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A PARADOX IN ADMISSIBILITY[1]

BY SHERALI MAVJIBHAI MAKANI

University of California, Berkeley

Let X_1, X_2 be two independent variables with means θ_1, θ_2. Then two examples are given (one binomial, one normal) in which an estimator, depending only on X_2 is admissible for estimating θ_1.

The purpose of this note is to exhibit the following phenomenon. Let X_1, X_2 be two independent random variables (normal, binomial, Poisson) with means θ_1, θ_2. Then there may exist a nontrivial estimator $\delta(X_2)$, not depending on X_1, which is admissible for estimating θ_1. In Example 1 below the distributions are binomial, the loss function is squared error, and the estimator $\delta(X_2)$ is linear in X_2. In Example 2, which is concerned with normal distributions, the loss function is quite general.

EXAMPLE 1. Let X_i $(i = 1, \cdots, k)$ be independent binomial random variables corresponding to n_i trials and with success probability p_i.

Then a necessary and sufficient condition for the linear estimator

$$\delta(X_1, \cdots, X_k) = \sum_{i=1}^k a_i \frac{X_i}{n_i} + c \tag{1}$$

to be admissible for p_1, when the loss function is squared error, is that either

$$0 \leqq a_1 < 1\,, \quad 0 \leqq c \leqq 1 \quad \text{and} \quad 0 \leqq \sum_{i=2}^k a_i + c \leqq 1\,, \quad 0 \leqq \sum_{i=1}^k a_i + c \leqq 1 \tag{2}$$

or

$$a_1 = 1 \quad \text{and} \quad a_2 = \cdots = a_k = c = 0\,. \tag{3}$$

This is easy to prove by the method of Cohen (1965) and Johnson (1971); a detailed proof is given in [4].

If we now put $a_1 = 0$ in (2), we find that a linear estimator (1) of p_1 which is a function only of $X_2, \cdots, X_k$ is admissible for estimating p_1 provided $0 \leqq c \leqq 1$ and $a_2, \cdots, a_k$ satisfy

$$0 \leqq \sum_{i=2}^k a_i + c \leqq 1\,. \tag{4}$$

A similar result holds in the Poisson case. On the other hand, if X_i $(i = 1, \cdots, k)$ are independent normal variables with mean μ_i and known variance σ_i^2, a necessary and sufficient condition for

$$\delta(X_1, \cdots, X_k) = \sum_{i=1}^k a_i X_i + c \tag{5}$$

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to be admissible for estimating μ_1 with squared error loss is that either

$$\sigma_1^2 a_1(a_1 - 1) + \sum_{i=2}^{k} \sigma_i^2 a_i^2 \leqq 0 \quad \text{and} \quad a_1 \neq 0 \tag{6}$$

or

$$a_1 = 1\,, \quad a_i = 0 \quad \text{for} \quad i > 1 \quad \text{and} \quad c = 0\,. \tag{7}$$

This follows easily from the results of Cohen (1965). Hence no nonconstant linear function of $X_2, \cdots, X_k$ is admissible for estimating μ_1.

However, even in the normal case the paradox continues to exist when the restriction to linear estimators is dropped. This is shown by the following example communicated by L. D. Brown.

EXAMPLE 2. Let X_1, X_2 be independent normal random variables with unknown means μ_1 and μ_2 and known variances σ_i^2. Let $L((\mu_1, \mu_2), \delta)$ be any loss function satisfying $L((\mu_1, \mu_2), \delta) = 0$ for $\delta = \mu_1$, and > 0 for $\delta \neq \mu_1$. (Of course, the usual quadratic loss $(\delta - \mu_1)^2$ for estimating μ_1 satisfies this condition.) Consider the estimator $\delta(x_1, x_2) = \operatorname{sgn} x_2$. Then this estimator is an admissible estimator for μ_1, even though it depends only on x_2.

PROOF. For convenience take $\sigma_i^2 \equiv 1$. Let δ' be any estimator such that $R((\mu_1, \mu_2), \delta') \leqq R((\mu_1, \mu_2), \delta)$. Let $r(\mu_2) = \int (\delta'(x_1, \mu_2) - 1)^2 \phi(x_1 - 1)\, dx_1 \geqq 0$. Then, for $\mu_2 > 0$,

$$\begin{aligned} 0 &\geqq R((1, \mu_2), \delta') - R((1, \mu_2), \delta) \\ &\geqq \textstyle\int_0^\infty r(x_2)\phi(x_2 - \mu_2)\, dx_2 - \int_{-\infty}^0 4\phi(x_2 - \mu_2)\, dx_2 \end{aligned} \tag{8}$$

since $(\delta(x_1, x_2) - 1)^2 = 0$ for $x_2 > 0$ and $(\delta - 1)^2 \leqq 4$. Well-known properties of exponential families yield that

$$\lim_{\mu_2 \to \infty} \frac{\int_0^\infty r(x_2)\phi(x_2 - \mu_2)\, dx_2}{\int_{-\infty}^0 \phi(x_2 - \mu_2)\, dx_2} \to \infty$$

unless $r(x_2) = 0$ for almost all $x_2 > 0$. (See, e.g., Theorem 3 of Birnbaum (1955).) Now, $r(\cdot)$ must satisfy this latter condition since (8) is equivalent to $\int_0^\infty r(x_2)\phi(x_2 - \mu_2)\, dx_2 / \int_{-\infty}^0 \phi(x_2 - \mu_2)\, dx_2 \leqq 4$.

The symmetric argument yields that $r(x_2) = 0$ for $x_2 \leqq 0$. Hence $\delta' = \delta$ a.e. This implies that δ is admissible, as claimed.

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DEPARTMENT OF STATISTICS
UNIVERSITY OF CALIFORNIA
BERKELEY, CALIFORNIA 94720