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A NOTE ON ORTHOGONAL PARTITIONS AND SOME WELL-KNOWN STRUCTURES IN DESIGN OF EXPERIMENTS[1]

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Finney has used orthogonal partitions in the context of the search for higher order (coarser) partitions of given Latin squares. Hedayat and Seiden use the term F-square to denote higher order partitions that are orthogonal to both rows and columns. This note is a short expository treatment of orthogonal partitions in general and is based on the identification of a partition with a vector subspace of Euclidean N-space R^N. This identification is not new as it is part of the usual vector space approach to analysis of variance. This approach puts the concept of orthogonal partitions in a simple light unencumbered by the language of design of experiments. Another advantage is that certain published bounds on the maximum number of orthogonal partitions of specified type are immediate from the dimensionality restriction imposed by R^N. In addition, some counting problems are identified which are of possible interest to researchers in design of experiments and combinatorics.

1. Introduction and summary. In design of experiments are found structures which consist of arrays of symbols. An array of N positions filled with symbols induces a partition of $S = \{1, 2, \cdots, N\}$ into subsets corresponding to the positions where the various symbols appear. For example, an $n \times n$ Latin square has $N = n^2$ positions. The n symbols partition S into n subsets of size n as do the row and column partitions. In this note (Section 2) we will identify partitions Π of S with subspaces $\mathscr{V}(\Pi)$ of R^N and define orthogonality of partitions by the orthogonality of the corresponding subspaces. This simple device immediately yields crude bounds on the maximum number of orthogonal F-squares of specified types and the maximum number of constraints k in an orthogonal array OA (N, k, s, t). This approach is not used to obtain sharp bounds, but it is conceptually simple and a convenient way to pose some counting problems (Section 3).

2. Orthogonal partitions and some structures in design of experiments. In this section we define orthogonality of partitions and in Remark 2 give a dimensionality constraint for a system of orthogonal partitions. Let

$$S = \{1, 2, \cdots, N\}$$

and for subsets $T \subset S$, let $|T|$ denote the cardinality of T. A partition Π of S is a collection of nonempty disjoint subsets of S called blocks whose union is

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S. Let Π be a partition of S and define

$$k(\Pi) \equiv \text{number of elements of } \Pi$$

and

$$c(\Pi) \equiv \text{smallest block size in } \Pi .$$

Of course, $1 \leqq k(\Pi)c(\Pi) \leqq N$.

Let R^N denote Euclidean N-space with the usual inner product denoted by $x'y$ and let $\mathbf{1}$ denote the vector of all 1's. The orthogonal projection of x onto $\mathbf{1}$ will be denoted by x_1. When it is convenient, subsets of S will be identified with their indicator functions, i.e., vectors of 0's and 1's in R^N. For example, if $\Pi = \{S_1, \cdots, S_k\}$ is a partition of S, then $\sum_{i=1}^k S_i = \mathbf{1}$ and $|S_i| = S_i'\mathbf{1}$, $i = 1, \cdots, k$.

For a partition $\Pi = \{S_1, \cdots, S_k\}$ let

$$\mathscr{V}(\Pi) \equiv \text{orthocomplement of } \mathbf{1} \text{ in the subspace spanned by } \{S_1, \cdots, S_k\} .$$

(If $\Pi \neq \Pi^*$, then $\mathscr{V}(\Pi) \neq \mathscr{V}(\Pi^*)$. See the proof of the remark in Gilliland (1972).) Let

$$d(\Pi) \equiv \text{dimension of } \mathscr{V}(\Pi) = k(\Pi) - 1 .$$

DEFINITION 1. If $\Pi = \{S_1, \cdots, S_k\}$ and $\Pi^* = \{S_1^*, \cdots, S_{k^*}^*\}$ are partitions of S, we say that they are orthogonal and write $\Pi \perp \Pi^*$ if and only if $\mathscr{V}(\Pi) \perp \mathscr{V}(\Pi^*)$ in R^N.

The following remark gives the known and simple characterization for orthogonality of Π and Π^*.

REMARK 1. $\Pi \perp \Pi^*$ if and only if

$$|S_i S_j^*| = \frac{|S_i|\,|S_j^*|}{N}, \qquad \text{for all} \quad i = 1, \cdots, k ; \quad j = 1, \cdots, k^* .$$

PROOF. Since $\{S_i - S_{i1} \mid i = 1, \cdots, k\}$ spans $\mathscr{V}(\Pi)$ and $\{S_j^* - S_{j1}^* \mid j = 1, \cdots, k^*\}$ spans $\mathscr{V}(\Pi^*)$, then $\mathscr{V}(\Pi) \perp \mathscr{V}(\Pi^*)$ if and only if $(S_i - S_{i1})'(S_j^* - S_{j1}^*) = 0$ for all i, j. The proof is completed by noting that $S_i'S_j^* = |S_i S_j^*|$, $S_{i1}' S_j^* = (|S_i|/N)\mathbf{1}'S_j^* = |S_i|\,|S_j^*|/N = S_i'S_{j1}^* = S_{i1}' S_{j1}^*$.

Remark 1 provides a useful guide in the search for orthogonal partitions. One consequence is that $\Pi \perp \Pi^*$ implies $c(\Pi) \geqq k(\Pi^*)$, $k(\Pi) \leqq c(\Pi^*)$.

The following remark is completely trivial, and, as we will see, it provides a method of obtaining a crude bound on the maximum number of orthogonal F-squares, the number of constraints k in OA ($N, k, s, t \geqq 2$) and, more generally, the number of partitions in any regular system of mutually orthogonal partitions.

REMARK 2. Let $\{\Pi_\alpha \mid \alpha \in I\}$ be a set of mutually orthogonal partitions of S. Since the $\mathscr{V}(\Pi_\alpha)$ are mutually orthogonal subspaces of R^N, all orthogonal to $\mathbf{1}$, then

$$1 + \textstyle\sum_{\alpha \in I} d(\Pi_\alpha) \leqq N .$$

The literature in design of experiments is replete with systems of orthogonal partitions of which we give some examples.

EXAMPLE 1. (Latin and F-squares). Let $N = n^2$ and identify each element of S with a position in an $n \times n$ square. The partition $\mathscr{R} = \{R_1, \cdots, R_n\}$ induced by rows is orthogonal to the partition $\mathscr{C} = \{C_1, \cdots, C_n\}$ induced by columns since $1 = |R_i C_j| = |R_i||C_j|/n^2$ for all $i, j = 1, \cdots, n$. A Latin square defines a partition $\mathscr{L} = \{L_1, \cdots, L_n\}$ through the positions of the n symbols and since each symbol occurs once in each row and each column, $\mathscr{L} \perp \mathscr{R}$, $\mathscr{L} \perp \mathscr{C}$. Again it is easy to check by definition and Remark 1 that orthogonal Latin squares induce orthogonal partitions. An F-square $F(n; \lambda_1, \cdots, \lambda_k)$ (see Hedayat and Seiden (1970) for the definitions of F-square and orthogonal F-squares) induces a partition $\mathscr{F} = \{F_1, \cdots, F_k\}$ with $|F_j| = \lambda_j n$, $j = 1, \cdots, k$ which is orthogonal to $\mathscr{R}$ since $\lambda_j \equiv |F_j R_i| = |F_j||R_i|/n^2 = (\lambda_j n)(n)/n^2$ for all i, j. Likewise, $\mathscr{F}$ is by definition orthogonal to $\mathscr{C}$. It is equally easy to check that F-squares $\mathscr{F}$ and $\mathscr{F}^*$ are orthogonal according to Hedayat and Seiden (1970, Definition 3.1) if and only if the induced partitions are orthogonal. It follows from Remark 2 that the maximum number t of mutually orthogonal F-squares of type $F(n; \lambda_1, \cdots, \lambda_k)$ satisfies $1 + (n - 1) + (n - 1) + t(k - 1) \leqq n^2$ from which $t \leqq (n - 1)^2/(k - 1)$.

Theorem 2.1 of Hedayat, Raghavarao and Seiden (1975) gives this bound for the special case where the system of mutually orthogonal F-squares consists of F-squares of type $F(n; \lambda)$, i.e., $\lambda_1 = \cdots = \lambda_k = \lambda$. The given proof does not expose the simple dimensionality considerations that render the result as an immediate corollary. The suggested alternative proof, which is essentially based on a dimensionality constraint, is for readers familiar with the properties of fractional factorial designs and is couched in the language of that area.

EXAMPLE 2. (Orthogonal arrays of strength $t \geqq 2$). See Raghavarao (1971, Definition 2.1.3) for the definition of an orthogonal array OA $(N = \lambda s^t, k, s, t)$. The s symbols in a row define a partition of S into s blocks of size λs^{t-1}. Let $t \geqq 2$. Then given any two rows, each pair of symbols occurs λs^{t-2} times across the $N = \lambda s^t$ columns. Hence, if the partitions corresponding to the two rows are denoted by $\Pi = \{S_1, \cdots, S_s\}$ and $\Pi^* = \{S_1^*, \cdots, S_s^*\}$, it follows that $\lambda s^{t-2} = |S_i S_j^*| = |S_i||S_j^*|/N = (\lambda s^{t-1})(\lambda s^{t-1})/(\lambda s^t)$ for all $i, j, = 1, \cdots, s$. Thus, the k rows of OA (N, k, s, t) give k mutually orthogonal partitions of S. It follows from Remark 2 that if $t \geqq 2$, then the number of constraints k in OA (N, k, s, t) satisfies $1 + k(s - 1) \leqq N$ which together with $N = \lambda s^t$ yields $k \leqq (\lambda s^t - 1)/(s - 1)$.

The above bound can be sharpened to the Rao (1947) bounds also given as Theorem 2.2.1 of Raghavarao (1971). For example, let $t = 3$ and define $\mathscr{V}(\Pi)\mathscr{V}(\Pi^*)$ to be the subspace corresponding to the product partition $\Pi\Pi^* = \{S_i S_j^* \mid S_i \in \Pi, S_j^* \in \Pi^*\}$. If $\Pi_1, \cdots, \Pi_k$ denote the partitions corresponding to the k rows of OA $(\lambda s^3, k, s, 3)$, then it easily follows that $\mathbf{1} \oplus \sum_1^k \mathscr{V}(\Pi_i) \oplus \sum_2^k \mathscr{V}(\Pi_1)\mathscr{V}(\Pi_i)/(\mathscr{V})\Pi_1) \oplus \mathscr{V}(\Pi_i))$ is an orthogonal decomposition of a subspace of R^N from which $1 + k(s-1) + (k-1)(s-1)^2 \leqq \lambda s^3$. For $t = 4$ one gets $\mathbf{1} \oplus \sum_1^k \mathscr{V}(\Pi_i) \oplus \sum_{i<j} \mathscr{V}(\Pi_i)\mathscr{V}(\Pi_j)/(\mathscr{V}(\Pi_i) \oplus \mathscr{V}(\Pi_j))$ is an orthogonal decomposition of a subspace of R^N from which $1 + k(s-1) + \frac{1}{2}k(k-1)(s-1)^2 \leqq \lambda s^4$. The general t odd and t even cases follow in an analogous way. Perhaps, readers will find this development easier than that of Rao (1947) where an orthogonal basis of factorial effects is defined.

Bose and Bush (1952) use an alternative (algebraic) method to get sharper bounds for the $t = 2, 3$ cases. Raghavarao (1971, Section 2.2) gives these and other bounds.

3. Some counting problems. From design of experiments there is interest in evaluating the maximum number of dimensions that can be extracted from R^N using systems of orthogonal partitions of certain types. Consider

DEFINITION 2. For each $c = 1, 2, \cdots, N$ and $N = 1, 2, \cdots$ let

$$\rho(N, c) \equiv \max \{1 + \textstyle\sum_{\alpha \in I} d(\Pi_\alpha) \mid \{\Pi_\alpha \mid \alpha \in I\} \text{ is a set of mutually orthogonal partitions of } S \text{ with } c(\Pi_\alpha) \geqq c \text{ for all } \alpha \in I\}\,.$$

Here the restriction is to partitions with all blocks of size at least c. The evaluation of ρ is an interesting problem. Of course, $\rho(N, c)$ is monotone in c with

$$1 = \rho(N, N) \leqq \rho(N, N-1) \leqq \cdots \leqq \rho(N, 1) = N\,.$$

Since there does not exist a partition Π with both $k(\Pi) > 1$ and $c(\Pi) > \frac{1}{2}N$, we see that $\rho(N, c) = 1$ for all $c > \frac{1}{2}N$. Furthermore, by taking a single partition Π with $c(\Pi) \geqq c$, we see that $\rho(N, c) \geqq N/c$.

We now give a few of the many properties of ρ which follow from known constructive techniques and existence theorems in design of experiments. (See e.g., Hall (1967), Raghavarao (1971) and Ryser (1963) for comprehensive coverage.) For example

$$\rho(s^a, s^{a-1}) = s^a\,, \qquad \text{if } a \geqq 1 \text{ and } s \text{ is a prime power.}$$

Since for each $n \geqq 1$ there exists an $n \times n$ Latin square and since the row partition is orthogonal to the column partition provided $n \geqq 2$,

$$\rho(n^2, n) \geqq 1 + 3(n-1)\,, \qquad \text{if } n = 2, 3, \cdots\,.$$

This result is improved for all $n \geqq 3$, $n \neq 6$ by using the fact that there exist orthogonal $n \times n$ Latin squares for such n. Thus,

$$\rho(n^2, n) \geqq 1 + 4(n-1)\,, \qquad \text{if } n = 3, 4, \cdots, n \neq 6\,.$$

Federer (1976) has constructed a complete set of orthogonal $F(n; \frac{1}{2}n, \frac{1}{2}n)$-squares using a Hadamard matrix of order n, H_n. Hence,

$$\rho(n^2, n) = n^2\,, \quad \text{provided } H_n \text{ exists.}$$

Since the last $n - 1$ rows of a (normalized) Hadamard matrix H_n yield a system of $n - 1$ orthogonal partitions, each partition consisting of 2 blocks of size $\frac{1}{2}n$,

$$\rho(n, \tfrac{1}{2}n) = n\,, \quad \text{provided } H_n \text{ exists.}$$

If $N = a \cdot b$ and $c \leqq a \wedge b$, where $\wedge$ denotes minimum, then by using the row and column partitions of an $a \times b$ array we see that

$$\rho(a \cdot b, a \wedge b) \geqq 1 + (a - 1) + (b - 1)\,, \qquad \text{if} \quad a, b = 1, 2, \cdots .$$

Other simple results follow directly from the Remark 1 characterization of orthogonality. For example, using Remark 1 it is easy to check that

$$\rho(2n, n) = 2\,, \qquad \text{if} \quad n \quad \text{is odd.}$$

Also, if N is a prime then no two partitions are orthogonal. Hence,

$$\rho(p, c) = \text{greatest integer in} \quad p/c\,, \qquad \text{if} \quad p \quad \text{is prime.}$$

Table 1 gives values for $N \leqq 13$ and $c \leqq \frac{1}{2}N$. The orthogonal array OA (12, 11, 2, 2) in Plackett and Burman (1946) establishes the fact $\rho(12, 6) = 12$. This also follows from the existence of a Hadamard matrix of order 12.

TABLE 1
Some values of $\rho(N, c)$

c	N: 4	5	6	7	8	9	10	11	12	13
2	4	2	3	3	8	9	6	5	12	6
3			2	2	8	9	3	3	12	4
4					8	2	3	2	12	3
5							2	2	12	2
6									12	2

Of particular interest is the evaluation of $\rho(36, c)$ for various c. The orthogonal array OA (36, 13, 3, 2) constructed by Seiden (1954) provides a system of 13 orthogonal partitions each consisting of 3 blocks of size 12. Therefore, $\rho(36, 12) \geqq 27$. (The Bose and Bush bound on k in OA $(36, k, 3, 2)$ is 16 indicating that at most 33 dimensions can be accounted for by **1** and orthogonal partitions each consisting of 3 blocks of size 12.) Recently, Federer and Seiden (1975) have found an $F(6; 3)$ orthogonal to a given set of 8 orthogonal $F(6; 2)$'s from which it follows that $\rho(36, 6) \geqq 28$. Constructive techniques for orthogonal F(even; 2) squares are given by Anderson, Federer and Seiden (1974).

We now consider extracting dimensions from R^N with a system of orthogonal partitions each containing no more than k blocks.

DEFINITION 3. For each $k = 1, 2, \cdots, N$ and $N = 1, 2, \cdots$ let

$$\sigma(N, k) \equiv \max \{1 + \textstyle\sum_{\alpha \in I} d(\Pi_\alpha) \mid \{\Pi_\alpha \mid \alpha \in I\} \text{ is a set of mutually orthogonal partitions of } S \text{ with } k(\Pi_\alpha) \leqq k \text{ for all } \alpha \in I\}\,.$$

Monotonicity in k is immediate with

$$1 \leqq \sigma(N, 1) \leqq \sigma(N, 2) \leqq \cdots \leqq \sigma(N, N) = N\,.$$

Since $c(\Pi) \geqq c$ implies $k(\Pi) \leqq [N/c] \equiv$ greatest integer in N/c,

$$\rho(N, c) \leqq \sigma(N, [N/c])\,.$$

There is strict inequality for some N and c, e.g., $\rho(6, 3) = 2$ and $\sigma(6, 2) = 3$.

There are many other constraints to place on the system of orthogonal partitions in the search for maximal extraction of dimensions from R^N. For example, for $N = n^2$ and requiring all partitions to be into n blocks of size n, the maximal extraction of dimension from R^N is $1 + t(n - 1)$ where $t - 2$ is the maximal number of mutually orthogonal $n \times n$ Latin squares. For $N = n^2$ and requiring two partitions to be into n blocks of size n and the rest to be into k blocks of size λn ($n = \lambda k$), the maximal extraction of dimension from R^N is $1 + 2(n - 1) + t(k - 1)$ where t is the maximal number of mutually orthogonal F-squares $F(n; \lambda)$. The construction of systems which achieve the maximal extraction subject to constraints remains a difficult problem in general.

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