and-black. This conforms to the intuitive notion that the utility of 1 is discounted by β at each play.

Dubins and Savage (1965) showed that the bold strategy is optimal for basic red-and-black with $w \leqq \frac{1}{2}$. The bold strategy may be defined by the stake-valued function $s(f) = \min(f, 1-f)$ for $0 \leqq f \leqq 1$ and $s(f) = 0$ otherwise. That is, when the gambler's current fortune is f the bold strategy requires that he stake $s(f)$ on the next play. It seems reasonable to expect that the bold strategy is also optimal for discounted red-and-black with $w \leqq \frac{1}{2}$ since this strategy concludes the basic game in the least expected number of plays (Ross (1974)).

Let $Q(f)$ be the utility of the bold strategy when the gambler's initial fortune is f. After one play the gambler's fortune will be $f + s(f)$ with probability $w\beta$, $f - s(f)$ with probability $\bar{w}\beta$, and 0 with probability $\bar{\beta}$. Thus

$$Q(f) = w\beta Q(2f) \qquad 0 \leqq f \leqq \tfrac{1}{2} \tag{2.1}$$

$$Q(f) = w\beta + \bar{w}\beta Q(2f-1) \qquad \tfrac{1}{2} \leqq f < 1\,. \tag{2.2}$$

Of course, $Q(f) = 1$ if $f \geqq 1$.

Call a fortune f a *binary rational* if it can be written in the form $f = k2^{-n}$ for some integer k and some nonnegative integer n. The order of a binary rational is the smallest value of n for which f can be written in this form. The binary expansion of any fortune is $f = \sum_{i=0}^{\infty} x_i 2^{-i}$ with x_0 an integer, $x_i = 0$ or 1 for each $i > 0$, and an infinite number of the x_i must be 0.

The following three lemmas establish some useful properties of Q.

LEMMA 2.1. *If* $0 < f < 1$ *is a binary rational of order* n, *then*

$$Q(f) = Q(f - 2^{-n}) + (w\beta)^{b+1}(\bar{w}\beta)^{a-1} \tag{2.3}$$

where a *is the number of* 1's *appearing in the binary expansion of* f *and* $b = n - a$.

PROOF. If $n = 1$ then $f = \frac{1}{2}$, $a = 1$, $b = 0$, and (2.3) becomes $Q(\frac{1}{2}) = w\beta$ which follows directly from (2.1). Assume (2.3) holds for all fortunes of order less than n and let f be a binary rational of order n. If $0 < f \leqq \frac{1}{2}$ use (2.1) to write

$$\begin{aligned} Q(f) &= w\beta Q(2f) = w\beta[Q(2f - 2^{-(n-1)}) + (w\beta)^{b}(\bar{w}\beta)^{a-1}] \\ &= Q(f - 2^{-n}) + (w\beta)^{b+1}(\bar{w}\beta)^{a-1} . \end{aligned}$$

If $\frac{1}{2} < f < 1$ a similar argument using (2.2) completes the induction.

LEMMA 2.2. Q *is strictly increasing on* $[0, 1]$.

PROOF. Let $0 \leqq f < g \leqq 1$ and $h = g - f$. If $f < \frac{1}{2} \leqq g$ write

$$Q(f) = w\beta Q(2f) < w\beta \leqq w\beta + \bar{w}\beta Q(2g - 1) = Q(g) . \tag{2.4}$$

If $g < \frac{1}{2}$ then (2.1) yields $Q(g) - Q(f) = w\beta[Q(2g) - Q(2f)]$ while if $f \geqq \frac{1}{2}$ then (2.2) yields $Q(g) - Q(f) = \bar{w}\beta[Q(2g - 1) - Q(2f - 1)]$. In both cases the arguments of the two functions on the right-hand side are separated by $2h$. Repeatedly applying either (2.1) or (2.2) to the resultant expression will eventually separate them by more than $\frac{1}{2}$ so that the argument used in (2.4) yields the desired result.

LEMMA 2.3. Q *is right-continuous at all* $f \geqq 0$. *In addition,* Q *is left-continuous at* f *if and only if* f *is a binary irrational.*

PROOF. Let $0 \leqq f \leqq 1$ and let $\sum_{i=1}^{\infty} x_i 2^{-i}$ be its binary expansion. Define $f_n^- = \sum_{i=1}^{n} x_i 2^{-i}$ and $f_n^+ = f_n^- + 2^{-n}$ for $n = 1, 2, \cdots$. Let $n_1 < n_2 < \cdots$ be the subsequence of all positive integers i for which $x_i = 0$. Use Lemma 2.1 to write

$$0 < Q(f_{n_j}^+) - Q(f) \leqq Q(f_{n_j}^+) - Q(f_{n_j}^-) = (w\beta)^{b_j+1}(\bar{w}\beta)^{a_j-1}$$

where a_j is the number of 1's in the set $\{x_1, \cdots, x_{n_j-1}, 1\}$ and $b_j = n_j - a_j$. By the definition of binary expansion $b_j \to \infty$ as $j \to \infty$ and therefore $Q(f_{n_j}^+) \to Q(f)$ as $j \to \infty$. Since $f_{n_j}^+$ decreases to f, right-continuity is established. If f is not a binary rational, $f_n^- < f$ for all n and the same argument shows left continuity.

Now let $f = \sum_{i=1}^{k} x_i 2^{-i}$ be a kth order binary rational less than 1. Let $f_n = f - 2^{-n}$ for $n = k, k+1, \cdots$. By Lemma 2.1, $Q(f_{n+1}) - Q(f_n) = (w\beta)^{b+2}(\bar{w}\beta)^{n-b-1}$ where b is the number of zeros in $\{x_1, \cdots, x_k\}$. From the same lemma, $Q(f) - Q(f_k) = (w\beta)^{b+1}(\bar{w}\beta)^{k-b-1}$. Combine these as a telescoping sum to obtain

$$Q(f) - Q(f_n) = (w\beta)^{b+1}(\bar{w}\beta)^{k-b-1} - (w\beta)^{b+2}(\bar{w}\beta)^{k-b-1}(1 - (\bar{w}\beta)^{n-k})/(1 - \bar{w}\beta) .$$

Then $\lim_{n\to\infty} [Q(f) - Q(f_n)] = (w\beta)^{b+1}(\bar{w}\beta)^{k-b-1} - (w\beta)^{b+2}(\bar{w}\beta)^{k-b-1}/(1 - \bar{w}\beta) > 0$. Noting that f_n increases to f completes the proof.

THEOREM 2.4. *For discounted red-and-black with* $0 \leqq w \leqq \frac{1}{2}$ *the bold strategy is optimal.*

PROOF. By Theorem 2.12.1 of Dubins and Savage (1965) it is sufficient to show that

$$Q(f) \geqq w\beta Q(f+s) + \bar{w}\beta Q(f-s) \qquad \text{for all } 0 < f < 1 \text{ and } 0 \leqq s \leqq f. \tag{2.5}$$

By Lemma 2.3 it is sufficient to show (2.5) holds for all binary rational values of f and s. The proof then follows by an induction on the order of f and s. The details are identical to the argument used in Theorem 5.3.1 of Dubins and Savage and will not be presented here.

The following corollary outlines the cases where inequality (2.5) is strict.

COROLLARY 2.5. $Q(f) > w\beta Q(f+s) + \bar{w}\beta Q(f-s)$ *if either*

(i) $w < \frac{1}{2}$ *and* $0 < f - s \leqq \frac{1}{2} < f + s < 1$, *or*
(ii) $\frac{1}{2} \leqq f - s \leqq f - s < 1$, *or*
(iii) $0 < f - s \leqq \frac{1}{2} \leqq f < f + s < 1$.

3. Other optimal strategies for $w < \frac{1}{2}$.

Determining all optimal strategies is the same as finding every stake which is *conserving* at each fortune. For any fortune $f < 1$ a stake is said to be conserving at f if $Q(f) = w\beta Q(f+s) + \bar{w}\beta Q(f-s)$.

The set of all conserving stakes can be expressed by a sequence of stake-valued functions. For $n = 0, 1, \cdots$ let

$$\begin{aligned} S_n(f) &= \min(f, 2^{-n} - f) & 0 \leqq f < 2^{-n} \\ &= \min(f, 1 - f) & 2^{-n} \leqq f \leqq 1 . \end{aligned}$$

Since $S_0(f)$ yields the bold stake, it is conserving. Now suppose $S_n(f)$ gives conserving stakes for $n = 0, 1, \cdots, k-1$. Let $Q_k(f)$ be the utility for the strategy that stakes $S_k(f)$. For $0 \leqq f < 2^{-k}$

$$Q_k(f) = Q(2^n f)Q(2^{-n}) = (w\beta)^{-n}Q(f)(w\beta)^n = Q(f) .$$

Thus this strategy is optimal and $S_k(f)$ is a conserving stake-valued function. It turns out that there are no other conserving stakes.

THEOREM 3.1. *For* $0 < w < \frac{1}{2}$ *and* $0 \leqq f < 1$, $Q(f) = w\beta Q(f+s) + \bar{w}\beta Q(f-s)$ *if and only if* $s = S_n(f)$ *for some* $n = 0, 1, \cdots$.

PROOF. It has already been shown that $S_n(f)$ is conserving for all n. To show that no other stakes are conserving let $\mathscr{S} = \{s : Q(f) = w\beta Q(f+s) + \bar{w}\beta Q(f-s)$ for some $0 \leqq f < 1$ and for at least one such f, $s \neq S_n(f)$ for all $n\}$. Assume that $\mathscr{S}$ is not empty. To obtain a contradiction choose $s \in \mathscr{S}$ such that $s > \frac{1}{2} \sup \mathscr{S}$ and let f be a fortune for which s is conserving and $s \neq S_n(f)$ for all n.

First assume $f + s \leqq \frac{1}{2}$. According to (2.2), $Q(2f) = w\beta Q(2f + 2s) + \bar{w}\beta Q(2f - 2s)$, so $2s$ is conserving for $2f$. This implies that $2s = S_n(2f)$ for some n and therefore $s = S_{n+1}(f)$ if $0 \leqq f < 2^{-(n+1)}$ or $s = S_1(f)$ if $f \geqq 2^{-(n+1)}$. In either case there is a contradiction. By Corollary 2.5 only the bold stake is conserving when $f + s > \frac{1}{2}$ so that $\mathscr{S}$ is empty and the theorem is established.

Theorem 3.1 along with an argument identical to that used in Theorem 5.4.2 of Dubins and Savage (1965) is sufficient to characterize all optimal strategies for $0 < w < \frac{1}{2}$.

THEOREM 3.2. *For $0 < w < \frac{1}{2}$ and $0 < \beta < 1$, a strategy σ is optimal if and only if, immediately following each of the at most denumerably many partial histories of positive σ-probability only stakes of the form $S_n(f)$ for some $n = 0, 1, \cdots$ are used.*

Thus the set of optimal strategies $D_{w,\beta}(f)$ is constant over $0 < w < \frac{1}{2}$ and $0 < \beta < 1$. In Chapter 5 of Dubins and Savage (1965), $R_w(f)$ is shown to be constant over $0 < w < \frac{1}{2}$. For future reference these two sets will be denoted by $D(f)$ and $R(f)$. Savage is said to have been "incredulous" (page ii, Dubins and Savage, 1976) to learn that $R(f)$ contains strategies other than bold. We expected $D(f)$ to be strictly contained in $R(f)$ but were equally surprised to find that $D(f)$ also contains nonbold strategies.

4. Discounted fair red-and-black. Although the case $w = \frac{1}{2}$ is subfair in the sense that $U(f) < f$ for all f, Corollary 2.5 suggests that there are more optimal strategies than when $w < \frac{1}{2}$. From Section 3 it is clear that any strategy optimal for $w < \frac{1}{2}$ is also optimal for $w = \frac{1}{2}$. In this section the remaining optimal strategies will be characterized.

Since the bold strategy is optimal, the utility is given by $Q(f)$. Once again the problem reduces to determining the set of conserving stakes. Let $f = \sum_{i=1}^{\infty} a_i 2^{-i}$ be the binary expansion of the gambler's initial fortune. Apply Lemmas 2.1 and 2.3 to obtain

$$Q(f) = \sum_{i=1}^{\infty} a_i(\beta/2)^i .$$

Let $0 \leqq s \leqq f$ be a stake available at f with $f + s < 1$ and write $f + s = \sum_{i=1}^{\infty} b_i 2^{-i}$ and $f - s = \sum_{i=1}^{\infty} c_i 2^{-i}$. This stake will be conserving if

$$\sum_{i=1}^{\infty} a_i(\beta/2)^i = \sum_{i=1}^{\infty} (b_i + c_i)(\beta/2)^{i+1} . \tag{4.1}$$

Also, except for the bold stake, which is known to be conserving, any stake-fortune combination for which (4.1) does not hold cannot be conserving. If $a_1 = 0$ and $b_i + c_i < 2$ for $i > 0$ then $a_{i+1} = b_i + c_i$ for $i > 0$ and (4.1) holds. There are no other conserving stakes.

THEOREM 4.1. *For discounted red-and-black with $w = \frac{1}{2}$ and $0 < \beta < 1$, s is a conserving stake for fortune $f \geqq \frac{1}{2}$ if and only if $s = 1 - f$. For $f < \frac{1}{2}$, s is conserving if and only if $b_i + c_i < 2$ for all $i > 0$, where b_i and c_i are the* ith *coefficients in the binary expansions of $f + s$ and $f - s$ respectively.*

PROOF. From parts (ii) and (iii) of Corollary 2.5 the bold stake is the only conserving stake for $f \geqq \frac{1}{2}$. For $f < \frac{1}{2}$, $a_1 = 0$ and the stakes given in the theorem are all conserving. Let $\mathscr{S} = \{s : s$ is conserving for some $f < \frac{1}{2}$ and f and s do not satisfy the hypothesis$\}$. Assume $\mathscr{S}$ is not empty and choose $s \in \mathscr{S}$ such that $s > \frac{1}{2} \sup \mathscr{S}$. If $f + s < \frac{1}{2}$ then $2s$ is conserving for $2f$ and they must satisfy the hypothesis. However, this implies that f and s also satisfy the hypothesis and thus $f + s > \frac{1}{2}$. Let $k = \inf\{i : b_i + c_i = 2\}$. The assumption implies that $k < \infty$. A contradiction will be obtained by an inductive argument. Generate a sequence of conserving fortunes and stakes in the following manner. From Case 3 in the proof of Theorem 2.4, $s^{(1)} = |2s - \frac{1}{2}|$ is conserving for $f^{(1)} = 2f - \frac{1}{2}$. As binary expansions $f^{(1)} = \sum_{i=2}^{\infty} a_{i+1}2^{-i}$ and $\{f^{(1)} + s^{(1)}, f^{(1)} - s^{(1)}\} = \{\sum_{i=1}^{\infty} b_{i+1}2^{-i}, \sum_{i=1}^{\infty} c_{i+1}2^{-i}\}$. If $k = 2$, then $(f^{(1)} + s^{(1)}) + (f^{(1)} - s^{(1)}) \geqq 1$ but $f^{(1)} < \frac{1}{2}$ and thus $k > 2$. After the $(n-1)$st step the process will have terminated or else $k > n$ will have been established and $s^{(n-1)}$ will be a conserving stake for $f^{(n-1)} = \sum a_{i+n-1}2^{-i}$ with

$$\{f^{(n-1)} + s^{(n-1)}, f^{(n-1)} - s^{(n-1)}\} = \{\textstyle\sum_{i=1}^{\infty} b_{i+n-1}2^{-i}, \sum_{i=1}^{\infty} c_{i+n-1}2^{-i}\}.$$

The nth step may be divided into three cases.

CASE 1. $f^{(n-1)} < \frac{1}{2} < f^{(n-1)} + s^{(n-1)}$. This is the same situation as in the first step. A similar argument will derive $f^{(n)}$ and $s^{(n)}$ and show that $k > n + 1$.

CASE 2. $f^{(n-1)} < \frac{1}{4}$. The stake $s^{(n)} = 2s^{(n-1)}$ is conserving for $f^{(n)} = 2f^{(n-1)}$. Since $b_n = c_n = 0$, $\{f^{(n)} + s^{(n)}, f^{(n)} - s^{(n)}\} = \{\sum_{i=1}^{\infty} b_{i+n}2^{-i}, \sum_{i=1}^{\infty} c_{i+n}2^{-i}\}$ and if $k = n + 1$ then $(f^{(n)} + s^{(n)}) + (f^{(n)} - s^{(n)}) \geqq 1$ contradicting the fact that $f^{(n)} < \frac{1}{2}$ and thus $k > n + 1$.

CASE 3. $\frac{1}{4} \leqq f^{(n-1)} \leqq f^{(n-1)} + s^{(n-1)} \leqq \frac{1}{2}$. Define $s^{(n)}$ and $f^{(n)}$ as in Case 2. Since $f^{(n)} \geqq \frac{1}{2}$ and $s^{(n)}$ is conserving, $s^{(n)} = 1 - f^{(n)}$ and $s^{(n-1)} = \frac{1}{2} - f^{(n-1)}$. But if $f^{(n-1)} + s^{(n-1)} = \frac{1}{2}$ then either $b_n = 1$ and $b_i = 0$ for $i > n$ or $c_n = 1$ and $c_i = 0$ for $i > n$. Either possibility contradicts the fact that $n < k < \infty$ and in this case the induction may be immediately terminated.

If Case 3 never occurs the induction shows that $k > n$ for all n and this contradiction completes the proof.

5. Rapid subfair red-and-black. Another method of encouraging the gambler to conclude the game quickly is to require him to choose a strategy that minimizes the expected number of plays. It turns out to be more interesting to condition the expectation on the event that he reaches the goal and also to restrict the gambler to those strategies optimal for the corresponding basic game. More formally, for any strategy σ, let $T(\sigma) = E_\sigma[t \mid f_t \geqq 1]$ and let the rapid utility of the game be $T_w(f) = \inf T(\sigma)$ with the infimum taken over all $\sigma \in R_w(f)$. Then σ available at f will be called rapidly optimal if $\sigma \in R_w(f)$ and $T(\sigma) = T_w(f)$.

It is useful to have a formula for evaluating $T(\sigma)$ in terms of the expected time for the game using the same strategy strating at the gambler's fortune after one play. For a strategy σ let $\sigma[f]$ denote the strategy available at f that is used when σ leads the gambler to fortune f after one play. For a stop rule τ and a fortune f such that $\tau(f, f_1, f_2, \cdots) \neq 1$ define a stop rule $\tau[f]$ by $\tau[f](f_1, f_2, \cdots) = \tau(f, f_1, f_2, \cdots) - 1$.

LEMMA 5.1. *Let σ be a strategy available at f and let s be the initial stake specified by σ. Then*

$$T(\sigma) = [wu(\sigma[f+s])(1 + T(\sigma[f+s])) + \bar{w}u(\sigma[f-s])(1 + T(\sigma[f-s]))]/u(\sigma). \tag{5.1}$$

If $\sigma \in R_w(f)$, then

$$T(\sigma) = 1 + [wU(f+s)T(\sigma[f+s]) + \bar{w}U(f-s)T(\sigma[f-s])]/U(f). \tag{5.2}$$

PROOF. For $0 < f < 1$, let $A(f) = \{(f_1, f_2, \cdots) : f_{t[f]} \geqq 1\}$. If $0 < f - s \leqq f + s < 1$,

$$\begin{aligned} T(\sigma) &= \frac{1}{u(\sigma)} \int\int_{A(f_1)} (t[f_1] + 1)\, d\sigma[f_1]\, d\sigma_0(f_1) \\ &= \frac{1}{u(\sigma)} [w \int_{A(f+s)} (t[f+s] + 1)\, d\sigma[f+s] \\ &\qquad + \bar{w} \int_{A(f-s)} (t[f-s] + 1)\, d\sigma[f-s]] \\ &= [wu(\sigma[f+s])(1 + T(\sigma[f+s])) + \bar{w}u(\sigma[f-s])(1 + T(\sigma[f-s]))]/u(\sigma). \end{aligned}$$

If $f - s = 0$ then $u(\sigma[0]) = 0$, $t = 1$ on all histories beginning with $f_1 = 0$, and a similar argument establishes (5.1). If $f + s = 1$, then $u(\sigma[1]) = 1$, $t = 1$ on all histories beginning with $f_1 = 1$, and (5.1) again follows.

If $\sigma \in R_w(f)$, then $u(\sigma) = U(f)$, $u(\sigma[f+s]) = U(f+s)$, and $u(\sigma[f-s]) = U(f-s)$ and (5.2) follows directly from (5.1).

Let $TB(f) = T(\sigma)$ when σ is the bold strategy and f is the initial fortune. Then Lemma 5.1 implies

$$TB(f) = 1 + wU(2f)TB(2f)/U(f) = 1 + TB(2f) \qquad 0 < f \leqq \tfrac{1}{2} \tag{5.3}$$

$$TB(f) = 1 + \bar{w}U(2f-1)TB(2f-1)/U(f) \qquad \tfrac{1}{2} \leqq f < 1. \tag{5.4}$$

LEMMA 5.2. *For $0 < w \leqq \frac{1}{2}$ and $\sigma \in D(f)$, $U(f)T(\sigma) \leqq \bar{w}/w^2$.*

PROOF. For any positive integer n, there is at most one partial history $(f_1, \cdots, f_n)$ of positive σ-probability such that $f_n \geqq 1$. If σ is the bold strategy then there is exactly one sequence of wins and losses that leaves the game unresolved after play $n-1$. To possibly end the game at play n with $f_n \geqq 1$, the gambler must win his stake at play n and thus at most one such partial history exists. For any $\sigma \in D(f)$, if the gambler is at a fortune for which σ does not specify a bold stake, he will have a fortune of the form 2^{-k} if he wins the stake and the only partial history that reaches the goal requires exactly k more plays. If he loses the stake, any partial history must take more than k more plays to reach the goal. Let $I(n) = 1$ if such a partial history of length n exists and

$I(n) = 0$ otherwise. Use the fact that $w \leqq \bar{w}$ to write

$$U(f)T(\sigma) \leqq \sum_{n=1}^{\infty} nI(n)\bar{w}^n \leqq \bar{w}/w^2 .$$

An immediate consequence of this lemma is that $T(\sigma)$ is finite for all $\sigma \in D(f)$.

6. For $0 < w < \frac{1}{2}$, $D(f)$ is the set of rapidly optimal strategies. First it will be shown that $T(\sigma)$ is constant over $\sigma \in D(f)$. This will be done by showing $T(\sigma) = TB(f)$ for all $\sigma \in D(f)$. Then it will be shown that $T(\sigma) > TB(f)$ for $\sigma \in R(f) - D(f)$.

Some definitions are now in order. Let σ be any strategy and τ a stop rule. Let the strategy σ^τ be one that follows σ until time τ and then switches to the bold strategy. If τ is identically n then σ^n will be used with σ^0 denoting the bold strategy. Let $\alpha(\tau)$, the *structure* of τ, be determined as follows. If τ is constant, then $\alpha(\tau) = 0$. For an ordinal $\alpha > 0$, τ is said to have structure at most α if for each $f \in F$ the stop rule τf given by $\tau f(f_1, f_2, \cdots) = \tau(f, f_1, f_2, \cdots)$ has structure at most $\lambda < \alpha$. If such an α exists, τ is said to be structured and $\alpha(\tau)$ is the smallest ordinal for which τ has structure at most α. All stop rules are structured (page 20, Dubins and Savage (1965)).

THEOREM 6.1. *For basic red-and-black with $0 < w < \frac{1}{2}$, let $\sigma \in D(f)$. Then $T(\sigma) = TB(f)$.*

PROOF. The proof will consist of four steps. In Step 1 it is shown that $T(\sigma^1) = TB(f)$. Step 2 shows $T(\sigma^n) = TB(f)$ for $n = 1, 2, \cdots$. Step 3 shows $T(\sigma^\tau) = TB(f)$ for all stop rules. In Step 4 a sequence of stop rules $(\tau_1, \tau_2, \cdots)$ is constructed so that $T(\sigma^{\tau_n}) \to T(\sigma)$, completing the proof.

To begin Step 1, note that there are at most two possible initial stakes for σ. One is the bold stake, in which case $\sigma^1 = \sigma^0$ and $T(\sigma^1) = TB(f)$. A second stake is possible if there exists a positive integer k such that $2^{-(k+1)} < f < 2^{-k}$ and then the stake is $s = 2^{-k} - f$. If this is the initial stake,

$$\begin{aligned} T(\sigma^1) &= 1 + [wU(2^{-k})TB(2^{-k}) + \bar{w}U(2f - 2^{-k})TB(2f - 2^{-k})]/U(f) && \text{(by (5.2))} \\ &= 1 + [w^{k+1}TB(2^{-k}) + \bar{w}w^k U(2^{k+1}f - 1)TB(2f - 2^{-k})]/w^k U(2^k f) && \text{(by (2.1))} \\ &= 1 + [wk + \bar{w}kU(2^{k+1}f - 1) + \bar{w}U(2^{k+1}f - 1)TB(2^{k+1}f - 1)]/U(2^k f) && \\ & && \text{(by (5.3))} \\ &= k + TB(2^k f) && \text{(by (5.4) and (2.2))} \\ &= TB(f) && \text{(by (5.3))} . \end{aligned}$$

Step 2 is established by an induction. Assume $T(\sigma^n) = TB(f)$ for any fortune f and any $\sigma \in D(f)$. Let σ^{n+1} be based on σ as given in the theorem and let s be its initial stake. Then,

$$\begin{aligned} T(\sigma^{n+1}) &= 1 + [wU(f + s)T(\sigma^{n+1}[f + s] + \bar{w}U(f - s)T(\sigma^{n+1}[f - s])]/U(f) \\ &= 1 + [wU(f + s)TB(f + s) + \bar{w}U(f - s)TB(f - s)]/U(f) \\ &= TB(f) . \end{aligned}$$

Step 3 requires that $T(\sigma^\tau) = TB(f)$ be established for all τ. This will be done by a transfinite induction on the structure of τ. Since the structure of τ cannot exceed the cardinality of $F \times F \times \cdots$ (page 15, Dubins and Savage (1965)), the induction will be sufficient to complete this step (Theorem 10.6, Goffman (1963)).

Let τ have structure 0. Then $\tau \equiv n$ and $T(\sigma^\tau) = TB(f)$ by Step 2. Now assume $T(\sigma^\tau) = TB(f)$ for all f, all $\sigma \in D(f)$, and all τ with $\alpha(\tau) < \kappa$ where κ is an ordinal greater than 0. Take σ and f as in the theorem and let τ be a stop rule with $\alpha(\tau) = \kappa$. Let s be the initial stake of σ and note that s is also the initial stake of σ^τ and $\sigma^\tau \in D(f)$. Then by (5.2), $T(\sigma^\tau) = 1 + [wU(f + s)T(\sigma^{\tau[f+s]}[f + s]) + \bar{w}U(f - s)T(\sigma^{\tau[f-s]}[f - s])]/U(f)$. The inductive hypothesis implies that $T(\sigma^{\tau[f+s]}[f + s]) = TB(f + s)$ and $T(\sigma^{\tau[f-s]}[f - s]) = TB(f - s)$ and then by (5.2), $T(\sigma^\tau) = TB(f)$.

Begin Step 4 by defining $\tau^* = \inf\{n : f_n = 0 \text{ or } f_n \geqq 1 \text{ or } (f_1, \cdots, f_n) \text{ has } 0$ σ-probability$\}$. If f is a binary rational, then τ^* is a stop rule. Clearly $T(\sigma^{\tau^*}) = T(\sigma)$ and then by Step 3, $T(\sigma) = TB(f)$. If f is a binary irrational, there is exactly one history $(f_1^*, f_2^*, \cdots)$ such that for all $n > 0$, f_n^* is a binary irrational and $(f_1^*, \cdots, f_n^*)$ has positive σ-probability. Define a sequence of subsets of $F \times F \times \cdots$ by $A_n = (f_1^*, \cdots, f_n^*) \times F \times F \times \cdots$ for $n = 1, 2, \cdots$. For each n let τ_n be a stop rule given by

$$\begin{aligned} \tau_n &= n \qquad \text{on} \quad A_n \\ &= \tau^* \qquad \text{otherwise.} \end{aligned}$$

Theorem 6.1 will be established if it can be shown that $T(\sigma^{\tau_n}) \to T(\sigma)$. To see this let $B_n = A_n \cap [f_t \geqq 1]$. Then,

$$T(\sigma^{\tau_n}) - T(\sigma) = \frac{1}{U(f)} [\textstyle\int_{B_n} t \, d\sigma^{\tau_n} - \int_{B_n} t \, d\sigma] .$$

By Lemma 5.2, $T(\sigma) < \infty$ and the dominated convergence theorem implies that $\int_{B_n} t \, d\sigma \to 0$. Again apply Lemma 5.2 to write

$$\begin{aligned} \textstyle\int_{B_n} t \, d\sigma^{\tau_n} &= \sigma(A_n)U(f_n^*)(n + TB(f_n^*)) \\ &\leqq n\bar{w}^n + \bar{w}^n U(f_n^*)TB(f_n^*) \\ &\leqq n\bar{w}^n + \bar{w}^{n+1}/w^2 \end{aligned}$$

and therefore $\int_{B_n} t \, d\sigma^{\tau_n} \to 0$, completing the proof.

THEOREM 6.2. *For basic red-and-black with* $0 < w < \frac{1}{2}$, *a strategy* σ *is rapidly optimal if and only if* $\sigma \in D(f)$.

PROOF. It is sufficient to show that if $\sigma \in R(f)$ but $\sigma \notin D(f)$ then $T(\sigma) > TB(f)$. This will be done by roughly following the same four steps used in the proof of Theorem 6.1.

Step 1 is to show that if the initial stake specified by σ is not conserving for discounted subfair red-and-black, then $T(\sigma^1) > TB(f)$. Dubins and Savage (1965) characterized all conserving stakes for subfair basic red-and-black. To find such a stake for fortune f, obtain nonnegative integers i and n such that $i2^{-n} < f < (i+1)2^{-n}$. Then $s = \min(f - i2^{-n}, (i+1)2^{-n} - f)$ is a conserving stake. Let s be the initial stake specified by σ. Let i and n be integers used to obtain s. For s to be not conserving for discounted subfair red-and-black $i \neq 0$ and if $f \geqq i2^{-n} + 2^{-(n+1)}$ then $i + 1 \neq 2^k$ for $k = 1, 2, \cdots, n$.

Step 1 will be established by an induction on n. The smallest value of n for which a required stake exists is $n = 1$. Then $i = 1$, $\frac{1}{2} < f < \frac{3}{4}$, $s = f - \frac{1}{2}$, and

$$\begin{aligned} T(\sigma^1) &= 1 + [wU(2f - \tfrac{1}{2})TB(2f - \tfrac{1}{2}) + \bar{w}U(\tfrac{1}{2})TB(\tfrac{1}{2})]/U(f) && \text{(by (5.2))} \\ &= 1 + \left[wU(2f - \tfrac{1}{2})\left(1 + \frac{\bar{w}U(4f-2)TB(4f-2)}{U(2f - \frac{1}{2})}\right) + w\bar{w}\right]\Big/U(f) \\ & && \text{(by (5.4))} \\ &= 1 + [w^2 + w\bar{w}U(4f-2) + w\bar{w}U(4f-2)TB(4f-2) + w\bar{w})]/U(f) \\ & && \text{(by (2.2))} \\ &> 1 + [w\bar{w}U(4f-2)(1 + TB(4f-2)) + w\bar{w}]/U(f) \\ &= 1 + [\bar{w}U(2f-1)TB(2f-1)]/U(f) && \text{(by (2.1) and (5.3))} \\ &= TB(f) && \text{(by (5.4))}\,. \end{aligned}$$

Now assume $T(\sigma^1) > TB(f)$ for $n = 1, 2, \cdots, k-1$. There are four cases to consider to show Step 1 holds for $n = k$.

CASE 1. $i = 1, 2, \cdots, 2^{k-1} - 1$ and $f < i2^{-k} + 2^{-(k+1)}$. The initial stake is $s = f - i2^{-k}$ and

$$\begin{aligned} T(\sigma^1) &= 1 + [wU(2f - i2^{-k})TB(2f - i2^{-k}) + \bar{w}U(i2^{-k})TB(i2^{-k})]/U(f) \quad \text{(by (5.2))} \\ &= 1 + [w^2U(4f - i2^{-(k-1)})\{1 + TB(4f - i2^{-(k-1)})\} \\ &\quad + w\bar{w}U(i2^{-(k-1)})\{1 + TB(4f - i2^{-(k-1)})\}]/U(f) \qquad \text{(by (2.1) and (5.3))} \\ &= 1 + [wU(2f) + w\{wU(4f - i2^{-(k-1)})TB(4f - i2^{-(k-1)}) \\ &\quad + \bar{w}U(i2^{-(k-1)})TB(i2^{-(k-1)})\}]/U(f) \\ &= 1 + [1 + \{wU(4f - i2^{-(k-1)})TB(4f - i2^{-(k-1)}) \\ &\quad + \bar{w}U(i2^{-(k-1)})TB(i2^{-(k-1)})\}/U(2f)] \qquad \text{(by (2.1))} \\ &= 1 + T(\sigma^*) \end{aligned}$$

where σ^* is the strategy available at $2f$ that uses an initial stake of $2f - i2^{-(k-1)}$ and then plays boldly. By the inductive hypothesis, $T(\sigma^*) > TB(2f)$ and thus $T(\sigma^1) > 1 + TB(2f) = TB(f)$.

CASE 2. $i = 1, 2, \cdots, 2^{k-1}$; $i + 1 \neq 2^m$ for all $m < k - 1$; and $f \geqq i2^{-k} + 2^{-(k+1)}$. The initial stake is $s = (i+1)2^{-k} - f$ and the argument is identical to that used in Case 1.

CASE 3. $i = 2^{k-1}, \cdots, 2^k - 1$ and $f < i2^{-k} + 2^{-(k+1)}$. The initial stake is

$s = f - i2^{-k}$ and

$$\begin{aligned} T(\sigma^1) &= 1 + [wU(2f - i2^{-k})TB(2f - i2^{-k}) + \bar{w}U(i2^{-k})TB(i2^{-k})]/U(f) \quad \text{(by (5.2))} \\ &= 1 + [wU(2f - i2^{-k}) + w\bar{w}U(4f - 1 - i2^{-(k-1)})TB(4f - 1 - i2^{-(k-1)}) \\ &\qquad + \bar{w}U(i2^{-k}) + \bar{w}^2U(i2^{-(k-1)} - 1)TB(i2^{-(k-1)} - 1)]/U(f) \quad \text{(by (5.4))} \\ &= 1 + [U(f) + \bar{w}U(2f - 1)\{T(\sigma^*) - 1\}]/U(f) \\ &= 1 + [w + \bar{w}U(2f - 1)T(\sigma^*)]/U(f) \quad \text{(by (2.2))} \end{aligned}$$

where σ^* is the strategy available at $2f - 1$ that uses an initial stake of $2f - i2^{-(k-1)}$ and then plays boldly. By the inductive hypothesis $T(\sigma^*) \geqq TB(2f - 1)$. The inequality is not strict since if $i = 2^{k-1}$, σ^* is the bold strategy and if $i + 1 - 2^{k-1} = 2^m$ for some $m < k - 2$, then $\sigma^* \in D(f)$. Then $T(\sigma^1) \geqq 1 + [w + \bar{w}U(2f - 1)TB(2f - 1)]/U(f) = 1 + [w + \{TB(f) - 1\}U(f)]/U(f) = TB(f) + w/U(f) > TB(f)$.

CASE 4. $i = 2^{k-1}, \cdots, 2^k - 2$ and $f \geqq i2^{-k} + 2^{-(k+1)}$. The initial stake is $s = (i + 1)2^{-k} - f$ and the argument is identical to that used in Case 3.

Step 2 is to show that $T(\sigma^{n+1}) \geqq T(\sigma^n)$ for $n \geqq 0$. Another induction will be used for this step. For $n = 0$, first assume that the initial stake s of σ is conserving for discounted subfair red-and-black. Step 1 in the proof of Theorem 6.1 shows that $T(\sigma^1) = T(\sigma^0)$. If s is not conserving for discounted subfair red-and-black then $T(\sigma^1) > T(\sigma^0)$ by Step 1 of this proof.

To complete the induction assume $T(\sigma^{n+1}) \geqq T(\sigma^n)$ for $n = 1, 2, \cdots, k - 1$. Let s be the initial stake of σ and write

$$\begin{aligned} T(\sigma^{k+1}) &= 1 + [wU(f + s)T(\sigma^{k+1}[f + s]) + \bar{w}U(f - s)T(\sigma^{k+1}[f - s])]/U(f) \\ &\geqq 1 + [wU(f + s)T(\sigma^k[f + s]) + \bar{w}U(f - s)T(\sigma^k[f - s])]/U(f) \\ &= T(\sigma^k) . \end{aligned}$$

For a history $(f_1, f_2, \cdots)$, let $g(f_1, f_2, \cdots) = \inf\{n : |f_n - f_{n-1}|$ is not conserving for f_{n-1} in discounted subfair red-and-black$\}$. Define the infimum of the empty set to be ∞. For a strategy σ let $\mathscr{T}(\sigma)$ be the set of stop rules given by $\mathscr{T}(\sigma) = \{\tau : \tau(f_1, f_2, \cdots) \geqq g(f_1, f_2, \cdots)$ for at least one history for which $(f_1, \cdots, f_\tau)$ has positive σ-probability$\}$. Step 3 is to show that $T(\sigma^\tau) > TB(f)$ for all $\tau \in \mathscr{T}(\sigma)$. This will be done by a transfinite induction on the structure of τ. Let $\tau \in \mathscr{T}(\sigma)$ have stucture 0. Then $\tau \equiv n$ for some n and there is at least one partial history of length less than or equal to n and positive σ-probability that involves a stake not conserving for discounted subfair red-and-black. From Steps 1 and 2, $T(\sigma^n) > TB(f)$.

Now assume $T(\sigma^\tau) > TB(f)$ for all f, all $\sigma \in R(f)$ with $\sigma \notin D(f)$, and all $\tau \in \mathscr{T}(\sigma)$ with $\alpha(\tau) < \kappa$ where κ is an ordinal greater than 0. Take σ and f as in the theorem and let $\tau \in \mathscr{T}(\sigma)$ with $\alpha(\tau) = \kappa$. Let s be the initial stake of σ. Since s is also the initial stake of σ^τ and $\sigma^\tau \in R(f)$, by (5.2), $T(\sigma^\tau) = 1 + [wU(f + s)T(\sigma^\tau[f + s]) + \bar{w}U(f - s)T(\sigma^\tau[f - s])]/U(f)$. If $\sigma^\tau[f + s] \in D(f + s)$

and $\sigma^\tau[f - s] \in D(f - s)$, then $T(\sigma^\tau[f + s]) = TB(f + s)$ and $T(\sigma^\tau[f - s]) = TB(f - s)$. Since $\sigma \notin D(f)$, s cannot be conserving for discounted subfair red-and-black and $T(\sigma^\tau) > TB(f)$ by Step 1. If $\sigma^\tau[f + s] \notin D(f + s)$ then $\tau[f + s]$ is well defined and $\tau[f + s] \in \mathscr{T}(\sigma[f + s])$. By the inductive hypothesis, $T(\sigma^\tau[f + s]) > TB(f + s)$. Similarly, if $\sigma^\tau[f - s] \notin D(f - s)$, then $T(\sigma^\tau[f - s]) > TB(f - s)$. If at least one of $\sigma^\tau[f + s]$ and $\sigma^\tau[f - s]$ is not optimal for discounted subfair red-and-black, then $T(\sigma^\tau) > 1 + [wU(f + s)TB(f + s) + \bar{w}U(f - s)TB(f - s)]/U(f) \geqq TB(f)$ with the second inequality following from Theorem 6.1 if s is conserving for discounted subfair red-and-black and from Step 1 otherwise.

Define τ^* as in Step 4 of the proof of Theorem 6.1. If f is a binary rational, then τ^* is a stop rule, $\tau^* \in \mathscr{T}(\sigma)$, and $T(\sigma^{\tau^*}) > TB(f)$. Clearly $T(\sigma^{\tau^*}) = T(\sigma)$ and the proof is completed for this case.

If f is a binary irrational define $(f_1^*, f_2^*, \cdots)$, $(\tau_1, \tau_2, \cdots)$, $(A_1, A_2, \cdots)$ and $(B_1, B_2, \cdots)$ as in Step 4 of the proof of Theorem 6.1. There exists an integer N such that $\tau_n \in \mathscr{T}(\sigma)$ for all $n \geqq N$. Then from Step 3, $T(\sigma^{\tau_n}) > TB(f)$ for all $n \geqq N$. By an argument similar to that used in Step 2, $T(\sigma^{\tau_n})$ is nondecreasing in n. It suffices to show that $T(\sigma^{\tau_n}) \to T(\sigma)$. Argue as in Step 4 of the proof of Theorem 6.1 to show $\int_{B_n} t\, d\sigma^{\tau_n} \to 0$. Complete the proof by noting that if $T(\sigma) = \infty$, Theorem 6.2 is trivially true, while if $T(\sigma) < \infty$, the dominated convergence theorem implies $\int_{B_n} t\, d\sigma \to 0$.

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