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# FUNCTIONS DECREASING IN TRANSPOSITION AND THEIR APPLICATIONS IN RANKING PROBLEMS

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Let $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_n)$, $\lambda_1 \leqq \cdots \leqq \lambda_n$, and $\mathbf{x} = (x_1, \cdots, x_n)$. A function $g(\boldsymbol{\lambda}, \mathbf{x})$ is said to be decreasing in transposition (DT) if (i) $g$ is unchanged when the same permutation is applied to $\boldsymbol{\lambda}$ and to $\mathbf{x}$, and (ii) $g(\boldsymbol{\lambda}, \mathbf{x}) \geqq g(\boldsymbol{\lambda}, \mathbf{x}')$ whenever $\mathbf{x}'$ and $\mathbf{x}$ differ in two coordinates only, say $i$ and $j$, $(x_i - x_j) \cdot (i - j) \geqq 0$, and $x_i' = x_j$, $x_j' = x_i$. The DT class of functions includes as special cases other well-known classes of functions such as Schur functions, totally positive functions of order two, and positive set functions, all of which are useful in many areas including stochastic comparisons. Many well-known multivariate densities have the DT property. This paper develops many of the basic properties of DT functions, derives their preservation properties under mixtures, compositions, integral transformations, etc. A number of applications are then made to problems involving rank statistics.

**1. Introduction and summary.** In this paper we study the concept of functions decreasing in transposition (DT). The DT concept allows us to make stochastic comparisons among multivariate distributions. In the bivariate case, a function $f(\lambda_1, \lambda_2; x_1, x_2)$ is said to have the DT property if (a) $f(\lambda_1, \lambda_2; x_1, x_2) = f(\lambda_2, \lambda_1; x_2, x_1)$ and (b) $\lambda_1 < \lambda_2$, $x_1 < x_2$ implies that $f(\lambda_1, \lambda_2; x_1, x_2) \geqq f(\lambda_1, \lambda_2; x_2, x_1)$; i.e., transposing from the natural order $(x_1, x_2)$ to $(x_2, x_1)$ decreases the value of the function. One deals with precisely such comparisons in multivariate ranking problems.

This paper explores some of the basic aspects of DT functions, their preservation properties and applications in ranking problems. In future papers we propose to study other concepts such as DT families of distributions, other preservation theorems, and their applications in statistics. The results of the present paper generalize some of those of Proschan and Sethuraman (1977) and Nevius, Proschan and Sethuraman (1977) and help unify the area of stochastic comparisons.

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We now present a summary of the rest of the paper. In Section 2 we show that the DT class of functions includes as special cases other well-known classes of functions such as Schur functions, $TP_2$ functions, and positive set functions, all of which are useful in many areas including stochastic comparisons. Section 3 deals with various operations under which the DT property is preserved. Notable are the composition theorem (Theorem 3.3) and the preservation theorem (Theorem 3.7). These theorems are then used to show that many of the common multivariate distributions have DT densities. Section 4 gives applications to rank statistics.

**2. Definition and basic properties of functions decreasing in transposition.** Let $S$ be the group of all permutations of $\{1, 2, \cdots, n\}$. A member of $S$ will be denoted by $\boldsymbol{\pi} = (\pi_1, \cdots, \pi_n)$. The product operation is the composition of $\boldsymbol{\pi}$, $\boldsymbol{\pi}' \in S$; i.e.,

$$\boldsymbol{\pi} \circ \boldsymbol{\pi}'(i) = \boldsymbol{\pi}(\boldsymbol{\pi}'(i)) , \qquad i = 1, \cdots, n , \quad \text{where} \quad \pi'(i) = \pi_i' .$$

Thus $S$ is a noncommutative group. The identity element is $\mathbf{e} = (1, \cdots, n)$.

Let $\boldsymbol{\pi}$ and $\boldsymbol{\pi}'$ be two members of $S$ such that $\boldsymbol{\pi}'$ contains exactly one inversion of a pair of coordinates which occur in the natural order in $\boldsymbol{\pi}$; e.g.,

$$\begin{aligned} \boldsymbol{\pi} &= (\pi_1, \cdots, \pi_i, \cdots, \pi_j, \cdots, \pi_n) \qquad \text{and} \\ \boldsymbol{\pi}' &= (\pi_1, \cdots, \pi_j, \cdots, \pi_i, \cdots, \pi_n) , \end{aligned}$$

where $i < j$ and $\pi_i < \pi_j$. We say that $\boldsymbol{\pi}'$ is a *simple transposition* of $\boldsymbol{\pi}$; in symbols, $\boldsymbol{\pi} >^{\mathrm{t}} \boldsymbol{\pi}'$. Note that $\boldsymbol{\pi} >^{\mathrm{t}} \boldsymbol{\pi}' \Leftrightarrow \boldsymbol{\pi}^{-1} >^{\mathrm{t}} \boldsymbol{\pi}'^{-1}$.

Let $\boldsymbol{\pi}$ and $\boldsymbol{\pi}'$ be two elements in $S$ such that there exists a finite number of elements $\boldsymbol{\pi}^0, \boldsymbol{\pi}^1, \cdots, \boldsymbol{\pi}^k$ in $S$ satisfying $\boldsymbol{\pi} = \boldsymbol{\pi}^0 >^{\mathrm{t}} \boldsymbol{\pi}^1 >^{\mathrm{t}} \cdots >^{\mathrm{t}} \boldsymbol{\pi}^k = \boldsymbol{\pi}'$; i.e., $\boldsymbol{\pi}'$ is obtained from $\boldsymbol{\pi}$ by a finite number of simple transpositions. We say that $\boldsymbol{\pi}'$ is a *transposition* of $\boldsymbol{\pi}$.

Note that the elements of $S$ are partially ordered by transposition.

We say that a function $f$ from $S$ into $R^1$ is *decreasing in transposition* (DT) on $S$ if $\boldsymbol{\pi} >^{\mathrm{t}} \boldsymbol{\pi}'$ implies that $f(\boldsymbol{\pi}) \geqq f(\boldsymbol{\pi}')$ for $\boldsymbol{\pi}, \boldsymbol{\pi}'$ in $S$. Note that if $\boldsymbol{\pi}'$ is a transposition of $\boldsymbol{\pi}$ and $f$ is a DT function, then $f(\boldsymbol{\pi}) \geqq f(\boldsymbol{\pi}')$.

The following are some examples of DT functions on $S$, as can easily be verified. See also Lemma 2.2. Throughout the paper, the indices of sums and products range from 1 to $n$ unless otherwise indicated.

1. $f_1(\boldsymbol{\pi}) = -(\pi_1 + \cdots + \pi_k)$, where $1 \leqq k \leqq n$;
2. $f_2(\boldsymbol{\pi}) = \sum a_i \pi_i$, where $a_1 \leqq \cdots \leqq a_n$;
3. $f_3(\boldsymbol{\pi}) = \prod g(\lambda_i, \pi_i)$, where $\lambda_1 \leqq \cdots \leqq \lambda_n$, and $g(\lambda, i)$ is *totally positive of order* 2 ($TP_2$) for $-\infty < \lambda < \infty$ and $i = 1, \cdots, n$, i.e., $g(\lambda, i)$ is a nonnegative function such that $-\infty < \lambda_1 < \lambda_2 < \infty$, $1 \leqq i_1 < i_2 \leqq n$ implies that $g(\lambda_1, i_1)g(\lambda_2, i_2) - g(\lambda_2, i_1)g(\lambda_1, i_2) \geqq 0$;
4. $f_4(\boldsymbol{\pi}) = \sum g(\lambda_i, \pi_i)$, where $\lambda_1 \leqq \cdots \leqq \lambda_n$, and $g(\lambda, i)$ is a *positive set function*; i.e., $-\infty < \lambda_1 < \lambda_2 < \infty$, $1 \leqq i_1 < i_2 \leqq n$ implies that $g(\lambda_1, i_1) - g(\lambda_1, i_2) - g(\lambda_2, i_1) + g(\lambda_2, i_2) \geqq 0$;

5. $$f_5(\boldsymbol{\pi}) = 1 \quad \text{if } f(\boldsymbol{\pi}) \geqq a$$
$$= 0 \quad \text{if } f(\boldsymbol{\pi}) < a\,,$$

where $f$ is a DT function on $S$.

Thus far we have considered functions of one vector argument. Next we consider functions of two vector arguments. Let $g(\boldsymbol{\lambda}, \mathbf{x})$ be a function from $R^{2n}$ into $R^1$. Let $\boldsymbol{\lambda} \circ \boldsymbol{\pi}$ denote $(\lambda_{\pi_1}, \cdots, \lambda_{\pi_n})$, where $\boldsymbol{\pi}$ is a permutation in $S$. We say that $g(\boldsymbol{\lambda}, \mathbf{x})$ is *permutation-invariant* if

$$g(\boldsymbol{\lambda} \circ \boldsymbol{\pi}, \mathbf{x} \circ \boldsymbol{\pi}) = g(\boldsymbol{\lambda}, \mathbf{x})$$

for all $\boldsymbol{\pi} \in S$; i.e., applying a common permutation to both vector arguments $\boldsymbol{\lambda}$ and $\mathbf{x}$ leaves the function $g$ unchanged.

Let $\Lambda$, $M$ be subsets of $R^1$. We say that $g(\boldsymbol{\lambda}, \mathbf{x})$ is *decreasing in transposition* (DT) on $\Lambda^n \times M^n$ if

(i) $g(\boldsymbol{\lambda}, \mathbf{x})$ is permutation-invariant, and

(ii) $\boldsymbol{\lambda} \in \Lambda^n$, $\mathbf{x} \in M^n$, $\lambda_1 \leqq \cdots \leqq \lambda_n$, $x_1 \leqq \cdots \leqq x_n$, $\boldsymbol{\pi} >^t \boldsymbol{\pi}'$ implies that $g(\boldsymbol{\lambda}, \mathbf{x} \circ \boldsymbol{\pi}) \geqq g(\boldsymbol{\lambda}, \mathbf{x} \circ \boldsymbol{\pi}')$.

(We shall see in Lemma 2.1 that $\boldsymbol{\lambda}$ and $\mathbf{x}$ play dual roles.)

Note that condition (ii) just above may be replaced by the equivalent condition:

(ii′) Define $f_{\boldsymbol{\lambda},\mathbf{x}}(\boldsymbol{\pi}) = g(\boldsymbol{\lambda}, \mathbf{x} \circ \boldsymbol{\pi})$, where $\lambda_1 \leqq \cdots \leqq \lambda_n$ and $x_1 \leqq \cdots \leqq x_n$. Then $f_{\boldsymbol{\lambda},\mathbf{x}}(\boldsymbol{\pi})$ is DT on $S$.

The following are some examples of DT functions on $R^{2n}$. This can be easily verified from Lemma 2.2. We need some definitions before we present Example 6.

DEFINITIONS. Let $x_{[1]} \geqq \cdots \geqq x_{[m]}$ be a decreasing rearrangement of the coordinates of vector $\mathbf{x}$. Let $\mathbf{x}$ and $\mathbf{x}'$ satisfy:

$$\textstyle\sum_{i=1}^{j} x_{[i]} \geqq \sum_{i=1}^{j} x'_{[i]}\,, \qquad j = 1, \cdots, n-1$$
$$\textstyle\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} x'_{[i]}\,.$$

Then $\mathbf{x}$ is said to *majorize* $\mathbf{x}'$.

A function $f$ from $R^n$ into $R^1$ is said to be *Schur-convex* (*Schur-concave*) if $\mathbf{x}$ majorizes $\mathbf{x}'$ implies $f(\mathbf{x}) \geqq (\leqq) f(\mathbf{x}')$.

6. $g_6(\boldsymbol{\lambda}, \mathbf{x}) = h(\boldsymbol{\lambda} - \mathbf{x})$, where $h$ is a Schur-concave function on $R^n$: $g_{6'}(\boldsymbol{\lambda}, \mathbf{x}) = h(\boldsymbol{\lambda} + \mathbf{x})$, where $h$ is a Schur-convex function on $R^n$.

7. $g_7(\boldsymbol{\lambda}, \mathbf{x}) = \prod \phi(\lambda_i, x_i)$, where $\phi(\lambda, x)$ is $TP_2$ in $-\infty < \lambda < \infty$, $-\infty < x < \infty$. Note that a converse also holds: if a DT function $g(\boldsymbol{\lambda}, \mathbf{x})$ is of the form $\prod \phi(\lambda_i, x_i)$ with $\phi \geqq 0$, then $\phi$ must be $TP_2$.

8. $g_8(\boldsymbol{\lambda}, \mathbf{x}) = \sum \phi(\lambda_i, x_i)$, where $\phi$ is a positive set function.

We can also define DT functions on $R^n$. A function $h$ defined on $M^n$ is said to be *decreasing in transposition* (DT) on $M^n$ if for every $\mathbf{x} \in M^n$ with $x_1 \leqq \cdots \leqq x_n$ and for every pair $\boldsymbol{\pi}, \boldsymbol{\pi}' \in S$ satisfying $\boldsymbol{\pi} >^t \boldsymbol{\pi}'$, we have

$$h(\mathbf{x} \circ \boldsymbol{\pi}) \geqq h(\mathbf{x} \circ \boldsymbol{\pi}')\,.$$

Note that the corresponding function defined by

$$f_{\mathbf{x}}(\boldsymbol{\pi}) = h(\mathbf{x} \circ \boldsymbol{\pi}) \quad \text{is} \quad \mathrm{DT} \quad \text{on} \quad S .$$

It is clear from the definitions above that DT is essentially a property of functions on $S$. In most situations we can put $\Lambda = R^1 = M$, though in some cases like Theorem 3.7 and in some applications, one has functions defined only on $\Lambda^n \times M^n$, where $\Lambda$ and $M$ are proper subsets of $R^1$. Thus it becomes more convenient for many theoretical and practical applications to formulate the DT property for functions on $R^{2n}$ and on $R^n$. We summarize the relationships among the various domains in the following lemma. From now on we put $\Lambda = M = R^1$, unless some essential generality is to be gained by doing otherwise.

LEMMA 2.1. *Let $g(\boldsymbol{\lambda}, \mathbf{x})$ be a permutation-invariant function on $R^{2n}$. Define*

(a) $g^*(\mathbf{x}, \boldsymbol{\lambda}) = g(\boldsymbol{\lambda}, \mathbf{x})$ *for* $\boldsymbol{\lambda}, \mathbf{x} \in R^n$,

(b) $h_{\boldsymbol{\lambda}}(\mathbf{x}) = g(\boldsymbol{\lambda}, \mathbf{x})$ *for* $\mathbf{x} \in R^n$, $\lambda_1 \leqq \cdots \leqq \lambda_n$,

(c) $f_{\boldsymbol{\lambda},\mathbf{x}}(\boldsymbol{\pi}) = g(\boldsymbol{\lambda}, \mathbf{x} \circ \boldsymbol{\pi})$ *for* $\lambda_1 \leqq \cdots \leqq \lambda_n$, $x_1 \leqq \cdots \leqq x_n$, *and* $\boldsymbol{\pi} \in S$.

*Then the following statements are equivalent*:

(1) $g$ *is* DT *on* $R^{2n}$.

(2) $g^*$ *is* DT *on* $R^{2n}$.

(3) $h_{\boldsymbol{\lambda}}$ *is* DT *on* $R^n$ *for each* $\boldsymbol{\lambda}$ *such that* $\lambda_1 \leqq \cdots \leqq \lambda_n$.

(4) $f_{\boldsymbol{\lambda},\mathbf{x}}$ *is* DT *on* $S$ *for each* $\boldsymbol{\lambda}$ *and* $\mathbf{x}$ *such that* $\lambda_1 \leqq \cdots \leqq \lambda_n$ *and* $x_1 \leqq \cdots \leqq x_n$.

The equivalences follow immediately from the definitions of the various types of DT.

The next lemma shows that the concept of a DT function yields as special cases such well-known and useful concepts as (a) Schur-concave and Schur-convex functions, (b) total positivity of order 2, and (c) positive set functions.

LEMMA 2.2. (a) *Let $g(\boldsymbol{\lambda}, \mathbf{x}) = h(\boldsymbol{\lambda} - \mathbf{x})$. Then $g$ is* DT *on $R^{2n}$ if and only if $h$ is Schur-concave on $R^n$.*

(b) *Let $g(\boldsymbol{\lambda}, \mathbf{x}) = h(\boldsymbol{\lambda} + \mathbf{x})$. Then $g$ is* DT *on $R^{2n}$ if and only if $h$ is Schur-convex on $R^n$.*

(c) *Let $g(\boldsymbol{\lambda}, \mathbf{x}) = \prod h(\lambda_i, x_i)$. Then $g$ is* DT *if and only if $h$ is* $\mathrm{TP}_2$ *in $\lambda$ and $x$.*

(d) *Let $g(\boldsymbol{\lambda}, \mathbf{x}) = \sum h(\lambda_i, x_i)$. Then $g$ is* DT *if and only if $h$ is a positive set function.*

PROOF. We give the proof of (a) only. The rest are proved similarly. Let $\lambda_1 \leqq \lambda_2$ and $x_1 \leqq x_2 \leqq \cdots \leqq x_n$. Now $g(\boldsymbol{\lambda}, x_1, x_2, \cdots, x_n) - g(\boldsymbol{\lambda}, x_2, x_1, \cdots, x_n) = h(\lambda_1 - x_1, \lambda_2 - x_2, \cdots, \lambda_n - x_n) - h(\lambda_1 - x_2, \lambda_2 - x_1, \cdots, \lambda_n - x_n)$ and $(\lambda_1 - x_2, \lambda_2 - x_1)$ majorizes $(\lambda_1 - x_1, \lambda_2 - x_2)$. This shows that $g$ is DT if and only if $h$ is Schur-concave. □

**3. Preservation properties of functions decreasing in transposition.** In this section we show that the DT property is preserved under a number of basic mathematical and statistical operations.

We begin with the following lemma which is sometimes useful in determining whether a function is DT.

LEMMA 3.1. *Let $g(\boldsymbol{\lambda}, \mathbf{x})$ be* DT *on $R^{2n}$. Let $f$ and $h$ be permutation-invariant and nonnegative functions on $R^n$. Then $k(\boldsymbol{\lambda}, \mathbf{x}) \equiv f(\boldsymbol{\lambda})g(\boldsymbol{\lambda}, \mathbf{x})h(\mathbf{x})$ is* DT *on $R^{2n}$.*

PROOF. This lemma follows immediately from the definition of a DT function. ☐

The DT *property is preserved under mixtures*; stated formally:

THEOREM 3.2. *Let $f_\alpha$ be* DT *on $S$ and integrable with respect to $\mu$, a positive measure. Then $f(\boldsymbol{\pi}) = \int f_\alpha(\boldsymbol{\pi})\, d\mu(\alpha)$ is* DT.

The proof is obvious and hence omitted.

A similar preservation under mixtures property holds for DT functions $g(\boldsymbol{\lambda}, \mathbf{x})$ on $R^{2n}$ and DT functions $h(\mathbf{x})$ on $R^n$.

We will find very useful the fact that the DT *property is preserved under composition*; stated formally:

THEOREM 3.3. *Let $g_i$ be* DT *on $R^{2n}$, $i = 1, 2$. Let $g(\mathbf{x}, \mathbf{z}) \equiv \int \cdots \int g_1(\mathbf{x}, \mathbf{y})g_2(\mathbf{y}, \mathbf{z})\, d\sigma(y_1, \cdots, y_n)$ be well defined, where $\int_A d\sigma(\mathbf{y}) = \int_A d\sigma(\mathbf{y} \circ \boldsymbol{\pi})$ for each permutation $\boldsymbol{\pi} \in S$ and Borel set $A$ in $R^n$. Then $g(\mathbf{x}, \mathbf{z})$ is* DT *on $R^{2n}$.*

PROOF. That $g(\mathbf{x}, \mathbf{z})$ is permutation-invariant is obvious.

To complete the proof, it will suffice to show that $g(\mathbf{x}, \mathbf{z}) - g(\mathbf{x}, \mathbf{z}') \geqq 0$ for $x_1 \leqq \cdots \leqq x_n$, $z_1 < z_2$, $z_1' = z_2$, $z_2' = z_1$, and $z_i' = z_i$ for $i = 3, \cdots, n$. Write

$$g(\mathbf{x}, \mathbf{z}) - g(\mathbf{x}, \mathbf{z}') = \int \cdots \int [g_1(\mathbf{x}; y_1, y_2, \cdots)g_2(y_1, y_2, \cdots; z_1, z_2, \cdots) \\ - g_1(\mathbf{x}; y_1, y_2, \cdots)g_2(y_1, y_2, \cdots; z_2, z_1, \cdots)]\, d\sigma(\mathbf{y}),$$

where the "$\cdots$" indicates standard ordering of the omitted arguments. Breaking up the region of integration into the two regions $y_1 < y_2$ and $y_1 \geqq y_2$, and making a change of variable in the second region yields:

$$\begin{aligned} g(\mathbf{x}, \mathbf{z}) - g(\mathbf{x}, \mathbf{z}') &= \int \cdots \int_{y_1<y_2} [g_1(\mathbf{x}; y_1, y_2, \cdots)g_2(y_1, y_2 \cdots; z_1, z_2, \cdots) \\ &\quad - g_1(\mathbf{x}; y_1, y_2, \cdots)g_2(y_1, y_2, \cdots; z_2, z_1, \cdots) \\ &\quad + g_1(\mathbf{x}; y_2, y_1, \cdots)g_2(y_2, y_1, \cdots; z_1, z_2, \cdots) \\ &\quad - g_1(\mathbf{x}; y_2, y_1, \cdots)g_2(y_2, y_1, \cdots; z_2, z_1, \cdots)]\, d\sigma(\mathbf{y}) \\ &= \int \cdots \int_{y_1<y_2} [g_1(\mathbf{x}; \mathbf{y})g_2(\mathbf{y}; \mathbf{z}) - g_1(\mathbf{x}; \mathbf{y})g_2(\mathbf{y}; z_2, z_1, \cdots) \\ &\quad + g_1(\mathbf{x}; y_2, y_1, \cdots)g_2(\mathbf{y}; z_2, z_1, \cdots) \\ &\quad - g_1(\mathbf{x}; y_2, y_1, \cdots)g_2(\mathbf{y}, \mathbf{z})]\, d\sigma(\mathbf{y}) \end{aligned}$$

by virtue of the permutation-invariance property of $g_2$ and of $\sigma$. The integrand may be rewritten as

$$[g_1(\mathbf{x}, \mathbf{y}) - g_1(\mathbf{x}; y_2, y_1, \cdots)][g_2(\mathbf{y}, \mathbf{z}) - g_2(\mathbf{y}, z_2, z_1, \cdots)].$$

Since $g_1(g_2)$ is DT, the first (second) square bracket is nonnegative. Thus the integrand is nonnegative, and so $g(\mathbf{x}, \mathbf{z}) - g(\mathbf{x}, \mathbf{z}') \geqq 0$. ☐

In a similar fashion, we may prove analogous composition theorems for DT functions on $S$ and on $R^n$:

THEOREM 3.3′. *Let $f_1$ and $f_2$ be* DT *functions on $S$. Define*

$$f(\boldsymbol{\pi}) = \sum_{\pi^0 \in S} f_1(\boldsymbol{\pi}^0 \circ \boldsymbol{\pi}^{-1}) f_2(\boldsymbol{\pi}^0) .$$

*Then $f$ is* a DT *function on $S$.*

THEOREM 3.3″. *Let $h_1$ and $h_2$ be* DT *functions on $R^n$. Suppose that*

$$h(\boldsymbol{\pi}) = \int \cdots \int h_1(\mathbf{x} \circ \boldsymbol{\pi}^{-1}) h_2(\mathbf{x})\, dx_1 \cdots dx_n$$

*is well defined for each $\boldsymbol{\pi}$ in $S$. Then $h$ is a* DT *function on $S$.*

An immediate application of the composition theorem (Theorem 3.3) and of Lemma 2.2(a) is the following corollary:

COROLLARY 3.4. *Let $h_i$ be Schur-concave on $R^n$, $i = 1, 2$. Let $h(\mathbf{x}) = \int \cdots \int h_1(\mathbf{x} - \mathbf{y}) h_2(\mathbf{y})\, dy_1 \cdots dy_n$ denote the convolution of $h_1$ and $h_2$. Then $h$ is also Schur-concave on $R^n$.*

Corollary 3.4 is equivalent to the main result, Theorem 2.1, of Marshall and Olkin (1974).

Given a multivariate density $f(\boldsymbol{\lambda}, \mathbf{x})$ with parameter vector $\boldsymbol{\lambda}$, let $F(\boldsymbol{\lambda}, \mathbf{x})$ denote the corresponding distribution function and $\bar{F}(\boldsymbol{\lambda}, \mathbf{x})$ denote the joint survival probability $\int_{x_1}^{\infty} \cdots \int_{x_n}^{\infty} f(\boldsymbol{\lambda}, \mathbf{y})\, dy_1 \cdots dy_n$. Then the next corollary shows that both $F(\boldsymbol{\lambda}, \mathbf{x})$ and $\bar{F}(\boldsymbol{\lambda}, \mathbf{x})$ inherit the DT property from $f(\boldsymbol{\lambda}, \mathbf{x})$.

COROLLARY 3.5. *Let $f(\boldsymbol{\lambda}, \mathbf{x})$ be* DT. *Then $F(\boldsymbol{\lambda}, \mathbf{x})$ and $\bar{F}(\boldsymbol{\lambda}, \mathbf{x})$ are* DT.

PROOF. Write $F(\boldsymbol{\lambda}, \mathbf{x}) = \int \cdots \int f(\boldsymbol{\lambda}, \mathbf{y}) H(\mathbf{x} - \mathbf{y})\, dy_1 \cdots dy_n$, where $H(\mathbf{u}) = 1$ if $u_i \geqq 0$, $i = 1, \cdots, n$, and 0 otherwise. Now $f(\boldsymbol{\lambda}, \mathbf{y})$ is DT by hypothesis, while $H(\mathbf{x} - \mathbf{y})$ is DT, as is readily verified. Thus $F(\boldsymbol{\lambda}, \mathbf{x})$ is DT by the composition theorem.

Writing $\bar{F}(\boldsymbol{\lambda}, \mathbf{x}) = \int \cdots \int f(\boldsymbol{\lambda}, \mathbf{y}) H(\mathbf{y} - \mathbf{x})\, dy_1 \cdots dy_n$, we may prove $\bar{F}(\boldsymbol{\lambda}, \mathbf{x})$ is DT by the same argument. □

The DT property of nonnegative functions is preserved under products; stated formally:

THEOREM 3.6. *Let $g_i(\mathbf{x}, \mathbf{y})$ be a nonnegative* DT *function on $R^{2n}$, $i = 1, 2$. Then $g(\mathbf{x}, \mathbf{y}) \equiv g_1(\mathbf{x}, \mathbf{y}) g_2(\mathbf{x}, \mathbf{y})$ is* DT *on $R^{2n}$.*

The proof is obvious and thus omitted.

A similar preservation under products property holds for DT functions $f(\boldsymbol{\pi})$ on $S$ and DT functions $h(\mathbf{x})$ on $R^n$.

To present the next preservation property of DT functions, we need an additional definition.

Let $\Lambda$ and $T$ be semigroups in $R^1$. Let $\mu$ be a measure on $T$. It is said to be *invariant*, if

$$\mu(A \cap T) = \mu((A + x) \cap T)$$

for each Borel set $A$ of $R^1$ and each $x \in T$. A measurable function $\phi(\boldsymbol{\lambda}, \mathbf{x})$ integrable with respect to $\mu$, defined on $\Lambda^n \times T^n$ is said to have the *semigroup property* with respect to $\mu$ if, for each $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2$ in $\Lambda^n$ and $\mathbf{x}$ in $T^n$, $\phi(\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2, \mathbf{x}) = \int_{T^n} \phi(\boldsymbol{\lambda}_1, \mathbf{x} - \mathbf{y})\phi(\boldsymbol{\lambda}_2, \mathbf{y})\, d\mu(y_1) \cdots d\mu(y_n)$.

The next theorem shows that the Schur-convex (Schur-concave) property of functions is preserved under an integral transform through a DT function possessing the semigroup property.

THEOREM 3.7. *Let $f(\mathbf{x})$ be Schur-convex (Schur-concave) on $R^n$. Let $\phi(\boldsymbol{\lambda}, \mathbf{x})$ defined on $\Lambda^n \times T^n$ have the semigroup property with respect to an invariant measure $\mu$ and be* DT. *Let $h(\boldsymbol{\lambda}) = \int_{T^n} \phi(\boldsymbol{\lambda}, \mathbf{x})f(\mathbf{x})\, d\mu(x_1) \cdots d\mu(\mathbf{x}_n)$ be well defined for $\boldsymbol{\lambda} \in \Lambda^n$. Then $h(\boldsymbol{\lambda})$ is Schur-convex (Schur-concave).*

PROOF. We write

$$\begin{aligned} h(\boldsymbol{\lambda} + \boldsymbol{\lambda}') &= \int_{T^n} \phi(\boldsymbol{\lambda} + \boldsymbol{\lambda}', \mathbf{x})f(\mathbf{x})\, d\mu(x_1) \cdots d\mu(x_n) \\ &= \int_{T^{2n}} \phi(\boldsymbol{\lambda}, \mathbf{x} - \mathbf{y})\phi(\boldsymbol{\lambda}', \mathbf{y})f(\mathbf{x})\, d\mu(y_1) \cdots d\mu(y_n)\, d\mu(x_1) \cdots d\mu(x_n)\,. \end{aligned}$$

Substituting $\mathbf{z} = \mathbf{x} - \mathbf{y}$ and using the fact that $\mu$ is invariant, we obtain

$$h(\boldsymbol{\lambda} + \boldsymbol{\lambda}') = \int_{T^n} \phi(\boldsymbol{\lambda}', \mathbf{y})\left[\int_{T^n} \phi(\boldsymbol{\lambda}, \mathbf{z})f(\mathbf{z} + \mathbf{y})\, d\mu(z_1) \cdots d\mu(z_n)\right] d\mu(y_1) \cdots d\mu(y_n)\,.$$

Since $\phi(\boldsymbol{\lambda}, \mathbf{z})$ is DT in $\boldsymbol{\lambda}, \mathbf{z}$ and $f(\mathbf{z} + \mathbf{y})$ is DT in $\mathbf{z}, \mathbf{y}$, the composition, appearing within the square brackets above, is DT in $\boldsymbol{\lambda}, \mathbf{y}$ from the composition theorem (Theorem 3.3). By a second application of the same theorem, $h(\boldsymbol{\lambda} + \boldsymbol{\lambda}')$ is DT in $\boldsymbol{\lambda}, \boldsymbol{\lambda}'$, and hence $h(\boldsymbol{\lambda})$ is Schur-convex (Schur-concave), from Lemma 2.2b(a). □

The following special case of Theorem 3.7, equivalent to Theorem 1.1 of Proschan and Sethuraman (1977), is obtained by restricting $\phi(\boldsymbol{\lambda}, \mathbf{x})$ to be of the form $\prod \psi(\lambda_i, x_i)$.

COROLLARY 3.8. *Let $f(x)$ be Schur-convex (Schur-concave). Let $\psi(\lambda, x)$ defined on $(0, \infty) \times [0, \infty)$ obey the semigroup property in $\lambda$ with respect to an invariant measure $\mu$ on $[0, \infty)$, and be* $\mathrm{TP}_2$ *in $(\lambda, x)$. Define*

$$h(\boldsymbol{\lambda}) = \int \cdots \int \prod \psi(\lambda_i, x_i) f(\mathbf{x})\, d\mu(x_1) \cdots d\mu(x_n)\,.$$

*Then $h(\boldsymbol{\lambda})$ is Schur-convex (Schur-concave).*

Interpreting $\phi(\boldsymbol{\lambda}, \mathbf{x})$ as a multivariate density function with vector parameter $\boldsymbol{\lambda}$, we may interpret Theorem 3.7 as stating that *the Schur property of a function on the sample space is transformed into a corresponding Schur property of the expected value of the function on the parameter space.* This type of preservation property is very useful in deriving inequalities and bounds for a variety of multivariate distributions, as shown in Proschan and Sethuraman (1977) and in Nevius, Proschan and Sethuraman (1977).

By application of the next theorem, we may demonstrate that a large number of well-known multivariate densities are DT.

THEOREM 3.9. *Let $g(\boldsymbol{\lambda}, \mathbf{x})$ be a* DT *density of random variables $X_1, \cdots, X_n$. Let $u(\mathbf{x})$ be a permutation-invariant function on $R^n$. Then the conditional density $g_u(\boldsymbol{\lambda}, \mathbf{x})$ of $\mathbf{X}$ given that $u(\mathbf{X}) = u$ is a* DT *density.*

PROOF.

$$g_u(\boldsymbol{\lambda}, \mathbf{x}) = g(\boldsymbol{\lambda}, \mathbf{x}) I_{[u(\mathbf{x})=u]}/h(\boldsymbol{\lambda}, u) ,$$

where $h(\boldsymbol{\lambda}, u)$ is the induced density of $u(\mathbf{x})$. By hypothesis, $g(\boldsymbol{\lambda}, \mathbf{x})$ is DT. Trivially, $I_{[u(\mathbf{x})=u]}$ is permutation-invariant, as is the denominator. Thus by Lemma 3.1, the desired result follows. □

EXAMPLES 3.10. The following multivariate densities are DT, as verified following the listing.

1. *Multinomial.*

$$g_1(\boldsymbol{\lambda}, \mathbf{x}) = N! \prod \frac{\lambda_i^{x_i}}{x_i!} ,$$

where $0 < \lambda_i$, $x_i = 0, 1, 2, \cdots, i = 1, \cdots, n$, $\sum \lambda_i = 1$, and $\sum x_i = N$.

2. *Negative multinomial.*

$$g_2(\boldsymbol{\lambda}, \mathbf{x}) = \frac{\Gamma(N + \sum x_i)}{\Gamma(N)} \{(1 + \sum \lambda_i)^{-N-\sum x_i}\} \prod \frac{\lambda_i^{x_i}}{x_i!} ,$$

where $\lambda_i > 0$, $x_i = 0, 1, \cdots, i = 1, \cdots, n$, and $N > 0$.

3. *Multivariate hypergeometric.*

$$g_3(\boldsymbol{\lambda}, \mathbf{x}) = \prod \genfrac{[}{]}{0pt}{}{\lambda_i}{x_i} \Big/ \genfrac{[}{]}{0pt}{}{\sum \lambda_i}{N} ,$$

where $\lambda_i > 0$, $x_i = 0, 1, \cdots, \sum x_i = N < \sum \lambda_i$.

4. *Dirichlet.*

$$g_4(\boldsymbol{\lambda}, \mathbf{x}) = \frac{\Gamma(\theta + \sum \lambda_i)}{\Gamma(\theta) \prod \Gamma(\lambda_i)} (1 - \sum x_i)^{\theta-1} \prod x_i^{\lambda_i - 1} ,$$

where $\lambda_i > 0$, $x_i \geqq 0$, $i = 1, \cdots, n$, $\sum x_i \leqq 1$, and $\theta > 0$.

5. *Inverted Dirichlet.*

$$g_5(\boldsymbol{\lambda}, \mathbf{x}) = \frac{\Gamma(\theta + \sum \lambda_i)}{\Gamma(\theta) \prod \Gamma(\lambda_i)} \times \frac{\prod x_i^{\lambda_i - 1}}{(1 + \sum x_i)^{\theta + \sum \lambda_i}} ,$$

where $\lambda_i > 0$, $x_i \geqq 0$, $i = 1, \cdots, n$, and $\theta > 0$.

6. *Negative multivariate hypergeometric.*

$$g_6(\boldsymbol{\lambda}, \mathbf{x}) = \frac{N!\, \Gamma(\sum \lambda_i)}{\prod x_i!\, \Gamma(N + \sum \lambda_i)} \prod \frac{\Gamma(x_i + \lambda_i)}{\Gamma(\lambda_i)} ,$$

where $\lambda_i > 0$, $x_i = 0, 1, \cdots, N$, $\sum x_i = N$, and $N = 1, 2, \cdots$.

7. *Dirichlet compound negative multinomial.*

$$g_7(\boldsymbol{\lambda}, \mathbf{x}) = \frac{\Gamma(N + \sum x_i)\Gamma(\theta + \sum \lambda_i)\Gamma(N + \theta)}{\prod x_i!\, \Gamma(N)\Gamma(\theta)\Gamma(N + \theta + \sum \lambda_i + \sum x_i)} \prod \frac{\Gamma(x_i + \lambda_i)}{\Gamma(\lambda_i)} ,$$

where $\lambda_i > 0$, $x_i = 0, 1, \cdots, i = 1, \cdots, n$, $\theta > 0$, and $N = 1, 2, \cdots$.

8. *Multivariate logarithmic series distribution.*

$$g_8(\boldsymbol{\lambda}, \mathbf{x}) = \frac{(\sum x_i - 1)!}{\log(1 + \sum \lambda_i)} (1 + \sum \lambda_i)^{-\sum x_i} \prod \frac{\lambda_i^{x_i}}{x_i!},$$

where $\lambda_i > 0$, $x_i = 0, 1, \cdots, i = 1, \cdots, n$, and $\sum x_i > 0$.

9. *Multivariate F distribution*:

$$g_9(\boldsymbol{\lambda}, \mathbf{x}) = \frac{\Gamma(\lambda) \prod_{j=0}^{n} (\lambda_j)^{\lambda_j} \prod x_j^{\lambda_j - 1}}{\prod_{j=0}^{n} \Gamma(\lambda_j)(\lambda_0 + \sum \lambda_j x_j)^{\lambda}},$$

where $\lambda_j > 0, j = 0, 1, \cdots, n, \lambda = \sum_0^n \lambda_j, x_j \geqq 0, j = 1, \cdots, n$.

10. *Multivariate Pareto distribution.*

$$g_{10}(\boldsymbol{\lambda}, \mathbf{x}) = a(a+1) \cdots (a+n-1)(\prod \lambda_j)^{-1}(\sum \lambda_j^{-1} x_j - n + 1)^{-(a+n)},$$

where $x_j > \lambda_j > 0, j = 1, \cdots, n, a > 0$.

11. *Multivariate normal distribution with common variance and common covariance.*

$$g_{11}(\boldsymbol{\lambda}, \mathbf{x}) = |(2\pi)^{\frac{1}{2}} \textstyle\sum|^{-1} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\lambda})\Sigma^{-1}(\mathbf{x}-\boldsymbol{\lambda})'},$$

where $\sum$ is the positive definite covariance matrix with elements $\sigma^2$ along the main diagonal and elements $\rho\sigma^2$ elsewhere, $\rho > -(1/(n-1))$.

To verify that $g_1, g_2, g_4, g_5$ and $g_8$ are DT, note that $\lambda^x$ is $TP_2$, and from Lemma 2.2c, the product $g(\boldsymbol{\lambda}, \mathbf{x}) = \prod \lambda_i^{x_i}$ of $TP_2$ functions is DT. The additional factors that appear are functions of $\sum x_i$ and are permutation-invariant. Thus by Lemma 3.1, the desired conclusion follows.

To verify that $g_3, g_6$ and $g_7$ are DT, we use a similar argument. We note that the functions $[{\lambda \atop x}]$ and $\Gamma(\lambda + x)$ are $TP_2$. The remainder of the argument is as just above.

To verify that $g_9$ is DT, we first note that $g_9$ is the joint density of $(X_j/\lambda_j)/(X_0/\lambda_0), j = 1, \cdots, n$, where $X_j$ has a $\chi^2$-distribution with $2\lambda_j$ degrees of freedom, $j = 0, 1, \cdots, n$. For fixed outcome $X_0 = x_0$ say, the conditional density of $(X_j/\lambda_j)/(X_0/\lambda_0)$ is $TP_2$ in $\lambda_j, x_j$. Thus the corresponding joint density of $(X_1/\lambda_1)/(X_0/\lambda_0), \cdots, (X_n/\lambda_n)/(X_0/\lambda_0)$ is DT. By unconditioning on $X_0$ and using the fact that the DT property is preserved under mixtures (Theorem 3.2), we conclude that $g_9$ is DT.

Note $g_{10}$ is DT since $(\sum \lambda_j^{-1} x_j - n + 1)^{-(a+n)}$ is DT .

Eaton (1967) and Marshall and Olkin (1974) show that $g_{11}$ is DT. (This can be verified directly from the definition of DT by showing that $(\mathbf{x} - \boldsymbol{\lambda}) \sum^{-1} (\mathbf{x} - \boldsymbol{\lambda})' \leqq (\mathbf{x}' - \boldsymbol{\lambda}) \sum^{-1} (\mathbf{x}' - \boldsymbol{\lambda})'$; where $x_1 \leqq x_2 \leqq \cdots \leqq x_n$, $\lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_n$, and $\mathbf{x}' = (x_2, x_1, x_3, \cdots, x_n)$.)

**4. Applications to ranking problems.** Given a set of real numbers $\{x_1, \cdots, x_n\}$, let $r_i$ denote the rank of $x_i$; i.e., $r_i = 1 + \sum_{j \neq i} I(x_i, x_j)$, where $I(a, b) = 1$ if $a > b$, $\frac{1}{2}$ if $a = b$, and 0 if $a < b$. If there are tied $x$'s, this definition yields the average ranks. Let $\mathbf{r} = (r_1, \cdots, r_n)$, the vector of ranks, or the *rank order*. Similarly, for random variables $X_1, \cdots, X_n$, let $R_i$ denote the rank of $X_i$, and $\mathbf{R} = (R_1, \cdots, R_n)$.

THEOREM 4.1. *Let* $X_1, \cdots, X_n$ *have joint density function* $\phi(\boldsymbol{\lambda}, \mathbf{x})$, *a* DT *function on* $R^{2n}$ *with vector parameter* $\boldsymbol{\lambda}$. *Let* $g(\boldsymbol{\lambda}, \mathbf{r}) = P_\lambda[\mathbf{R} = \mathbf{r}]$ *for* $\mathbf{r} \in R^n$, *denote the probability of rank order* $\mathbf{r}$. *Then* $g(\boldsymbol{\lambda}, \mathbf{r})$ *is a* DT *function on* $R^{2n}$.

PROOF. We may write $g(\boldsymbol{\lambda}, \mathbf{r})$ as:

$$g(\boldsymbol{\lambda}, \mathbf{r}) = \int \phi(\boldsymbol{\lambda}, \mathbf{x}) J(\mathbf{x}, \mathbf{r}) \, d\sigma(x_1, \cdots, x_n) \tag{4.1}$$

where $\sigma$ is a permutation-invariant measure and where $J(\mathbf{x}, \mathbf{r}) = 1$ if $x_i$ has rank $r_i$, $i = 1, \cdots, n$, and $= 0$ otherwise. Since $\phi(\boldsymbol{\lambda}, \mathbf{x})$ is DT by hypothesis and $J(\mathbf{x}, \mathbf{r})$ is DT by construction, it follows that the composition $g(\boldsymbol{\lambda}, \mathbf{r})$ given in (4.1) is DT by Theorem 3.3. □

Thus if a set of random variables has a DT density, the corresponding rank order has a DT frequency function.

COROLLARY 4.2. *Let* $f$ *be a* DT *function on* $R^n$. *Let* $\mathbf{R}$ *be the rank order of vector* $\mathbf{X}$ *where* $\mathbf{X}$ *has the* DT *density* $\phi(\boldsymbol{\lambda}, \mathbf{x})$. *For real-valued* $a$, *define*

$$h_a(\boldsymbol{\lambda}) = P_\lambda[f(\mathbf{R}) \geqq a] .$$

*Then for each real fixed* $a$, $h_a(\boldsymbol{\lambda})$ *is a* DT *function on* $R^n$.

PROOF. $h_a(\boldsymbol{\lambda}) = \sum_{\mathbf{r}} I_{[f(\mathbf{r}) \geqq a]} g(\boldsymbol{\lambda}, \mathbf{r})$. By Theorem 4.1, $g(\boldsymbol{\lambda}, \mathbf{r})$ is DT on $R^{2n}$. Since $f(\mathbf{r})$ is DT on $R^n$, it follows that $I_{[f(\mathbf{r}) \geqq a]}$ is DT on $R^n$. Thus by Theorem 3.3, the composition $h_a(\boldsymbol{\lambda})$ is DT on $R^n$. □

REMARK 1. Thus if $\lambda_1 \leqq \cdots \leqq \lambda_n$ and $\boldsymbol{\pi} >^t \boldsymbol{\pi}'$, then the distribution of $f(\mathbf{R})$ when $\mathbf{X}$ has parameter $\boldsymbol{\lambda} \circ \boldsymbol{\pi}$ is stochastically larger than the distribution of $f(\mathbf{R})$ when $\mathbf{X}$ has parameter $\boldsymbol{\lambda} \circ \boldsymbol{\pi}'$.

REMARK 2. Note that Theorem 4.1 and Corollary 4.2 do not require that the DT density of $\mathbf{X}$ be absolutely continuous. Our theory easily covers "ties"; we simply use average ranks and thus do not insist that $\mathbf{r}$ be restricted to the set $S$. The reader should thus be aware that the subsequent applications discussed in this section also apply to multivariate discrete DT densities such as $g_1$, $g_2$, $g_3$, $g_6$, $g_7$ and $g_8$ of Section 3.

APPLICATION 4.3. (The trend problem). Let $X_i$ have $\mathrm{TP}_2$ density $f(\lambda_i, x)$ and let $\lambda_1 \leqq \cdots \leqq \lambda_n$. Then Theorem 1 of Savage (1957) states essentially that $g(\boldsymbol{\lambda}, \mathbf{r}) = P_\lambda[\mathbf{R} = \mathbf{r}]$ is a DT function. Savage's result follows from the application of Theorem 4.1 to the function $g_7$ of Section 2. As a further application, put $U(\mathbf{r}) = -\sum_{i=1}^m r_i$, where $1 \leq m \leq n$, and note that $U(\mathbf{r})$ is DT on $R^n$. From Corollary 4.2, it follows that if $\lambda_1 \leqq \cdots \leqq \lambda_n$ and $\boldsymbol{\pi} >^t \boldsymbol{\pi}'$, then the distribution of $U(\mathbf{R})$ under $\boldsymbol{\lambda} \circ \boldsymbol{\pi}$ is stochastically larger than the distribution of $U(\mathbf{R})$ under $\boldsymbol{\lambda} \circ \boldsymbol{\pi}'$. Restricting $\lambda_1 = \cdots = \lambda_m = 1$ and $\lambda_{m+1} = \cdots = \lambda_n = \lambda > 1$ in the above, we obtain a stochastic comparison result for the Wilcoxon statistic in the two-sample problem if the experimenter mistakenly counts observations from the second distribution as arising from the first distribution. These ideas are generalized and summarized in the following theorem.

THEOREM 4.4. *Let the random vector* $\mathbf{X}$ *have a density* $\phi(\boldsymbol{\lambda}, \mathbf{x})$ *which is* DT *on* $R^{2n}$. *Let* $\mathbf{R}$ *denote the vector of ranks of* $X_1, \cdots, X_n$. *Let* $E_{n1} \leqq E_{n2} \leqq \cdots \leqq E_{nn}$ *be numbers (scores) and let* $T(\mathbf{r}) = \sum_{i=1}^{m} E_{nr_i}$, *where* $1 \leqq m \leqq n$. *Finally, let* $\lambda_1 \leqq \cdots \leqq \lambda_n$ *and* $\boldsymbol{\pi} >^{\mathrm{t}} \boldsymbol{\pi}'$. *Then the distribution of* $T(\mathbf{R})$ *under* $\boldsymbol{\lambda} \circ \boldsymbol{\pi}$ *is stochastically smaller than the distribution under* $\boldsymbol{\lambda} \circ \boldsymbol{\pi}'$.

PROOF. The proof follows directly from Theorem 4.1, Corollary 4.2 and the easily verified fact that $T(\mathbf{r})$ is DT. □

Theorem 4.4 is applicable to many two-sample rank statistics including the Wilcoxon statistic ($E_{ni} = i$) and the normal scores statistic ($E_{ni}$ = the expected value of the $i$th order statistic in a random sample of size $n$ from a standard normal distribution).

There is an open question in this connection that we have not solved, and that does not follow merely from the DT concept. In Theorem 4.4, set $\lambda_1 = \cdots = \lambda_m = 1$, $\lambda_{m+1} = \cdots = \lambda_n = \lambda > 1$, and $\phi(\boldsymbol{\lambda}, \mathbf{x}) = \prod \psi(\lambda_i, x_i)$. Can it be shown that the distribution of $T(\mathbf{r})$ has a monotone likelihood ratio in $\lambda$? This is closely related to the conjecture of Saxena and Savage (1969).

REMARK 4.5. In Application 4.3, Savage's result for the trend case, the $X$'s are assumed to be mutually independent. However, Theorem 4.1 is applicable even when the $X$'s are dependent, as long as $\phi(\boldsymbol{\lambda}, \mathbf{x})$ is DT. Thus Theorem 4.1 gives conditions under which one rank order is at least as likely as another, under densities corresponding to dependent variables. In the spirit of Savage's paper, these results are readily translated into conditions for admissible rank tests in dependency situations. Examples of densities corresponding to nonindependent $X$'s are given in Section 3.

Similarly, Theorem 4.4 is applicable in two-sample cases where the assumptions of independence within each sample, and between samples, can be relaxed to DT densities corresponding to nonindependent $X$'s such as those given in Section 3. In this sense, Theorem 4.4 generalizes results of Savage (1956) to dependency situations.

APPLICATION 4.6 (Randomized blocks with ordered alternatives). Consider a randomized block experiment with $n$ treatments and $N$ blocks. Let $\mathbf{X}_i = (X_{i1}, \cdots, X_{in})$, $i = 1, \cdots, N$, be $N$ mutually independent vectors. From Corollary 4.2 and the independence of the $\mathbf{X}_i$'s, we can state:

COROLLARY 4.7. *Let* $\mathbf{X}_i$ *have density* $\phi_i(\boldsymbol{\lambda}, \mathbf{x})$, *where each* $\phi_i$ *is* DT *on* $R^{2n}$. *Let* $f$ *be a* DT *function on* $S$. *Let* $\mathbf{R}_i = (r_{i1}, \cdots, r_{in})$, *where* $r_{ij}$ *is the rank of* $X_{ij}$ *among* $X_{i1}, \cdots, X_{in}$. *If* $\lambda_1 \leqq \cdots \leqq \lambda_n$ *and* $\boldsymbol{\pi} >^{\mathrm{t}} \boldsymbol{\pi}'$, *then the distribution of* $\sum_1^N f(\mathbf{R}_i)$ *when each* $\mathbf{X}_i$ *has parameter* $\lambda \circ \boldsymbol{\pi}$ *is stochastically larger than the distribution of* $\sum_1^N f(\mathbf{R}_i)$ *when each* $\mathbf{X}_i$ *has parameter* $\boldsymbol{\lambda} \circ \boldsymbol{\pi}'$.

Corollary 4.7 gives power results about certain rank tests of $H_0$: $\lambda_1 = \lambda_2 = \cdots = \lambda_n$ versus ordered alternatives $\lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_n$, since many such tests are based on statistics of the form $\sum_{i=1}^{N} T(\mathbf{R}_i)$, where $T(\mathbf{R}_i)$ is a DT function of the form $T(\mathbf{R}_i) = \sum_{j=1}^{n} c_j E_{nr_{ij}}$, where $c_1 \leqq c_2 \leqq \cdots \leqq c_n$ are "regression" constants and $E_{n1} \leqq E_{n2} \leqq \cdots \leqq E_{nn}$ are scores. Ordered alternative test statistics of this form, for which Corollary 4.7 is applicable, include those due to Page (1963) ($c_j = j$, $E_{nj} = j$) and Pirie and Hollander (1972) ($c_j = j$, $E_{nj} =$ the expected value of the $j$th order statistic in a random sample of size $n$ from a normal distribution). Note here that the blocks can have different densities $\phi_i$, and, once again, the $\phi_i$'s need *not* be joint densities of independent random variables.

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