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SOME EXTENSIONS OF A THEOREM OF STEIN ON CUMULATIVE SUMS

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Let $u, u_1, u_2, \cdots$ be a sequence of i.i.d. random k-vectors and $a_1, a_2, \cdots$ be a sequence of k-vectors. Let $S_n = \sum_1^n a_i'u_i$. For any positive L, let $N = \min\{n \geqq 1 : |S_n| \geqq L\}$. In case $k = 1$ and all a_n's are equal and nonzero, Stein [5] showed that N is exponentially bounded provided that u is nondegenerate at 0. In this paper, conditions on the a_n's and on u which guarantee the exponential boundedness of N defined above are obtained. The exponential boundedness of $N' = \min\{n \geqq 1 : |S_n + C_n| \geqq L\}$, where $C_1, C_2, \cdots$ is an arbitrary sequence of real numbers, is also considered. Some applications are given.

1. Introduction. Let $u, u_1, u_2, \cdots$ be i.i.d. random k-vectors and $a_1, a_2, \cdots$ a sequence of k-vectors. (Vectors will be column vectors and prime denotes transpose.) Let

$$S_n = \sum_1^n a_i'u_i \tag{1.1}$$

and for any given positive L, let

$$N = \min\{n \geqq 1 : |S_n| \geqq L\}\,. \tag{1.2}$$

The objective of this research is to establish conditions for the exponential boundedness of N, i.e., for any $L > 0$ there is $c > 0$ and $0 < \rho < 1$ such that

$$P[N > n] < c\rho^n\,, \qquad n = 1, 2, \cdots\,. \tag{1.3}$$

Stein [5] showed that in case $k = 1$, all a_n's are equal and nonzero and u is not degenerate at 0, (1.3) holds. This result has been an invaluable tool in proving exponential boundedness of the stopping time of SPRT's both in the case of simple hypotheses (Wald SPRT) [5] and in the case of composite hypotheses, e.g., [1], [4], [6] and [7]. In this paper, we shall investigate the condition under which (1.3) holds for a general sequence $\{a_n\}$.

We shall also be interested in the exponential boundedness of

$$N' = \min\{n \geqq 1 : |S_n + C_n| \geqq L\}\,, \tag{1.4}$$

for any sequence $\{C_n\}$ of real numbers. The main results for the one dimensional case are contained in the Theorems 3.1 and 3.2. Some extensions to the case that the a_n and u_n are vectors are given in Theorems 5.1, 5.2 and 5.4–5.6. Examples to justify the need of some assumptions are given in Section 4. The results in this paper can be used to prove the exponential boundedness of the stopping time of SPRT involving non-i.i.d. random variables, especially those tests of

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parameters in (sequential) linear models. Three examples are considered in Section 6 as illustrations.

2. Preliminaries. For any positive integer r to be chosen later, we shall write for $j = 1, 2, \cdots$

$$b_j = \sum_{i=1}^{r} a_{(j-1)r+i} , \tag{2.1}$$

$$\beta_j = \sum_{i=1}^{r} \alpha_{(j-1)r+i} , \tag{2.2}$$

$$s_j = \sum_{i=1}^{r} a'_{(j-1)r+i} u_{(j-1)r+i} , \tag{2.3}$$

where, for $a_n' = (a_{1n}, \cdots, a_{kn})$, we write

$$\alpha_n = |a_n| = \sum_{j=1}^{k} |a_{jn}| . \tag{2.4}$$

For any sequence $\{x_i\}$, we write systematically $\bar{x}_n = (1/n) \sum_1^n x_i$.

It follows from the definition (1.2) of N that

$$[N > n] \subset [|s_j| < 2L, j = 1, 2, \cdots, [n/r]]$$

where $[x]$ denotes the largest integer which is not greater than x. Thus, in order to prove (1.3), by the independence of $s_1, s_2, \cdots$, it suffices to show that for any $L > 0$, there exists a positive integer r such that $\prod_1^n P[|s_j| < L] < c\rho^n$ for some $c > 0$ and $0 < \rho < 1$ and hence, equivalently, it suffices to show that for any $L > 0$, there is a positive integer r such that

$$\limsup\, (1/n) \sum_1^n \log P[|s_j| < L] < 0 . \tag{2.5}$$

Following a similar argument as above to show the exponential boundedness of N' defined in (1.4), it suffices to show

$$\limsup\, (1/n) \sum_1^n \log P[|s_j + \gamma_j| < L] < 0 , \tag{2.6}$$

where $\gamma_j = \sum_{i=1}^{r} c_{(j-1)r+i}$ and $\{c_i\}$ is any sequence such that $C_n = \sum_1^n c_i$.

Let $\{n_j\}$ be a subsequence of the sequence of positive integers of at most exponential growth, i.e., $\liminf n_j/n_{j+1} > 0$; for short we shall call such a sequence *steady*.

We shall prove that (2.5) and (2.6) hold under certain of the following assumptions. An assumption made throughout is that there is $A > 0$ such that $|a_n| < A$ for all n.

ASSUMPTION (a). $\bar{a}_n \to a$, for some $a \neq 0$.

ASSUMPTION (b). There is a steady sequence $\{n_j\}$ along which $\bar{a}_n \to a$, for some $a \neq 0$.

ASSUMPTION (c). There is a steady sequence $\{n_j\}$ along which $\bar{\alpha}_n \to \alpha$, for some $\alpha > 0$, where α_n is defined in (2.4).

When the a_n's are real, we specify

ASSUMPTION (b′). There is a positive number a and a steady sequence $\{n_j\}$ along which $\bar{a}_n \geqq 2a$ (or $\leqq -2a$).

LEMMA 2.1. *Let $\{d_n\}$ be a bounded sequence of real numbers such that $\bar{d}_n \to d > 0$ (<0) and let n' be the number of $d_1, \cdots, d_n$ which are larger (less) than $d/2$. Then* $\liminf n'/n = \rho_0 > 0$.

The proof of this lemma is elementary.

COROLLARY 2.1.1. *The lemma holds if $\bar{d}_n \to d$ is replaced by $\bar{d}_n \geqq d > 0$* ($\bar{d}_n \leqq d < 0$).

LEMMA 2.2. *If $\bar{a}_n$ converges to a along a steady sequence $\{n_j\}$, then there is a steady sequence $\{n_j'\}$ along which $\bar{b}_n$ converges to ra, where b_n is defined in* (2.1).

PROOF. Let n_j' be the first multiple of r which is not less than n_j. Then

$$(2.7) \qquad (1/n_j') \sum_1^{n_j'} a_i = (n_j/n_j')(1/n_j) \sum_1^{n_j} a_i + (1/n_j') \sum_{n_j}^{n_j'} a_i \,.$$

Since $1 \geqq (n_j/n_j') \geqq n_j/(n_j + r) \to 1$ as $j \to \infty$ and the second sum on the right-hand side of (2.7) is bounded by rA in absolute value, we have the right-hand side of (2.7) converging to a as $j \to \infty$. Noting that $\bar{b}_n = r((nr)^{-1} \sum_1^{nr} a_i)$, we have $\bar{b}_n$ converging to ra along $\{n_j'\}$. Further, noting that $n_j/(n_{j+1} + r) \leqq n_j'/n_{j+1}' \leqq (n_j + r)/n_{j+1}$ and taking lim inf as $j \to \infty$, we see that $\{n_j'\}$ is steady.

We can similarly prove

COROLLARY 2.2.1. *Under Assumption* (b′), *there is a steady sequence along which* $\bar{b}_n \geqq ra$ (or $\leqq -ra$).

It is recalled that a k-vector λ is called a *point of increase* (of the distribution) of u if for any neighborhood V of λ, $P[u \in V] > 0$.

LEMMA 2.3. *Let a be any k-vector. If $a'u$ is nondegenerate at* 0, *then there is a point of increase λ of u such that $a'\lambda \neq 0$.*

The proof of this lemma is elementary and will be omitted.

COROLLARY 2.3.1. *If u is a random variable such that $P[u>0]>0$ ($P[u<0]>0$), then there is at least one positive (negative) point of increase of u.*

3. Univariate case. In this section as well as in the next one, we assume $k = 1$, i.e., both the u_n's and the a_n's are real.

THEOREM 3.1. *If $P[u < 0] > 0$ and $P[u > 0] > 0$* and *$\{a_n\}$ satisfies Assumption* (c), *then both N and N' are exponentially bounded.*

PROOF. We divide the proof into several steps.

(i) By Corollary 2.3.1, there are positive numbers λ and ε such that both $P[u > \lambda] > \varepsilon$ and $P[u < -\lambda] > \varepsilon$ hold. Therefore, for $i = 1, 2, \cdots, P[(\operatorname{sgn} a_i)u_i > \lambda] > \varepsilon$, where $\operatorname{sgn} x = 1$ if $x \geqq 0$ and -1 if $x < 0$. Recall that $u, u_1, u_2, \cdots$ are i.i.d. Thus, multiplying by $|a_i|$ both sides of $(\operatorname{sgn} a_i)u_i > \lambda$, and summing, we get

$$(3.1) \qquad P[s_j \geqq \beta_j \lambda] \geqq P[(\operatorname{sgn} a_i)u_i > \lambda, i = (j-1)r + 1, \cdots, jr] > \varepsilon^r > 0 \,,$$

where β_j and s_j are defined respectively in (2.2) and (2.3). By Assumption (c) and Lemma 2.2 applied to $\{\alpha_n\}$, there is a steady sequence $\{n_j'\}$ along which $\bar{\beta}_n$ goes to $r\alpha > 0$. Choose r so large that $r\alpha > 2L/\lambda$ and let n_k'' be the number of β_i, $i = 1, 2, \cdots, n_k'$, which are larger than L/λ. Then, by Lemma 2.1,

$$\liminf n_k''/n_k' = \rho_0 > 0 . \tag{3.2}$$

Thus if $\beta_j > L/\lambda$, by (3.1), we have

$$P[s_j \geqq L] \geqq P[s_j \geqq \beta_j \lambda] > \varepsilon^r . \tag{3.3}$$

(ii) Therefore, when $\beta_j > L/\lambda$,

$$P[|s_j| \geqq L] > \varepsilon^r \tag{3.4}$$

and hence

$$(1/n_k') \sum_1^{n_k'} \log P[|s_j| < L] \leqq (n_k''/n_k') \log (1 - \varepsilon^r) .$$

Taking lim sup over k on both sides and observing (3.2), we get

$$\limsup (1/n_k') \sum_{j=1}^{n_k'} \log P[|s_j| < L] \leqq \rho_0 \log (1 - \varepsilon^r) < 0 . \tag{3.5}$$

(iii) Now, for any n, $n_k' \leqq n \leqq n'_{k+1}$ for some k, we have

$$(1/n) \sum_1^n \log P[|s_j| < L] \leqq (n_k'/n'_{k+1})(1/n_k') \sum_1^{n_k'} \log P[|s_j| < L] .$$

Taking lim sup on both sides and observing that $\{n_j'\}$ is steady, (2.5) follows from (3.5). Therefore, N is exponentially bounded.

(iv) Similar to the proof of (3.3), we can show that if $\beta_j > L/\lambda$, we get

$$P[s_j \leqq -L] > \varepsilon^r . \tag{3.6}$$

Now if $\gamma_j \geqq 0$ and $\beta_j > L/\lambda$, by (3.3) we have $P[s_j + \gamma_j \geqq L] \geqq P[s_j + \gamma_j \geqq L + \gamma_j] > \varepsilon^r$. Similarly, if $\gamma_j < 0$ and $\beta_j > L/\lambda$, using (3.6) we have $P[s_j + \gamma_j \leqq -L] > \varepsilon^r$. Therefore, for $\beta_j > L/\lambda$, we have $P[|s_j + \gamma_j| \geqq L] > \varepsilon^r$. This is the analogue of (3.4) and the rest of the argument is the same as in steps (ii) and (iii).

Since Assumption (a) implies (b) which, in turn, implies (c), we have the following corollary.

COROLLARY 3.1.1. *If* $P[u > 0] > 0$ *and* $P[u < 0] > 0$ *and* $\{a_n\}$ *satisfies either of Assumptions* (a) *and* (b), *then both* N *and* N' *are exponentially bounded.*

Note that if the expectation $\mu = E_P(u)$ is finite and u is nondegenerate, then $P[u > \mu] > 0$ and $P[u < \mu] > 0$. With γ_j being replaced by $\gamma_j + \mu \sum_{i=1}^r a_{(j-1)r+i}$ in (2.6), we have

COROLLARY 3.1.2. *If* $\mu = E_P(u)$ *is finite and* u *is nondegenerate, then* N' *is exponentially bounded.*

By using Corollaries 2.1.1 and 2.2.1 following the same argument as in the proof of Theorem 3.1, we can show

COROLLARY 3.1.3. *If* $P[u > 0] > 0$ *and* $P[u < 0] > 0$ *and* $\{a_n\}$ *satisfies Assumption* (b′), *then both N and N′ are exponentially bounded.*

Under Assumption (c), the assumption that u has positive probability on both sides of 0 in Theorem 3.1 cannot be relaxed. As a counterexample, we consider $a_n = (-1)^{n-1}$, $n = 1, 2, \cdots$, and $u = 1$ with probability one, then $\{a_n\}$ satisfies Assumption (c), but $|s_j| \leqq 1$ with probability one and hence (2.5) cannot be true for $L > 1$.

We now decrease the restrictions on the distribution of u and increase the restrictions on $\{a_n\}$.

THEOREM 3.2. *If* $P[u = 0] < 1$ *and* $\{a_n\}$ *satisfies Assumption* (b), *then N is exponentially bounded.*

PROOF. (i) The assumption on u implies that either $P[u>0]>0$ or $P[u<0]>0$. Without loss of generality, we may assume the former case. By Corollary 2.3.1 there is a positive point of increase λ of u. Thus, for any $\delta > 0$ there is $\varepsilon = \varepsilon(\delta) > 0$ such that $P[\lambda - \delta < u < \lambda + \delta] > \varepsilon$. Thus, for $a_i \geqq 0$, $P[a_i(\lambda - \delta) \leqq a_i u_i \leqq a_i(\lambda + \delta)] > \varepsilon$ and for $a_i < 0$, $P[a_i(\lambda + \delta) \leqq a_i u_i \leqq a_i(\lambda - \delta)] > \varepsilon$. Therefore, recalling the definitions of β_j, b_j and s_j defined in (2.1)—(2.3), we get

$$P[\lambda b_j - \delta\beta_j \leqq s_j \leqq \lambda b_j + \delta\beta_j] > \varepsilon^r . \tag{3.7}$$

(ii) We consider the case that $a > 0$ (the case $a < 0$ can be reduced to the case $a > 0$ by reversal of the signs of the a_n). Choose r so large that $r\lambda a > 4L$. By Assumption (b) and Lemma 2.2., there is a steady sequence $\{n_j'\}$ along which $\bar{b}_n$ converges to ra. Let n_k'' be the number of b_i, $i = 1, \cdots, n_k'$, which are larger than $2L/\lambda$; then by Lemma 2.1, $\liminf n_k''/n_k' = \rho_0 > 0$. Note that, since $\beta_j < rA$, we have $\lambda b_j - \delta\beta_j \geqq \lambda b_j - \delta rA$. Choose δ so small that $\delta rA < L$; then if $b_j > 2L/\lambda$, we have $\lambda b_j - \delta rA > L$ and thus $\lambda b_j - \delta\beta_j > L$. Therefore, by (3.7), if $b_j > 2L/\lambda$, we have

$$P[s_j \geqq L] > \varepsilon^r \tag{3.8}$$

which is the analogue of (3.3) in the proof of Theorem 3.1.

(iii) Following the same argument as (ii) and (iii) in the proof of Theorem 3.1, we have (2.5). Hence N is exponentially bounded.

COROLLARY 3.2.1. *If* $P[u = 0] < 1$ *and* $\{a_n\}$ *satisfies Assumption* (a), *then N is exponentially bounded.*

In the following section, we give two examples to justify the inclusion of the boundedness of $\{a_n\}$ and the steadiness of $\{n_j\}$ respectively, in the assumptions.

4. Remarks and examples.

1. Example 1 shows that if the boundedness condition of the sequence $\{a_n\}$ is removed, then the results in Section 3 are no longer true in general.

EXAMPLE 1. Let $a_1 = 1$, $a_i = i/2$ if $i = 2^j$ for some positive integer j and all other a_i's be 0, so that $\{a_n\}$ is not bounded. Observing that $\sum_{i=1}^{2k} a_i = 2^k$, it is readily seen that $\limsup \bar{a}_n = 1$ and $\liminf \bar{a}_n = \frac{1}{2}$. Thus $\{a_n\}$ satisfies Assumption (b).

Now let $P[u = -1] = P[u = 0] = P[u = 1] = \frac{1}{3}$. For any fixed positive integer r, choose l so large that $2^l - 2^{l-1} > r$ and for any given $L > 0$, choose m so large that $2^m > L$. Let $j_0 = \max(l, m)$. Then for $j \geqq j_0$, $P[|s_j| < L] < 1$ if one (and only one) of $(j-1)r + i$, $i = 1, \cdots, r$, is of the form 2^p for some positive integer p and for such j, $1 > P[|s_j| < L] \geqq (\frac{1}{3})^r$; for all other $j \geqq j_0$ we have $P[|s_j| < L] = 1$ (s_j is defined in (2.3)). Therefore

$$\begin{aligned} 0 &\geqq 2^{-k} \textstyle\sum_1^{2k} \log P[|s_j| < L] \\ &\geqq 2^{-k} \textstyle\sum_1^{j_0-1} \log P[|s_j| < L] + (k - j_0 + 1)2^{-k} \log (\frac{1}{3})^r \\ &\to 0 \qquad \text{as} \quad k \to \infty \end{aligned}$$

and hence $\limsup (1/n) \sum_1^n \log P[|s_j| < L] = 0$, i.e., (2.5) is not true.

2. It might be suspected that in Assumptions (b) and (c), the steadiness condition, i.e., $\liminf n_j/n_{j+1} > 0$, can be removed. This is not the case in general as can be seen in the following counterexample.

EXAMPLE 2. The sequences $\{n_k'\}$ and $\{n_k\}$ in this example have no connection with those used in Section 2. They are defined successively as the following: $n_1' = 0$ and $n_1 = 1$; for $k \geqq 2$, let $n_k' = n_{k-1} + k^k$ and $n_k = n_k' + k^k$. Thus

$$n_k' = 2 \textstyle\sum_1^{k-1} j^j + k^k - 1 \,, \tag{4.1}$$

$$n_k = 2 \textstyle\sum_1^k j^j - 1 \,. \tag{4.2}$$

Let $a_i = 0$ for $n_k < i \leqq n'_{k+1}$ and $a_i = 1$ if $n_k' < i \leqq n_k$. For $n_k \leqq n \leqq n'_{k+1}$,

$$\textstyle\sum_1^n a_i = \sum_1^k (n_j - n_j') = \sum_1^k j^j \,.$$

In particular, if $n = n_k$, noting (4.2), we get

$$(1/n_k) \textstyle\sum_1^{n_k} a_k \to \frac{1}{2} \qquad \text{as} \quad k \to \infty \,.$$

Note that

$$n_k/n_{k+1} = n_k/(n_k + 2(k+1)^{k+1}) \to 0 \qquad \text{as} \quad k \to \infty \,.$$

Thus all conditions of Assumptions (b) and (c) are satisfied except the steadiness of subsequence $\{n_k\}$ along which $\bar{a}_n$ ($= \bar{\alpha}_n$, in this example) converges to its nonzero limit $\frac{1}{2}$.

Let u be defined as in Example 1 and b_j and s_j as in (2.1) and (2.3). Note that $P[|s_j| \leqq b_j] = 1$ and hence for $L > b_j$ we have $P[|s_j| < L] = 1$. If $L \leqq b_j$ (which implies $r \geqq L$), then

$$1 > P[|s_j| < L] \geqq P[u_i = 0, i = 1, 2, \cdots, r] = (\tfrac{1}{3})^r$$

and hence

$$\begin{aligned} 0 &\geqq (1/n'_{k+1}) \textstyle\sum_1^{n'_{k+1}} \log P[|s_j| < L] \\ &\geqq (1/n'_{k+1})(1/r) \textstyle\sum_1^k (i^i + 2r) \log (\frac{1}{3})^r \,. \end{aligned} \tag{4.3}$$

Combining (4.1) and (4.3), we get

$$\limsup\, (1/n) \textstyle\sum_1^n \log P[|s_j| < L] = 0 \,,$$

i.e., the conclusions of Theorems 3.1 and 3.2 are not true.

5. Multidimensional case. We now extend the previous results (Theorems 3.1 and 3.2) to more than one dimension. We write $a_n' = (a_{1n}, \cdots, a_{kn})$. Similar notation applies to u_n.

THEOREM 5.1. *Let* $\{a_n\}$ *satisfy Assumption* (b) *in Section* 2. *If* $a'u$ *is nondegenerate at* 0, *then* N *is exponentially bounded.* (*It also suffices that* $\{a_n'\lambda\}$ *satisfies Assumption* (b′).)

PROOF. (i) By Lemma 2.3, there is a point of increase λ of u such that $a'\lambda \neq 0$. Either $a'\lambda > 0$ or < 0. Without loss of generality we may assume $a'\lambda > 0$ (the other case can be reduced to this case by reversing the signs of a_n). By definition of λ, for any $\delta > 0$ there is $\varepsilon > 0$ such that

$$P[\lambda_p - \delta < u_{p1} < \lambda_p + \delta, p = 1, 2, \cdots, k] > \varepsilon \,,$$

where we write $\lambda' = (\lambda_1, \cdots, \lambda_k)$. Hence, for $i = 1, 2, \cdots$

$$P[a_i'\lambda - \delta\alpha_i \leqq a'u_i \leqq a_i'\lambda + \delta\alpha_i] > \varepsilon \,,$$

where α_i is defined in (2.4). We thus have

$$P[b_j'\lambda - \delta\beta_j \leqq s_j \leqq b_j'\lambda + \delta\beta_j] > \varepsilon^r \,, \tag{5.1}$$

where b_j, β_j and s_j are defined in (2.1)—(2.3).

(ii) Note that we assume $a'\lambda > 0$. From (5.1) we have

$$P[|s_j| \geqq b_j'\lambda - \delta\beta_j] > \varepsilon^r \,. \tag{5.2}$$

Using Assumption (b) and Lemma 2.2 applied to the sequence $a_n'\lambda$, there is a steady sequence $\{n_j'\}$ along which $\bar{b}_n'\lambda$ converges to $ra'\lambda$. Choose r so large that $ra'\lambda > 4L$ and $\delta > 0$ so small that $\delta\beta_j < L$ for $j = 1, 2, \cdots$. The latter is possible due to the assumption that the a_n are bounded. Let n_k'' be the number of $b_j'\lambda$, $j = 1, 2, \cdots, n_k'$, which are larger than $2L$. Then by Lemma 2.1, $\liminf n_k''/n_k' = \rho_0 > 0$. Thus, if $b_j'\lambda - \delta\beta_j > L$, then by (5.2), we have

$$P[|s_j| \geqq L] > \varepsilon^r \,. \tag{5.3}$$

This is the equation analogous to (3.4). The rest of the proof is similar to (ii) and (iii) in the proof of Theorem 3.1.

COROLLARY 5.1.1. *Theorem* 5.1 *is true in particular if* $\{a_n\}$ *satisfies Assumption* (a).

THEOREM 5.2. *Let* $\{a_n\}$ *satisfy Assumption* (b). *If* $a'u$ *has positive probability on both sides of* 0, *then* N' *is exponentially bounded.*

PROOF. Since $a'u$ has positive probability on both sides of 0, there are points of increase λ_1 and λ_2 of u such that $a'\lambda_1 > 0$ and $a'\lambda_2 < 0$ (cf. Lemma 2.3).

Similar to (5.1), we obtain for $p = 1, 2$

$$P[b_j'\lambda_p - \delta\beta_j \leqq s_j \leqq b_j'\lambda_p + \delta\beta_j] > \varepsilon^r$$

and hence

$$P[s_j \geqq b_j'\lambda_1 - \delta\beta_j] > \varepsilon^r \tag{5.4}$$

and

$$P[s_j \leqq b_j'\lambda_2 + \delta\beta_j] > \varepsilon^r . \tag{5.5}$$

By Assumption (b) and Lemma 2.2 applied to the sequence $a_n'\lambda_p$, there is a steady sequence $\{n_j'\}$ along which $\bar{b}_n'\lambda_p$ converges to $ra'\lambda_p$, $p = 1, 2$. Note that in the proof of Lemma 2.2, $\{n_j'\}$ depends only on $\{n_j\}$ and r. Choose r so large that $r \min(a'\lambda_1, -a'\lambda_2) > 4L$ and δ so small that $\delta\beta_j < L$. Let $n_k''(1)$ $(n_k''(2))$ be the number of $b_j'\lambda_1(-b_j'\lambda_2)$'s, $j = 1, 2, \cdots, n_k'$, which exceed $2L$. Then by Lemma 2.1, we have $\liminf n_k''(p)/n_k' = \rho_0(p)$ for $p = 1, 2$. Let $n_k'' = \min(n_k''(1), n_k''(2))$. Then

$$\liminf n_k''/n_k' = \rho_0 = \min(\rho_0(1), \rho_0(2)) > 0 .$$

Now, if $\min(b_j'\lambda_1, -b_j'\lambda_2) > 2L$, by (5.4) and (5.5), respectively, we have

$$P[s_j \geqq L] > \varepsilon^r \tag{5.6}$$

and

$$P[s_j \leqq -L] > \varepsilon^r . \tag{5.7}$$

For $\min(b_j'\lambda_1, -b_j'\lambda_2) > 2L$, we have (if $c_j \geqq 0$, use (5.6) and if $c_j < 0$, use (5.7))

$$P[|s_j + c_j| \geqq L] > \varepsilon^r . \tag{5.8}$$

This is an analogue of (3.4) and the rest of the proof is similar to (ii) and (iii) in Theorem 3.1.

COROLLARY 5.2.1. *Theorem 5.2 is true in particular if $\{a_n\}$ satisfies Assumption* (a).

LEMMA 5.3. *Let $T = \{t \in R^k : |t| = 1\}$ ($|\cdot|$ defined in (2.4)). If $t'u$ has positive probability on both sides of 0 for every $t \in T$, then there are $\delta > 0$ and $\varepsilon > 0$ such that for all $t \in T$, $P[t'u > \delta] > \varepsilon$ and $P[t'u < -\delta] > \varepsilon$.*

PROOF. Note that T is compact and that for any $t \in T$, if $s \to t$, then $s'u \to t'u$ everywhere. Therefore this lemma is a special case of Lemma 5.2 in [6], page 119.

THEOREM 5.4. *Let $\{a_n\}$ satisfy Assumption* (c) *and suppose that $c'u$ has positive probability on both sides of 0 for every $c \neq 0$. Then N' is exponentially bounded (and hence so is N).*

PROOF. By assumption

$$(1/n) \textstyle\sum_1^n \alpha_i \to \alpha > 0 \tag{5.9}$$

along a steady sequence $\{n_j\}$, where α_i is defined in (2.4). By Lemma 5.3, there are $\delta > 0$ and $\varepsilon > 0$ such that

$$P[a_i'u_i \geqq \delta\alpha_i] > \varepsilon , \tag{5.10}$$

and

$$P[a_i'u_i \leqq -\delta\alpha_i] > \varepsilon .$$

(Note that the above inequalities are trivially true if $a_i = 0$.) Hence

$$P[s_j \geqq \delta\beta_j] > \varepsilon^r , \tag{5.11}$$

and

$$P[s_j \leqq -\delta\beta_j] > \varepsilon^r ,$$

where β_j and s_j are defined in (2.2) and (2.3) respectively. Thus, for any sequence $\{\gamma_j\}$,

$$P[|s_j + \gamma_j| \geqq \delta\beta_j] > \varepsilon^r . \tag{5.12}$$

Note that, by (5.9) and Lemma 2.2 applied to $\{\alpha_n\}$, $\bar{\beta}_n \to r\alpha$ along a steady $\{n_j'\}$. Choose r so large that $r\alpha > 2L$ and let n_k'' be the number of β_j, $j = 1, 2, \cdots, n_k'$, which are larger than L; then, by (5.12), if $\beta_j > L$, we have

$$P[|s_j + \gamma_j| \geqq L] > \varepsilon^r . \tag{5.13}$$

This is the analogue of (3.4) and the rest of the proof is similar to steps (ii) and (iii) in the proof of Theorem 3.1.

THEOREM 5.5. *Let* $u' = (u^{(1)'}, u^{(2)'})$ *with* $u^{(1)}$ *and* $u^{(2)}$ *independent. Partition* a_n *accordingly. If* $\{a_n^{(1)}\}$ *satisfies Assumption* (b) *and* $a^{(1)'}u^{(1)}$ *has positive probability on both sides of* 0, *then* N' *is exponentially bounded* (*and so is* N).

PROOF. Proceeding as in the proof of Theorem 5.2, analoguously to (5.6) and (5.7), we get for $j \in J$, a set of positive integers (corresponding to the set of integers of which (5.6) and (5.7) hold),

$$P[s_j^{(1)} \geqq L] > \varepsilon^r , \quad \text{and} \tag{5.14}$$
$$P[s_j^{(1)} \leqq -L] > \varepsilon^r$$

where $s_j^{(1)} = \sum_{i=1}^r a_{(j-1)r+i}^{(1)'} u_{(j-1)r+i}^{(1)}$. Let $s_j^{(2)} = s_j - s_j^{(1)}$, s_j defined in (2.3). Then by assumption $s_j^{(1)}$ and $s_j^{(2)}$ are independent. Now for any sequence $\{\gamma_j\}$,

$$\begin{aligned} [|s_j + \gamma_j| \geqq L] &= [|s_j^{(1)} + s_j^{(2)} + \gamma_j| \geqq L] \\ &\supset [\operatorname{sgn} s_j^{(1)} = \operatorname{sgn}(s_j^{(2)} + \gamma_j) \text{ and } |s_j^{(1)}| \geqq L] \\ &= F , \quad \text{say}. \end{aligned} \tag{5.15}$$

Note that

$$\begin{aligned} P[F] = P[s_j^{(2)} + \gamma_j \geqq 0]P[s_j^{(1)} \geqq 0; |s_j^{(1)}| \geqq L \mid s_j^{(2)} + \gamma_j \geqq 0] \\ + P[s_j^{(2)} + \gamma_j < 0]P[s_j^{(1)} < 0; |s_j^{(1)}| \geqq L \mid s_j^{(2)} + \gamma_j < 0] . \end{aligned}$$

By independence, we may drop the conditioning and, by (5.14), we get for $j \in J$

$$(5.16)\quad P[F] \geqq P[s_j^{(2)} + \gamma_j \geqq 0]P[s_j^{(1)} \geqq L] + P[s_j^{(2)} + \gamma_j < 0]P[s_j^{(1)} \leqq -L] \geqq \varepsilon^r .$$

By (5.15) and (5.16), we have for $j \in J$

$$P[|s_j + \gamma_j| > L] \geqq \varepsilon^r .$$

The rest of the proof is similar to that of Theorem 5.2.

THEOREM 5.6. *Let $u' = (u^{(1)'}, u^{(2)'})$ with $u^{(1)}$ and $u^{(2)}$ independent. Partition a_n accordingly. If $\{a_n^{(1)}\}$ satisfies Assumption* (c) *and for every $c \neq 0$, $c'u^{(1)}$ has positive probability on both sides of* 0, *then N' is exponentially bounded* (*and so is* N).

The proof is a combination of those of Theorems 5.5 and 3.1.

6. Applications. We now consider three examples as applications of results in previous sections. Examples 6.3 is rather abstract. It provides only hints of the connection between the results in this paper and how they can be applied. The detail is too lengthy to be included and will be discussed elsewhere along with other similar problems.

EXAMPLE 6.1. Consider the sequential model

$$(6.1)\qquad y_i = \alpha a_i + u_i , \qquad i = 1, 2, \cdots ,$$

where $\{a_i\}$ is a sequence of real numbers, $u, u_1, u_2, \cdots$ are i.i.d. $N(0, 1)$ random variables and α is an unknown parameter. Suppose that we are interested in testing sequentially $H_1: \alpha = \alpha_1$ vs. $H_2: \alpha = \alpha_2$, with $\alpha_1 \neq \alpha_2$. If, under the true distribution P, α is the true value of the parameter, the logarithm of probability ratio at stage n can be written as

$$L_n = -(\alpha_1 - \alpha_2) \textstyle\sum_1^n a_i u_i + C_n ,$$

where

$$C_n = (\tfrac{1}{2})[(\alpha_1^2 - \alpha_2^2) - \alpha(\alpha_1 - \alpha_2)] \textstyle\sum_1^n a_i^2 .$$

Thus, if $\{a_n\}$ satisfies Assumption (c) and if $P[u > 0] > 0$ and $P[u < 0] > 0$ (by Theorem 3.1) or if u is nondegenerate and has finite mean (by Corollary 3.1.2), the stopping time of the SPRT is exponentially bounded.

EXAMPLE 6.2. Suppose that $u_1, u_2, \cdots$ are independent and that u_i has $N(0, \sigma_i^2)$ distribution, $i = 1, 2, \cdots$. Suppose that we are interested in discriminating sequentially between $H_1: \{\sigma_i\} = \{\sigma_{1i}\}$ vs. $H_2: \{\sigma_i\} = \{\sigma_{2i}\}$. Consider the SPRT test. At stage n, the logarithm of probability ratio is given by

$$L_n = \textstyle\sum_1^n g_i u_i^2 + \sum_1^n b_i ,$$

where $g_i = -\frac{1}{2}(1/\sigma_{2i}^2 - 1/\sigma_{1i}^2)$ and $b_i = \log \sigma_{2i}/\sigma_{1i}$, $i = 1, 2, \cdots, n$. Now if, under the true distribution, u_i has distribution (not necessarily normal) with c_i as (true) scale parameter, $i = 1, 2, \cdots$, letting $v_i = (u_i/c_i)^2$, we can write

$$L_n = \textstyle\sum_1^n a_i v_i + \sum_1^n bi ,$$

where $v, v_1, \cdots$ are i.i.d. and $a_i = g_i c_i^2$. Thus, by Theorem 3.1 (or Corollary 3.1.2), the stopping time of the SPRT is exponentially bounded provided that $P[v < 0] > 0$ and $P[v > 0] > 0$ (or v has finite expectation) and $\{a_n\}$ satisfies Assumption (c). It is noted that similar technique can be applied to test of parameter in other distributions such as negative exponential (one or two tail), Weibull and gamma distributions. Examples of this kind in the real world could be: to determine sequentially whether the variance of weighting errors (assume normal distributions) is proportional to the volume of container or its square root (see [2]); or to determine whether or not the expected life of a radiative substance depends on its volume.

EXAMPLE 6.3. Under the framework (6.1), suppose that $u_1, u_2, \cdots$ are i.i.d. $N(0, \sigma^2)$ and α and σ^2 are unknown parameters. Let $\lambda = \alpha/\sigma$ and consider the sequential test of $H_1: \lambda = \lambda_1$ vs. $H_2: \lambda = \lambda_2$ with $\lambda_1 \neq \lambda_2$. The problem is invariant under the linear transformation: $y_i \to cy_i$, $i = 1, 2, \cdots$, where $c > 0$. At stage n, (s_n^2, z_n) is a sufficient statistic and $T_n = z_n/s_n$ is maximal invariant, where $z_n = \sum_1^n a_i y_i/k_n$, $s_n^2 = \sum_1^n y_i^2 - z_n^2/k_n$ and $k_n = \sum_1^n a_i^2$. Therefore $\{T_n\}$ is an invariantly sufficient sequence and the logarithm L_n of the probability ratio of the invariant SPRT for H_1 vs. H_2 depends only on T_n at stage n (see [3]). Now if α and σ^2 are the true values of the parameters (under the true distribution), by (6.1), $T_n = (\alpha k_n + \sum_1^n a_i u_i/k_n)/(\sum_1^n u_i^2 - (\sum_1^n a_i u_i/k_n)^2)^{\frac{1}{2}}$. It can be shown that, for the purpose of proving the exponential boundedness of the stopping time of the test, L_n can be locally and uniformly approximated by a linear function of $\sum_1^n u_i^2$ and $\sum_1^n a_i u_i$. When $y_1, y_2, \cdots$ are i.i.d. (i.e., $a_i = 1$ for all i), Wijsman used the technique that L_n can be locally and uniformly approximated by linear function of $\sum_1^n y_i$ and $\sum_1^n y_i^2$ and applied Stein's original result to prove the exponential boundedness of the stopping time for sequential t-test [6]. A similar idea can be used to prove the same result of the test in this example with the results in previous sections playing the role of Stein's result.

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