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ON THE EXPONENTIAL BOUNDEDNESS OF STOPPING TIMES OF INVARIANT SPRT'S

BY HOLGER ROOTZÉN[1] AND GORDON SIMONS[2]

University of Lund and University of North Carolina at Chapel Hill

It is shown, under conditions which include invariant sequential probability ratio tests, that the stopping time is always expontially bounded when the null or alternative hypothesis holds, except in a trivial instance.

1. Introduction and theorem. This paper is primarily concerned with showing that, for invariant sequential probability ratio tests, the stopping time is exponentially bounded under the null and alternative hypotheses, with the exception of trivial situations. This work is the by-product of a largely unsucessful attempt to obtain general results under nonmodel distributions. Since we view the lack of these general results as a major deficiency in the theory of sequential analysis, we shall suggest what we believe is a reasonable conjecture and shall discuss in Section 2 some of the past literature on the subject.

Let $(\Omega, \mathscr{B})$ be a measurable space and $\{\mathscr{B}_n, n \geqq 1\}$ be a nondecreasing sequence of sub-σ-fields of $\mathscr{B}$. A stopping time N adapted to $\{\mathscr{B}_n, n \geqq 1\}$[3] is said to be *exponentially bounded* for a family of probability measures $\mathscr{R}$ on $(\Omega, \mathscr{B})$ if for each probability measure $R \in \mathscr{R}$ there exist constants $c > 0$ and $\rho < 1$ such that $R(N > n) < c\rho^n$, $n = 1, 2, \cdots$. Such a condition implies that N is finite a.s. $(\mathscr{R})$ and that N has a moment generating function in some neighborhood of the origin for each $R \in \mathscr{R}$.

Let P and Q be elements in $\mathscr{R}$ and let L_n denote the $P - Q$ likelihood ratio for $\mathscr{B}_n$.[4] Sequential probability ratio tests are defined in terms of a stopping time N of the general form

$$\begin{aligned} (1) \qquad N &= \text{the first} \quad n \geqq 1 \quad \text{such that} \quad L_n \notin (A, B) \\ &= \infty \quad \text{if no such} \quad n \quad \text{exists,} \end{aligned}$$

where $0 < A < B < \infty$. At this level of generality, it is easy to construct examples for which N is exponentially bounded for $\{P, Q\}$ and others for which it is not. All that is known, in general, can be derived from the following set of

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[3] Here, N is said to be adapted to $\{\mathscr{B}_n, n \geqq 1\}$ if $[N = 1, 2, \cdots \text{ or } \infty] = \Omega$ and $[N = n] \in \mathscr{B}_n$ for $n = 1, 2, \cdots$.

[4] For a precise definition of this concept see Eisenberg, Ghosh and Simons (1976). Essentially, L_n is the Radon–Nikodym derivative dQ/dP relative to $\mathscr{B}_n$.

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elementary inequalities (see Eisenberg, Ghosh and Simons (1976)):

$$AP(N > n) \leqq Q(N > n) \leqq BP(N > n)\,, \qquad n = 1, 2, \cdots .$$

It follows immediately that

(a) $P(N < \infty) = 1$ iff $Q(N < \infty) = 1$,
(b) for each $r > 0$, $\int N^r\, dP < \infty$ iff $\int N^r\, dQ < \infty$, and
(c) N is exponentially bounded for $\{P\}$ iff it is exponentially bounded for $\{Q\}$.

Now suppose $X_1, X_2, \cdots$ is a sequence of random elements on $(\Omega, \mathscr{B})$ (to the statistician, "potential data") and $\mathscr{B}_n = \sigma(X_1, \cdots, X_n)$ (the σ-field generated by $X_1, \cdots, X_n$). If $X_1, X_2, \cdots$ is an i.i.d. sequence under P and Q, then $\{\log L_n, n \geqq 1\}$ is a random walk under both P and Q. It follows that N is exponentially bounded for $\{P, Q\}$ unless $L_1 = 1$ a.s. (P, Q). This is a well-known result due to Stein (1946). We shall now describe a similar result for invariant sequential probability ratio tests.

Consider the following mathematical structure:

(i) $\mathscr{P}$ and $\mathscr{Q}$ are two disjoint families of probability measures on $(\Omega, \mathscr{B})$.

(ii) $X_1, X_2, \cdots$ is an i.i.d. sequence of random elements for each probability measure $R \in \mathscr{P} \cup \mathscr{Q}$.

(iii) For $n \geqq 1$, $\mathscr{B}_n$ is a sub-σ-field of $\sigma(X_1, \cdots, X_n)$ which is symmetric in $X_1, \cdots, X_n$ in the sense that if B is a measurable set in the range of $(X_1, \cdots, X_n)$ and $\mathscr{B}_n$ contains the event $[(X_1, \cdots, X_n) \in B]$, then, for each permutation $(i_1, \cdots, i_n)$ of the indices $1, \cdots, n$, it contains the event $[(X_{i_1}, \cdots, X_{i_n}) \in B]$.

(iv) $\mathscr{B}_1 \subset \mathscr{B}_2 \subset \mathscr{B}_3 \subset \cdots$.

(v) For each $n \geqq 1$, the probability space $(\Omega, \mathscr{B}_n, R)$ is the same for every $R \in \mathscr{P}$, and is the same for every $R \in \mathscr{Q}$.

This is the structure one encounters when invariant sequential probability ratio tests are being considered. (For instance, for the sequential t-test [cf. T. L. Lai (1975)], $X_1, X_2, \cdots$ is a sequence of independent normal random variables with common mean μ and common variance σ^2. The object is to test whether the ratio μ/σ equals γ_0 or some other value γ_1. The members of $\mathscr{P}$ and $\mathscr{Q}$ correspond to pairs (μ, σ^2) for which the ratio is γ_0 and γ_1, respectively. The appropriate σ-field $\mathscr{B}_n$ can be expressed as $\sigma(X_1/|X_1|, X_2/|X_1|, \cdots, X_n/|X_1|)$, $n \geqq 1$.) If one wishes to test the (composite) hypothesis that the true probability measure R belongs to $\mathscr{P}$ against the (composite) alternative that it belongs to $\mathscr{Q}$, the following sequential probability ratio test suggests itself: Let $P \in \mathscr{P}$ and $Q \in \mathscr{Q}$ be chosen in an arbitrary manner, and let L_n be the $P - Q$ likelihood ratio for $\mathscr{B}_n$ $(n \geqq 1)$. Because of property (v), each possible pair (P, Q) leads to the same sequence of likelihood ratios $L_1, L_2, \cdots$. Let N be the stopping time described in (1) and choose $\mathscr{P}$ or $\mathscr{Q}$ according as $N < \infty$ and $L_N \leqq A$ or $N < \infty$ and $L_N \geqq B$, respectively. (If $N = \infty$, no decision is made.) We have the following theorem:

THEOREM. *Either* $L_n = 1$ a.s. $(\mathscr{P} \cup \mathscr{Q})$ *for* $n \geqq 1$, *or* N *is exponentially bounded for* $\mathscr{P} \cup \mathscr{Q}$.

PROOF. In view of (c) and (v), N is exponentially bounded for $\mathscr{P} \cup \mathscr{Q}$ iff it is exponentially bounded for $\{P\}$. Likewise, $L_n = 1$ a.s. $(\mathscr{P} \cup \mathscr{Q})$ for $n \geqq 1$ iff $P(L_n = 1) = 1$ for $n \geqq 1$. Suppose the latter is false, i.e., that for some k, $P(L_k = 1) < 1$. For simplicity, we shall assume that k can be taken to be unity. If a larger value of k is required, the proof we shall give that N is exponentially bounded for $\{P\}$ can be easily modified. Express L_1 as $l(X_1)$ and observe that $EL_1 \leqq 1$ and (since $P(L_1 = 1) < 1$) that $E \log L_1 < 0$. Set $\mathscr{B}_n' = \sigma(l(X_1), \cdots, l(X_n))$ and observe that $L_n' = \prod_{i=1}^n l(X_i)$ is the $P - Q$ likelihood ratio for $\mathscr{B}_n'$ in view of condition (ii) $(n \geqq 1)$. (For the sequential t-test referred to above, $\mathscr{B}_n'$ can be expressed in the somewhat simpler appearing form $\mathscr{B}_n' = \sigma(X_1/|X_1|, X_2/|X_2|, \cdots, X_n/|X_n|)$. The function $l_1(X_1)$ equals $\Phi(\gamma_1)/\Phi(\gamma_0)$ when $X_1 > 0$ and equals $\Phi(-\gamma_1)/\Phi(-\gamma_0)$ when $X_1 < 0$, where Φ denotes the standard normal distribution function.) Moreover, conditions (iii) and (iv) imply that $\mathscr{B}_n' \subset \mathscr{B}_n$ and, hence, $E^{\mathscr{B}_n'} L_n \leqq L_n'$[5] for $n \geqq 1$. Thus, for $\rho < 1$, to be chosen later,

$$P^{\mathscr{B}_n'}(L_n > A, L_n' \leqq \rho^n) \leqq A^{-1} I_{[L_n' \leqq \rho^n]} E^{\mathscr{B}_n'} L_n \leqq A^{-1}\rho^n$$

and, hence,

$$\begin{aligned} P(N > n) &\leqq P(L_n > A) = EP^{\mathscr{B}_n'}(L_n > A) \\ &\leqq A^{-1}\rho^n + P(L_n' > \rho^n) . \end{aligned}$$

Thus it suffices to show, for properly chosen $\rho < 1$ and $\rho' < 1$, that

$$P(L_n' > \rho^n) \leqq (\rho')^n .$$

Since $EL_1 \leqq 1$, $E(L_1)^t \leqq 1$ for $0 < t \leqq 1$. In fact, if ρ is chosen so that $E \log L_1 < \log \rho < 0$, then there exists a small positive t_0 such that

$$\rho' \equiv E e^{t_0(\log L_1 - \log \rho)} < 1 .$$

Then (cf. H. Chernoff (1952), equation (3.6)),

$$\begin{aligned} P(L_n' > \rho^n) &= P(\textstyle\prod_{i=1}^n l(X_i) > \rho^n) \\ &\leqq \rho^{-nt_0} E[\textstyle\prod_{i=1}^n l(X_i)]^{t_0} \\ &= \rho^{-nt_0} E^n(L_1)^{t_0} = (\rho')^n . \end{aligned}$$ □

REMARK 1. The case "$L_n = 1$ a.s. $(\mathscr{P} \cup \mathscr{Q})$ for $n \geqq 1$," appearing in the theorem, is of no importance since it corresponds to a testing situation in which no "information" is available with which to distinguish between the families $\mathscr{P}$ and $\mathscr{Q}$, i.e., all of the probability measures $R \in \mathscr{P} \cup \mathscr{Q}$ agree on $\mathscr{B}_n$ for every $n \geqq 1$.

REMARK 2. Robert Berk has informed us of an earlier (unpublished) use of the σ-fields $\mathscr{B}_n'$ (appearing in our proof), namely in his Ph. D. thesis (1964,

[5] Equality holds if P and Q are equivalent probability measures on $\mathscr{B}_n$.

pages 65–67). He uses them to show (the weaker result) that invariant sequential probability ratio tests stop with probability one under the null and alternative hypotheses.

REMARK 3. Our proof shows that the "single boundary" stopping time $N^* = \inf\{n \geqq 1 : L_n \geqq A\}$ is exponentially bounded for $\{P\}$.

REMARK 4. We suspect that our theorem would still hold if condition (iii) above were replaced by the weaker condition:

(iii′) For some $P \in \mathscr{P}$ and $Q \in \mathscr{Q}$ and each $n \geqq 1$, there is a version of the $P - Q$ likelihood ratio L_n which is a symmetric function of $X_1, \cdots, X_n$.

With this replacement (and with the help of the Hewitt–Savage zero-one law and the martingale convergence theorem), we have been able to establish the weaker result that N is finite a.s. ($\mathscr{P} \cup \mathscr{Q}$) (except, of course, when $L_n = 1$ a.s. ($\mathscr{P} \cup \mathscr{Q}$), $n \geqq 1$). This represents a mild improvement on Berk's result, referred to in Remark 2, but we can suggest no practical testing situation where the additional strength is useful.

2. Discussion. It is reasonable to ask whether the theorem above can be extended to nonmodel situations. Specifically, it would be nice to know whether the stopping time N of an invariant sequential probability ratio test is either (i) exponentially bounded, or (ii) almost surely finite, when the observations $X_1, X_2, \cdots$ are i.i.d. under a probability measure R *not* in $\mathscr{P} \cup \mathscr{Q}$. While the answer for neither (i) nor (ii) is an unqualified "yes," there still remains hope that there are useful general theorems yet to be discovered.

R. Wijsman (1972) discusses an interesting example for which (ii) holds but (i) does not. However, this example is defective in the sense that his probability measure R is orthogonal to the probability measures in $\mathscr{P} \cup \mathscr{Q}$ on each of the σ-fields $\mathscr{B}_n$. The difficulty is that a $P - Q$ likelihood ratio is uniquely defined up to a P and Q equivalence but not necessarily uniquely with respect to another probability measure such as R. Indeed, there exist other versions of the likelihood ratios L_n, appropriate to his example (albeit less natural from a topological viewpoint), for which (i) and (ii) both hold.

Recently, Wijsman (1976a) has been working with more revealing examples. His probability measures R are not equivalent to the probability measures in $\mathscr{P} \cup \mathscr{Q}$, but at least his R's are dominated by them (on each σ-field $\mathscr{B}_n$), which implies that the likelihood ratios of interest are unique up to R-equivalences. For these examples, again (ii) holds but (i) does not.

In our investigations, we have found even more discouraging examples. We have examples for which R is dominated but not equivalent to the probability measures in $\mathscr{P} \cup \mathscr{Q}$ and for which (i) and (ii) both fail in a nontrivial sense, i.e., $R(L_n = 1) \neq 1$ for some $n \geqq 1$. In these examples, $R(N = \infty) = 1$.

With such examples in mind, we have attempted, without success, to show (ii) (except in trivial situations) for probability measures R which are *equivalent* to those in $\mathscr{P} \cup \mathscr{Q}$. We still conjecture that such a result holds but have recently been informed by Wijsman that (i) cannot be demonstrated under the same set of circumstances. The counterexample he has in mind is based upon a variation (due to T. L. Lai) of Lai's (1975) Lemma 5 (page 588).

We do not want to leave the reader with the impression that there are no results of a general nature concerning $R \notin \mathscr{P} \cup \mathscr{Q}$. R. Berk (1970) has obtained results for what he calls *parametric* sequential prabability ratio tests and R. Wijsman (1976b) has recently obtained a theorem concerned with exponential boundedness. Their results exploit the fact that, in many useful examples of invariant sequential probability ratio tests, the log-likelihood ratios $\{\log L_n, n \geqq 1\}$ become a random walk asymptotically, in a certain sense, (under R) as $n \rightarrow \infty$. Nevertheless, there are practical examples where no asymptotic random walk is discernable. Therefore, if our conjecture above has validity, some other approach will be required to show it. We have tried to show the conjecture by using the auxiliary σ-fields referred to in Remark 2 above, feeling this might be the "other approach" required, but, as noted, without success.

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Department of Statistics
University of North Carolina
Chapel Hill, North Carolina 27514