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BAHADUR EFFICIENCY OF LINEAR RANK STATISTICS FOR SCALE ALTERNATIVES[1]

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Exact limiting Bahadur efficiencies of linear rank tests for scale are derived using a minor extension of the results of Hoadley. Efficiency curves are derived for the linear rank statistics of Mood, Freund and Ansari, and Capon and Klotz relative to the F test for normal scale alternatives when the null hypothesis is that of common normality.

1. Introduction and notation. Let $X_1, X_2, \cdots, X_m$ and $Y_1, Y_2, \cdots, Y_n$ be samples from distributions F_0 and G_0 respectively and let $0 < \lambda_0 \leqq \lambda_N < 1 - \lambda_0$ where $N = m + n$, $N\lambda_N = m$, $\lambda_N \to \lambda$ as $N \to \infty$. Define

$$(1.1) \qquad Z_{Ni} = 1 \text{ if the } i\text{th smallest observation in the pooled sample is an } X, \\ = 0 \text{ otherwise.}$$

We consider Chernoff–Savage (1958) statistics

$$(1.2) \qquad mT_N = \textstyle\sum_{i=1}^N E_{Ni} Z_{Ni}$$

where E_{Ni} are given constants and use the equivalent convenient representation of Govindarajulu, Le Cam, and Raghavachari (1966)

$$(1.3) \qquad T_N(F_m, G_n) = \textstyle\int_{-\infty}^{\infty} J_N(NH_N(x)/(N+1))\, dF_m(x)$$

where F_m and G_n are the respective X and Y sample cdf's, $H_N = \lambda_N F_m + (1 - \lambda_N)G_n$, and J_N is a score function.

For the problem of scale we consider distributions of the form $F_0(x) = \Psi((x - \nu)/\sigma)$, $G_0(x) = \Psi((x - \nu)/\tau)$ with the hypothesis $H: \sigma = \tau$ equivalent to $H: F_0 = G_0$ because of the assumption that X and Y have the same location parameter ν. Statistics of the form (1.3) for scale that we consider are determined by the following score functions:

$$(1.4) \qquad \begin{array}{ll} \text{Absolute scores} & J_N(u) = |(i/(N+1)) - (\frac{1}{2})| \\ \text{Quadratic scores} & J_N(u) = ((i/(N+1)) - (\frac{1}{2}))^2 \\ \text{Normal scores} & J_N(u) = [\Phi^{-1}(i/(N+1))]^2 \end{array}$$

where $u \in [i/(N+1), (i+1)/(N+1))$ and Φ is the standard normal cdf.

We derive the limiting large deviation Bahadur efficiency relative to the F test for normal scale alternatives and because of large sample similarity, the results for the absolute scores cover results for the tests of Barton and David

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(1958), Freund and Ansari (1957), and Siegel and Tukey (1960), with the normal scores covering the tests of Capon (1961), and Klotz (1962). The quadratic scores are those of the test of Mood (1954).

2. Large deviation probabilities. Sanov (1957) showed that if F_N is the sample cdf of a sample of size N from a population with distribution F_0 and if Ω is a certain well-behaved set of cdf's not containing F_0 then

$$\lim_{N\to\infty} -N^{-1} \ln P[F_N \in \Omega] = \inf_{F\in\Omega} \int \ln (dF/dF_0)\, dF .$$

Hoadley (1967) extended this theory to cover c-sample statistics for bounded score functions and one of us relaxed the conditions of Hoadley to statistics such as (1.3), (1.4) which satisfy the following

Conditions B.

(i) For each N, J_N is constant over the intervals $[i/(N+1), (i+1)/(N+1))$ for $i = 0, 1, 2, \cdots, N$.

(ii) There exists a score function J on (0, 1) such that

$$\sum_{i=1}^{N} |J_N(i/(N+1)) - J(i/(N+1))|/(N+1) \to 0 \qquad \text{as } N \to \infty .$$

(iii) $J(\cdot)$ is continuous, integrable on (0, 1) and $|J(\cdot)|$ is nonincreasing on $[0, \frac{1}{2}]$ nondecreasing on $(\frac{1}{2}, 1)$.

Hwang (1972) proved the following theorem essentially equivalent to Theorem 4 of Woodworth (1970):

THEOREM 1. *If T_N is a sequence of linear rank statistics* (1.3) *satisfying conditions* B *and $I_H(c^*; J) < \infty$ for some $c^* > c_0 = \int_{-\infty}^{\infty} J(H_0(x)\, dF_0(x))$, then for every $c^* > c > c_0$ we have*

$$\lim -N^{-1} \ln P_H[T_N(F_m, G_N) \geqq c] = I_H(c; J) \tag{2.1}$$

where

$$I_H(c; J) = 2h(c - v) - \int_0^1 \ln \{(1-\lambda) + \lambda \exp[2h(J(x) - v)/\lambda]\}\, dx \tag{2.2}$$

and (h, v) is the unique solution of the equations.

$$\int_0^1 \frac{\exp[2h(J(x) - v)/\lambda]}{(1-\lambda) + \lambda \exp[2h(J(x) - v)/\lambda]}\, dx = 1 \tag{2.3}$$

$$\int_0^1 \frac{J(x) \exp[2h(J(x) - v)/\lambda]}{(1-\lambda) + \lambda \exp[2h(J(x) - v)/\lambda]}\, dx = c . \tag{2.4}$$

For the absolute scores statistic (2.2), (2.3), (2.4) reduces to the equations

$$\begin{aligned} I_H(c; J) = {} & 4hc - \lambda \ln \lambda - (1-\lambda) \ln (1-\lambda) \\ & + (1-\lambda) \ln \left(\exp\left(\frac{h}{\lambda} - h\right) - 1\right) \\ & + \lambda \ln (1 - \exp(-h)) - \ln (\exp(h/\lambda) - 1) , \end{aligned} \tag{2.5}$$

using $J(x) = |x - \frac{1}{2}|$, where h is the unique solution to

$$(2.6) \qquad 2c = 1 - \frac{4}{\lambda} \int_0^{\frac{1}{2}} \frac{u \exp(2h(\frac{1}{2} - u)/\lambda)}{(1 - e^{h/\lambda - h})(e^{-h} - 1)^{-1} + e^{2h(\frac{1}{2}-u)/\lambda}}\, du .$$

For quadratic scores, using $J(x) = (x - \frac{1}{2})^2$, we obtain

$$(2.7) \qquad I_H(c, J) = 6hc - 2hv - \ln\{(1 - \lambda) + \lambda \exp[2h((1/4) - v)/\lambda]\}$$

where h, v uniquely satisfy

$$(2.8) \qquad \int_0^{\frac{1}{2}} \frac{\exp[2h(y^2 - v)/\lambda]}{(1 - \lambda) + \lambda \exp[2h(y^2 - v)/\lambda]}\, dy = \tfrac{1}{2}$$

and

$$(2.9) \qquad \int_0^{\frac{1}{2}} \frac{y^2 \exp[2h(y^2 - v)/\lambda]}{(1 - \lambda) + \lambda \exp[2h(y^2 - v)/\lambda]}\, dy = c/2 .$$

The normal scores evaluation uses

$$(2.10) \qquad I_H(c, J) = 2h(c - v) - 2 \int_0^\infty \ln\{(1 - \lambda) + \lambda \exp\{2h(x^2 - v)/\lambda]\}\phi(x)\, dx$$

where $\phi(x) = d\Phi(x)/dx$ and h, v is the unique solution to

$$(2.11) \qquad \int_0^\infty \frac{\exp[2h(x^2 - v)/\lambda]}{(1 - \lambda) + \lambda \exp[2h(x^2 - v)/\lambda]}\, \phi(x)\, dx = \tfrac{1}{2}$$

and

$$(2.12) \qquad \int_0^\infty \frac{x^2 \exp[2h(x^2 - v)/h]}{(1 - \lambda) + \lambda \exp[2h(x^2 - v)/\lambda]}\, \phi(x)\, dx = c/2 .$$

To evaluate the large deviation probability for the F statistic when the X's and Y's have a common normal distribution, routine calculations using the incomplete beta integral give

$$(2.13) \qquad \lim_{N\to\infty} -N^{-1} \ln P[S_x^2/S_y^2 \geqq c_{mn}] = 2^{-1}[\ln\{\lambda\theta^2 + (1 - \lambda)\} - \lambda \ln \theta^2]$$

for fixed $\theta > 1$, $c_{mn} \to \theta^2$ as $N \to \infty$, where $S_x^2 = \sum_i (x_i - \bar{x})^2/(m - 1)$ etc.

3. Bahadur efficiency calculations. For a fixed one-sided alternative (σ, τ) $\sigma > \tau$ the Bahadur efficiency of two tests is obtained by comparing the exponential rate of convergence of type I errors to zero while equating the large sample type II errors β subject to the condition $0 < \beta < 1$. Following Bahadur (1960) we note that the statistics $N^{\frac{1}{2}}(T_N(F_m, G_n) - \int_0^1 J(x)\, dx)$ and $N^{\frac{1}{2}}(S_x^2/S_y^2 - 1)$ each constitute standard sequences because of null asymptotic normality, (2.1), (2.13), and the following theorem of Hwang (1972) (Theorem 3.1, page 35).

THEOREM 3. *If J_N, J satisfy conditions* B *then*

$$T_N(F_m, G_n) \to_{\text{a.s.}} \int_{-\infty}^{\infty} J(H_0(x))\, d F_0(x)$$

as $N \to \infty$.

As pointed out by the referee, Theorem 3 can also be obtained under weaker conditions (not required for our application) from Theorem 1 of Hájek (1974). As a consequence the Bahadur efficiency is given by the ratio of the asymptotic

TABLE 1.1

Values of c, $I_H(c; J)$ for the absolute scores statistic under normal scale alternatives

σ/τ	$\lambda = \frac{1}{2}$		$\lambda = \frac{1}{4}$		$\lambda = \frac{1}{8}$	
	c	$I_H(c, J)$	c	$I_H(c; J)$	c	$I_H(c; J)$
1.05	0.12694	0.00072	0.06396	0.00054	0.03210	0.00032
1.25	0.13381	0.01496	0.06910	0.01122	0.03510	0.00655
1.50	0.14071	0.04807	0.07428	0.03610	0.03812	0.02109
1.75	0.14619	0.08855	0.07839	0.06657	0.04052	0.03895
2.00	0.15060	0.13103	0.08170	0.09863	0.04245	0.05780
2.25	0.15422	0.17292	0.08441	0.13033	0.04403	0.07650
2.50	0.15722	0.21304	0.08667	0.16078	0.04535	0.09454
2.75	0.15975	0.25088	0.08856	0.18958	0.04645	0.11167
3.00	0.16190	0.28628	0.09018	0.21661	0.04740	0.12781
3.25	0.16375	0.31922	0.09156	0.24184	0.04820	0.14296
3.50	0.16535	0.34979	0.09277	0.26534	0.04890	0.15710
3.75	0.16675	0.37818	0.09381	0.28721	0.04952	0.17033
4.00	0.16799	0.40460	0.09474	0.30763	0.05006	0.18274
4.25	0.16908	0.42928	0.09556	0.32677	0.05054	0.19542
4.50	0.17007	0.45245	0.09630	0.34478	0.05097	0.20546
4.75	0.17096	0.47430	0.09697	0.36181	0.05136	0.21594
5.00	0.17178	0.49497	0.09758	0.37798	0.05171	0.22593

TABLE 1.2

Values of c, $I_H(c; J)$ for the quadratic scores statistic under normal scale alternatives

σ/τ	$\lambda = \frac{1}{2}$		$\lambda = \frac{1}{4}$		$\lambda = \frac{1}{8}$	
	c	$I_H(c, J)$	c	$I_H(c; J)$	c	$I_H(c; J)$
1.05	0.04278	0.00090	0.02167	0.00067	0.01091	0.00039
1.25	0.04673	0.01864	0.02470	0.01408	0.01269	0.00826
1.50	0.05065	0.05948	0.02780	0.04528	0.01454	0.02664
1.75	0.05370	0.10856	0.03027	0.08314	0.01604	0.04908
2.00	0.05610	0.15829	0.03227	0.13249	0.01726	0.07258
2.25	0.05803	0.20829	0.03390	0.16093	0.01827	0.09566
2.50	0.05959	0.25460	0.03525	0.18739	0.01911	0.11766
2.75	0.06087	0.29744	0.03638	0.23131	0.01981	0.13828
3.00	0.06195	0.33715	0.03733	0.26292	0.02042	0.15756
3.25	0.06286	0.37376	0.03816	0.29216	0.02094	0.17548
3.50	0.06365	0.40780	0.03888	0.31939	0.02140	0.19228
3.75	0.06434	0.43760	0.03951	0.34488	0.02180	0.20816
4.00	0.06495	0.46967	0.04009	0.36926	0.02217	0.22328
4.25	0.06550	0.49827	0.04061	0.39265	0.02251	0.23797
4.50	0.06601	0.52617	0.04110	0.41532	0.02282	0.25224
4.75	0.06648	0.55354	0.04155	0.43761	0.02312	0.26650
5.00	0.06693	0.58060	0.04199	0.45970	0.02340	0.28057

TABLE 1.3
Values of c, $I_H(c, J)$ for the normal scores statistic under normal scale alternatives

σ/τ	$\lambda = \frac{1}{2}$		$\lambda = \frac{1}{4}$		$\lambda = \frac{1}{8}$	
	c	$I_H(c; J)$	c	$I_H(c; J)$	c	$I_H(c; J)$
1.05	0.52433	0.00118	0.26871	0.00090	0.13605	0.00053
1.25	0.66641	0.02388	0.33923	0.00192	0.18087	0.00118
1.50	0.67798	0.07311	0.40992	0.06150	0.23112	0.03927
1.75	0.72627	0.12880	0.46203	0.11047	0.27120	0.07223
2.00	0.76021	0.18367	0.50069	0.15901	0.30238	0.10545
2.25	0.78505	0.23521	0.53007	0.20449	0.32682	0.13678
2.50	0.80388	0.28273	0.55295	0.24613	0.34629	0.16554
2.75	0.81856	0.32623	0.57121	0.28396	0.36205	0.19162
3.00	0.83028	0.36591	0.58610	0.31827	0.37506	0.21532
3.25	0.83982	0.40200	0.59842	0.34935	0.38603	0.23698
3.50	0.84769	0.43479	0.60871	0.37741	0.39536	0.25677
3.75	0.85429	0.46461	0.61736	0.40262	0.40328	0.27469
4.00	0.85992	0.49190	0.62467	0.42527	0.40999	0.29070
4.25	0.86481	0.51715	0.63094	0.44575	0.41566	0.30489
4.50	0.86916	0.54089	0.63642	0.46453	0.42048	0.31750
4.75	0.87310	0.56357	0.64133	0.48212	0.42467	0.32884
5.00	0.87674	0.58561	0.64584	0.49896	0.42839	0.33929

TABLE 2.1
Bahadur efficiency of the absolute scores statistic relative to the F test for normal scale alternatives
$e^{(B)}(J, F)$

σ/τ	$\lambda = \frac{1}{2}$	$\lambda = \frac{1}{4}$	$\lambda = \frac{1}{8}$
1.00	0.60792	0.60792	0.60792
1.05	0.60781	0.59736	0.59258
1.25	0.60565	0.56091	0.53834
1.50	0.60057	0.52163	0.48434
1.75	0.59416	0.48941	0.44735
2.00	0.58720	0.46295	0.39813
2.25	0.58018	0.44109	0.36791
2.50	0.57335	0.42208	0.34316
2.75	0.56678	0.40752	0.32264
3.00	0.56042	0.39434	0.30541
3.25	0.55422	0.38295	0.29074
3.50	0.54819	0.37271	0.27010
3.75	0.54234	0.36369	0.26714
4.00	0.53676	0.35864	0.25757
4.25	0.53151	0.34816	0.24919
4.50	0.52663	0.34205	0.24183
4.75	0.52214	0.33633	0.23534
5.00	0.51801	0.33121	0.22961

TABLE 2.2

Bahadur efficiency of the quadratic scores statistic relative to the F test for normal scale alternatives

$e^{(B)}(J, F)$

σ/τ	$\lambda = \frac{1}{2}$	$\lambda = \frac{1}{4}$	$\lambda = \frac{1}{8}$
1.00	0.75991	0.75991	0.75991
1.05	0.76138	0.74758	0.74139
1.25	0.75497	0.70391	0.67945
1.50	0.74316	0.65435	0.60799
1.75	0.72845	0.61127	0.54862
2.00	0.71384	0.57497	0.50002
2.25	0.69920	0.54466	0.46009
2.50	0.68522	0.51919	0.42712
2.75	0.67183	0.49722	0.39954
3.00	0.66002	0.47864	0.37650
3.25	0.64892	0.46252	0.35691
3.50	0.63911	0.44864	0.34040
3.75	0.63043	0.43671	0.32647
4.00	0.62309	0.42689	0.31475
4.25	0.61693	0.41871	0.30501
4.50	0.61244	0.41203	0.29690
4.75	0.60939	0.40678	0.29045
5.00	0.60763	0.40282	0.28515

TABLE 2.3

Bahadur efficiency of the normal scores statistic relative to the F test for normal scale alternatives

$e^{(B)}(J, F)$

σ/τ	$\lambda = \frac{1}{2}$	$\lambda = \frac{1}{4}$	$\lambda = \frac{1}{8}$
1.00	1.00000	1.00000	1.00000
1.05	0.99836	0.99770	0.99848
1.25	0.96749	0.96478	0.97293
1.50	0.91346	0.88884	0.89526
1.75	0.86422	0.81217	0.80740
2.00	0.82313	0.74638	0.72644
2.25	0.78919	0.69210	0.65785
2.50	0.76092	0.64741	0.60090
2.75	0.73707	0.61038	0.55366
3.00	0.71631	0.57940	0.51453
3.25	0.69797	0.55305	0.48201
3.50	0.68139	0.53014	0.45457
3.75	0.66629	0.50984	0.43081
4.00	0.65258	0.49164	0.40974
4.25	0.64031	0.47534	0.39079
4.50	0.62958	0.46086	0.37371
4.75	0.62042	0.44816	0.35840
5.00	0.61288	0.43722	0.34483

slopes (2.2), (2.13):

$$e^{(B)}(J, F) = \frac{2h(c-v) - \int_0^1 \ln\{(1-\lambda) + \lambda \exp[2h(J(u)-v)/\lambda]\}\, du}{[\ln\{\lambda\theta^2 + 1 - \lambda\} - \lambda \ln \theta^2]/2} \tag{3.1}$$

where h, v uniquely satisfy (2.3), (2.4) and

$$c = \int_{-\infty}^{\infty} J(H_0(x))\, dF_0(x)\,, \tag{3.2}$$

$H_0 = \lambda F_0 + (1-\lambda)G_0$, $F_0(x) = \Phi((x-\nu)/\sigma)$, $G_0(x) = \Phi((x-\nu)/\tau)$, and $\theta^2 = \sigma^2/\tau^2 > 1$ for one-sided alternatives, $\theta = 1$ under H.

Numerical values were computed at the University of Wisconsin computing center on the Univac 1108. Integrals were evaluated by Ganss–Legendre integration, and roots to equations (2.3), (2.4) were obtained using monotonicity and a half interval search algorithm. Accuracy of integrals was checked by comparison with known values and found accurate to 10^{-6}. Tables 2.1, 2.2, and 2.3 give values of c, $I_H(c; J)$ from (2.5), (2.7), (2.10) and (3.2) for selected σ/τ values and $\lambda = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$. *For the null hypothesis of common normality*, Tables 2.1, 2.2, and 2.3 give ratios (3.1) for the same λ values and σ/τ alternative values.

We note that the efficiency drops off with large scale ratios which may reflect a lack of efficiency for the nonparametric methods when alternatives are greatly different from the hypothesis. As the parameters approach the null hypothesis it can be shown with two applications of L'Hospital's rule that the Bahadur efficiency values converge to the Pitman efficiency values of $6/\pi^2$, $15/2\pi^2$, and 1 for the absolute, quadratic, and normal scores tests relative to the F test (Klotz (1962)). These results are quite similar in nature to those obtained for the location parameter problems (Klotz (1965), Hoadley (1967), Stonė (1967), and Woodworth (1970)).

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