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DUALS OF BALANCED INCOMPLETE BLOCK DESIGNS DERIVED FROM AN AFFINE GEOMETRY

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It is well known that by identifying the points of an affine geometry AG (t, q) with treatments and identifying the μ-flats $(1 \leqq \mu < t)$ of AG (t, q) with blocks, a BIB design denoted by AG $(t, q): \mu$ is derived from AG (t, q) where q is a prime or a prime power. In this paper, we introduce a new association scheme called an affine geometrical association scheme and show that the dual of the BIB design AG $(t, q): \mu$ is an affine geometrical type PBIB design with $m = \min\{2\mu + 1, 2(t - \mu)\}$ associate classes. It is also shown that in the case $\mu = 1$ and $t \geqq 3$, the number of the associate classes of this dual design can be reduced from three to two but it is not reducible except for the above case. From those results, we can get a new series of PBIB designs.

1. Introduction and summary. The design D^*, which is obtained from a design D by interchanging treatments and blocks in D, is said to be the dual design of D. Dualization of known designs sometimes yields new designs and the duals of BIB designs [12] or PBIB designs [2] have been investigated by several authors [4, 5, 6, 7, 8, 9, 10]. Shrikhande [9] proved that the duals of asymmetrical BIB designs with parameters $v, b, r, k, \lambda = 1$ or $v = \binom{r-1}{2}$, $b = \binom{r}{2}$, r, $k = r - 2$, $\lambda = 2$ are PBIB designs with two associate classes. But the dual of a BIB design with $\lambda \geqq 3$ is not always a PBIB design. For example, let us consider a BIB design with parameters $v = 8$, $b = 14$, $r = 7$, $k = 4$, $\lambda = 3$. It is known [11] that there are two non-isomorphic designs such as

$$D_1 = \begin{Bmatrix} 1248, 2358, 3468, 4578, 5618, 6728, 7138 \\ 3567, 4671, 5712, 6123, 7234, 1345, 2456 \end{Bmatrix}$$

$$D_2 = \begin{Bmatrix} 1234, 1256, 1278, 5678, 3478, 3456, 1357 \\ 2457, 2458, 1358, 1467, 1468, 2367, 2368 \end{Bmatrix}$$

where each of the numbers $1, 2, \cdots, 8$ represents each of the eight treatments and each set of four numbers $c_1 c_2 c_3 c_4$ represents a block which contains four treatments c_1, c_2, c_3 and c_4. The dual of the design D_1, which is isomorphic with the BIB design AG $(3, 2): 2$, is a group divisible type PBIB design but the dual of the design D_2 is not a PBIB design. This shows that in the case $\lambda \geqq 3$, the dual of a BIB design is not always a PBIB design and it depends not only on parameters (v, b, r, k, λ), but also on the block structure of the BIB design in general. Recently, Hamada [4] showed that the dual of the BIB design

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PG $(t, q): \mu$, which is derived from a finite projective geometry PG (t, q) by identifying the points of PG (t, q) with treatments and identifying the μ-flats of PG (t, q) with blocks, is a PBIB design with $\min\{\mu + 1, t - \mu\}$ associate classes for any integers t and μ such that $1 \leqq \mu < t - 1$. The purpose of this paper is to introduce a new association scheme called an affine geometrical association scheme and to show that the dual of the BIB design AG $(t, q): \mu$ is an affine geometrical type PBIB design with $m = \min\{2\mu + 1, 2(t - \mu)\}$ associate classes. Since the number of distinct coincidence numbers $\lambda_{(i,j)}$ in this dual design is $\min\{\mu + 1, t - \mu + 1\}$ $(< m)$, it seems that the number of the associate classes of this dual design can be reduced to associate classes less than m. But it is shown that it is not reducible except for the case $\mu = 1$ and $t \geqq 3$. From those results, we can get a new series of PBIB designs.

2. Points and μ-flats in PG (t, q) and AG (t, q). With the help of the Galois field GF (q), where q is a prime or a prime power, we can define a finite projective geometry PG (t, q) of t dimensions as a set of points satisfying the following conditions:

(a) A point in PG (t, q) is represented by (ν) where ν is a nonzero element of GF (q^{t+1}).

(b) Two points (ν_1) and (ν_2) represent the same point when and only when there exists a nonzero element σ of GF (q) such that $\nu_1 = \sigma\nu_2$.

(c) A μ-flat, V, $(0 \leqq \mu \leqq t)$ in PG (t, q) is defined as a set of points

$$V = \{(a_0\nu_0 + a_1\nu_1 + \cdots + a_\mu\nu_\mu)\}$$

where a's run independently over the elements of GF (q), not all zero, and $(\nu_0), (\nu_1), \cdots, (\nu_\mu)$ are linearly independent over the coefficient field GF (q), in other words, they do not lie on a $(\mu - 1)$-flat. These $\mu + 1$ linearly independent points $(\nu_0), (\nu_1), \cdots, (\nu_\mu)$ are called the defining points of the μ-flat V. For the sake of convenience, we denote the empty set $\emptyset$ by (-1)-flat. Using $t + 1$ elements $x_0, x_1, \cdots, x_t$ of GF (q) not all zero, any point in PG (t, q) is also represented by $((x_0, x_1, \cdots, x_t))$.

Let U_0 be the $(t - 1)$-flat composed of all points $((x_0, x_1, \cdots, x_t))$ in PG (t, q) such that $x_0 = 0$ and let us denote by $P_0(t, q)$, the set of points in PG (t, q) not contained in U_0 and $\mathscr{B}_0(t, \mu, q)$, the set of μ-flats in PG (t, q) not contained in U_0 (i.e., the set of μ-flats V in PG (t, q) such that $V \cap U_0$ is a $(\mu - 1)$-flat).

A point in the t-dimensional affine geometry AG (t, q) (or EG (t, q)) is represented by (ν) where ν is an element of GF (q^t) and each element represents a unique point. A μ-flat $(0 \leqq \mu \leqq t)$ in AG (t, q) may be defined as a set of points $\{((x_1, x_2, \cdots, x_t)) : ((1, x_1, x_2, \cdots, x_t)) \in V\}$ for some μ-flat V in $\mathscr{B}_0(t, \mu, q)$. It is well known that (i) there exists a one-to-one correspondence between points of AG (t, q) and points of $P_0(t, q)$, and between μ-flats of AG (t, q) and μ-flats of $\mathscr{B}_0(t, \mu, q)$, respectively and (ii) the number of points in AG (t, q) is equal to q^t and the number of μ-flats in AG (t, q) is equal to $\phi(t, \mu, q) - \phi(t - 1, \mu, q)$

where

$$\phi(t, \mu, q) = \frac{(q^{t+1} - 1)(q^{t} - 1)\cdots(q^{t-\mu+1} - 1)}{(q^{\mu+1} - 1)(q^{\mu} - 1)\cdots(q - 1)} \tag{2.1}$$

for any integers t and μ such that $0 \leqq \mu \leqq t$. For the sake of convenience, we make a promise that $\phi(t, -1, q) = 1$ for $t \geqq -1$ and $\phi(t, \mu, q) = 0$ for $\mu \leqq -2$ or $\mu > t$.

3. An affine geometrical association scheme. Let us denote $v = \phi(t, \mu, q) - \phi(t-1, \mu, q)$ μ-flats in $\mathscr{B}_0(t, \mu, q)$ by V_α ($\alpha = 1, 2, \cdots, v$) and let

$$V_\alpha^* = \{((x_1, x_2, \cdots, x_t)) : ((1, x_1, x_2, \cdots, x_t)) \in V_\alpha\}$$

for $\alpha = 1, 2, \cdots, v$. Among those v μ-flats V_α^* ($\alpha = 1, 2, \cdots, v$) in AG (t, q), we define a relation of association, called an affine geometrical (AG) association scheme, as follows:

DEFINITION 3.1. If $V_\alpha \cap V_\beta$ is a $(\mu - i)$-flat and $V_\alpha \cap V_\beta \cap U_0$ is a $(\mu - i - \varepsilon)$-flat for some integers i and ε such that $1 \leqq i \leqq \min\{\mu, t - \mu\}$ and $0 \leqq \varepsilon \leqq 1$, two μ-flats V_α^* and V_β^* in AG (t, q) are said to be (i, ε)th associates. In the special case $V_\alpha \cap V_\beta = \emptyset$, two μ-flats V_α^* and V_β^* in AG (t, q) are said to be $(\mu + 1, 0)$th associates.

Note that (i) if V_α and V_β are μ-flats in $\mathscr{B}_0(t, \mu, q)$, $\dim(V_\alpha \cap V_\beta) \geqq \max\{-1, 2\mu - t\}$ where "$\dim W = m$" means that W is an m-flat and (ii) if $V_\alpha \cap V_\beta$ is an m-flat ($\mu > m \geqq \max\{0, 2\mu - t\}$), $\dim(V_\alpha \cap V_\beta \cap U_0) = m$ or $m - 1$ since U_0 is a $(t-1)$-flat in PG (t, q).

THEOREM 3.1. *The association defined above is an association scheme with* $m = \min\{2\mu + 1, 2(t - \mu)\}$ *associate classes and parameters*

$$n_{(i,\varepsilon)} = q^{(i+\varepsilon)(i+\varepsilon-1)}\phi(\mu - 1, \mu - i - \varepsilon, q)\phi(t - \mu - 1, i + \varepsilon - 2, q) \times \{(1 - \varepsilon)(q^{t-\mu-i+1} - 1) + \varepsilon\}, \tag{3.1}$$

$$\begin{aligned} p^{(i,\varepsilon)}_{(j,\zeta)(k,\xi)} = {} & \textstyle\sum_{r=0}^{1-\varepsilon} \sum_{n=u}^{w} \sum_{l=0}^{z} q^{c(r,n,l;\varepsilon)}\phi(\mu - i - \varepsilon, n, q) \\ & \times \phi(i + \varepsilon - 2, \mu - j - \zeta - n - 1, q) \\ & \times \phi(i + \varepsilon - 2, \mu - k - \xi - n - 1, q) \\ & \times \phi(t - \mu - i - 1, n + j + \zeta + k + \xi - r - \mu - l - 2, q) \\ & \times \chi(n + i + \varepsilon + j + \zeta - \mu - 1, \\ & \quad n + i + \varepsilon + k + \xi - \mu - 1, l; q) \\ & \times \{(1 - \zeta)(1 - \xi)q^{t-\mu} - (-1)^{\zeta}(1 - \xi)q^{j+\zeta-1} \\ & - (-1)^{\xi}(1 - \zeta)q^{k+\xi-1} + (-1)^{\zeta+\xi}(r + \varepsilon)q^{\mu+l-i-\varepsilon-n}\} \end{aligned} \tag{3.2}$$

for

$$\varepsilon, \zeta, \xi = 0, 1, \quad i = 1, 2, \cdots, \gamma_\varepsilon, \quad j = 1, 2, \cdots, \gamma_\zeta, \quad k = 1, 2, \cdots, \gamma_\xi$$

where

$$
\begin{aligned}
&\gamma_0 = \min\{\mu+1, t-\mu\}, \qquad \gamma_1 = \min\{\mu, t-\mu\},\\
&u = \max\{-1, \mu+1-i-\varepsilon-j-\zeta, \mu+1-i-\varepsilon-k-\xi,\\
&\qquad \mu+1-j-\zeta-k-\xi\},\\
&w = \min\{\mu-i-\varepsilon, \mu-j-\zeta, \mu-k-\xi\},\\
&z = \min\{n+i+\varepsilon+j+\zeta-\mu-1, n+i+\varepsilon+k+\xi-\mu-1,\\
&\qquad n+j+\zeta+k+\xi-r-\mu-1\},\\
&c(r, n, l; \varepsilon) = (n+j+\zeta+k+\xi-\mu-l-1)\\
&\qquad \times (n+i+j+\zeta+k+\xi-r-\mu-l-1)\\
&\qquad + (\mu-n-i-\varepsilon)(l+2\mu-2n-j-\zeta-k-\xi)
\end{aligned} \tag{3.3}
$$

and $p^{(i,\varepsilon)}_{(j,\zeta)(k,\xi)} = 0$ *if* $u > w$ *or* $z < 0$ *and* $\chi(\omega_1, \omega_2, l; q)$ *is defined by*

$$
\chi(\omega_1, \omega_2, l; q) = \frac{\prod_{r=0}^{l-1}(q^{\omega_1}-q^r)(q^{\omega_2}-q^r)}{\prod_{r=0}^{l-1}(q^l-q^r)} \tag{3.4}
$$

for any positive integers ω_1, ω_2, l *and* $\chi(\omega_1, \omega_2, 0; q) = 1$ *for* $\omega_1, \omega_2 \geqq 0$.

In order to prove Theorem 3.1, we prepare several lemmas. Let U be any $(t-1)$-flat in PG (t, q) and let $\mathscr{B}(t, \mu, q)$ be the set of μ-flats in PG (t, q) not contained in U and let t, μ_1, μ_2 and m be any integers satisfying the following conditions:

$$
\mu_1, \mu_2 \geqq 0, \qquad -1 \leqq m \leqq \min\{\mu_1, \mu_2\}, \qquad \mu_1+\mu_2-m \leqq t. \tag{3.5}
$$

In the following lemmas, we denote by $T(V, W)$, the minimum flat of flats which contain both V and W, and by V_{μ_1} and V_{μ_2}, any μ_1-flat and μ_2-flat in PG (t, q) such that $V_{\mu_1} \cap V_{\mu_2}$ is an m-flat.

LEMMA 3.1. *Let* V_{μ_2+1} *be the* (μ_2+1)*-flat generated by the defining points of* V_{μ_2} *and a point* (δ) $(\notin V_{\mu_2})$ *in* PG (t, q), *i.e.,* $V_{\mu_2+1} = T(V_{\mu_2}, (\delta))$. *Then,* $V_{\mu_1} \cap V_{\mu_2+1}$ *is an* $(m+1)$*-flat or an* m*-flat according as the point* (δ) *belongs to* $T(V_{\mu_1}, V_{\mu_2})$ *or not.*

PROOF. If $(\delta) \in T(V_{\mu_1}, V_{\mu_2})$, it follows from $V_{\mu_2} \subset V_{\mu_2+1} \subset T(V_{\mu_1}, V_{\mu_2})$ that $T(V_{\mu_1}, V_{\mu_2}) \supset T(V_{\mu_1}, V_{\mu_2+1}) \supset T(V_{\mu_1}, V_{\mu_2})$. Therefore, we have

$$
\dim(T(V_{\mu_1}, V_{\mu_2})) \geqq \dim(T(V_{\mu_1}, V_{\mu_2+1})) \geqq \dim(T(V_{\mu_1}, V_{\mu_2})).
$$

Since $\dim(T(V_1, V_2)) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$ for any flats V_1 and V_2, it follows that $\dim(V_{\mu_1} \cap V_{\mu_2+1}) = m+1$, i.e., $V_{\mu_1} \cap V_{\mu_2+1}$ is an $(m+1)$-flat. If $(\delta) \notin T(V_{\mu_1}, V_{\mu_2})$, $T(V_{\mu_1}, V_{\mu_2+1}) \supset T(T(V_{\mu_1}, V_{\mu_2}), (\delta))$. Therefore, we have $\dim(V_{\mu_1} \cap V_{\mu_2+1}) = m$. This completes the proof.

From Lemma 3.1, we have the following lemma.

LEMMA 3.2. *Let* $V_{\mu_i+1} = T(V_{\mu_i}, (\delta))$ *for* $i = 1, 2$ *where* (δ) *is a point in* PG (t, q) *not contained in* $V_{\mu_1} \cup V_{\mu_2}$. *Then,* $V_{\mu_1+1} \cap V_{\mu_2+1}$ *(or* $T(V_{\mu_1}, (\delta)) \cap T(V_{\mu_2}, (\delta))$*) is an* $(m+2)$*-flat or an* $(m+1)$*-flat according as the point* (δ) *belongs to* $T(V_{\mu_1}, V_{\mu_2})$ *or not.*

In the following, let m_1, m_2 and n be integers such that

$$(3.6)\qquad -1 \leqq m_1 < \mu_1\,, \qquad -1 \leqq m_2 < \mu_2\,, \qquad -1 \leqq n \leqq \min\{m, m_1, m_2\}\,.$$

LEMMA 3.3. *If W is a μ-flat ($\mu \geqq m_1 + m_2 - n$) in PG (t, q) such that*

$$(3.7)\qquad \begin{aligned} &\dim(W \cap V_{\mu_1} \cap V_{\mu_2}) = n\,, \qquad \dim(W \cap V_{\mu_i}) = m_i \quad (i = 1, 2)\,,\\ &\dim(W \cap T(V_{\mu_1}, V_{\mu_2})) = (m_1 + m_2 - n) + l \end{aligned}$$

for some integer $l\,(\geqq 0)$ and for V_{μ_1} and V_{μ_2} such that $V_{\mu_1} \cap V_{\mu_2}$ is an m-flat, $\dim(T(W, V_{\mu_1}) \cap T(W, V_{\mu_2})) = (m + m_1 + m_2 - 2n) + 2l + s$ where $s = \mu - (m_1 + m_2 - n + l)$.

Note that l and n in Lemma 3.3 must be integers such that

$$(3.8)\qquad \begin{aligned} &0 \leqq l \leqq \min\{\mu_1 - m - m_2 + n, \mu_2 - m - m_1 + n, \mu - m_1 - m_2 + n\}\,,\\ &\max\{-1, m + m_1 - \mu_1, m + m_2 - \mu_2, m_1 + m_2 - \mu\}\\ &\qquad \leqq n \leqq \min\{m, m_1, m_2\}\,. \end{aligned}$$

PROOF. Since $\dim(T(W, V_{\mu_1}) \cap T(W, V_{\mu_2})) = m + m_1 + m_2 - 2n$ in the special case $l = 0$ and $s = 0$ (i.e., $\mu = m_1 + m_2 - n$), we have the required result from Lemma 3.2.

The following lemma is due to Hamada [4].

LEMMA 3.4. *The number of μ_2-flats W such that $W \cap V_{\mu_1} = V_m$ for V_{μ_1} and V_m (in V_{μ_1}) is equal to*

$$(3.9)\qquad \eta(\mu_2 : \mu_1, m, t, q) = q^{(\mu_1 - m)(\mu_2 - m)}\phi(t - \mu_1 - 1, \mu_2 - m - 1, q)\,.$$

LEMMA 3.5. *Let V_{m_i} ($i = 1, 2$) be an m_i-flat in V_{μ_i} such that $V_{m_1} \cap V_{m_2} = V_n$ for a given n-flat V_n in $V_{\mu_1} \cap V_{\mu_2}$. Then, the number of $(l + m_1 + m_2 - n)$-flats W in the $(\mu_1 + \mu_2 - m)$-flat $T(V_{\mu_1}, V_{\mu_2})$ such that*

$$(3.10)\qquad W \cap V_{\mu_1} = V_{m_1}\,, \qquad W \cap V_{\mu_2} = V_{m_2}\,, \qquad W \cap V_{\mu_1} \cap V_{\mu_2} = V_n$$

is equal to $q^{(m-n)l}\chi(\mu_1 - m - m_1 + n, \mu_2 - m - m_2 + n, l; q)$ where l is an integer such that $1 \leqq l \leqq \min\{\mu_1 - m - m_2 + n, \mu_2 - m - m_1 + n\}$ and $\chi(\omega_1, \omega_2, l; q)$ is given by (3.4).

PROOF. Let $V_n = L(\alpha_0, \alpha_1, \cdots, \alpha_n)$, $V_{m_i} = L(\alpha_0, \alpha_1, \cdots, \alpha_n, \beta_{i1}, \cdots, \beta_{i,m_i-n})$ ($i = 1, 2$) and $W = L(\alpha_0, \alpha_1, \cdots, \alpha_n, \beta_{11}, \cdots, \beta_{1,m_1-n}, \beta_{21}, \cdots, \beta_{2,m_2-n}, \gamma_1, \gamma_2, \cdots, \gamma_l)$ where $L(\delta_0, \delta_1, \cdots, \delta_\nu)$ denotes a ν-flat generated by the defining points (δ_0), $(\delta_1), \cdots, (\delta_\nu)$. Then, it follows from Lemma 3.1 that the first point (γ_1) must be chosen in $T(V_{\mu_1}, V_{\mu_2}) - \{T(V_{\mu_1}, W_0) \cup T(V_{\mu_2}, W_0)\}$, the second in $T(V_{\mu_1}, V_{\mu_2}) - \{T(V_{\mu_1}, W_1) \cup T(V_{\mu_2}, W_1)\}$, the third in $T(V_{\mu_1}, V_{\mu_2}) - \{T(V_{\mu_1}, W_2) \cup T(V_{\mu_2}, W_2)\}$ and so on where $W_0 = L(\alpha_0, \alpha_1, \cdots, \alpha_n, \beta_{11}, \cdots, \beta_{1,m_1-n}, \beta_{21}, \cdots, \beta_{2,m_2-n})$ and $W_{k+1} = T(W_k, (\gamma_{k+1}))$ for $k = 0, 1, \cdots, l-1$. Since $\dim(T(V_{\mu_1}, W_k) \cap T(V_{\mu_2}, W_k)) = (m + m_1 + m_2 - 2n) + 2k$ from Lemma 3.3, it follows that the first point (γ_1) can be chosen in $\phi(\mu_1 + \mu_2 - m, 0, q) - \phi(\mu_1 + m_2 - n, 0, q) - \phi(\mu_2 + m_1 - n, 0, q) + \phi(m + m_1 + m_2 - 2n, 0, q)$ ways, the second in $\phi(\mu_1 + \mu_2 - m,$

$0, q) - \phi(\mu_1 + m_2 - n + 1, 0, q) - \phi(\mu_2 + m_1 - n + 1, 0, q) + \phi(m + m_1 + m_2 - 2n + 2, 0, q)$ ways and the third in $\phi(\mu_1 + \mu_2 - m, 0, q) - \phi(\mu_1 + m_2 - n + 2, 0, q) - \phi(\mu_2 + m_1 - n + 2, 0, q) + \phi(m + m_1 + m_2 - 2n + 4, 0, q)$ ways and so on. Therefore, the total number of ways of choosing a set of l linearly independent points $(\gamma_1), (\gamma_2), \cdots, (\gamma_l)$ is equal to $q^{(m+1)l}\{\prod_{r=0}^{l-1}(q^{\mu_1-m} - q^{m_1-n+r})(q^{\mu_2-m} - q^{m_2-n+r})\}/(q-1)^l$. While, each $(l + m_1 + m_2 - n)$-flat W satisfying the condition (3.10) can be generated by any one of $\prod_{r=0}^{l-1}\{\phi(l + m_1 + m_2 - n, 0, q) - \phi(m_1 + m_2 - n + r, 0, q)\}$ sets of l independent points $(\gamma_1), (\gamma_2), \cdots, (\gamma_l)$. Hence, the number of $(l + m_1 + m_2 - n)$-flats W satisfying the condition (3.10) is equal to $q^{(m-n)l}\chi(\mu_1 - m - m_1 + n, \mu_2 - m - m_2 + n, l; q)$.

Note that Lemma 3.5 is valid for $l = 0$ if we define as $\chi(\omega_1, \omega_2, 0; q) = 1$ for $\omega_1, \omega_2 \geqq 0$.

LEMMA 3.6. *The number of μ-flats $W (\mu \geqq m_1 + m_2 - n + l)$ in* PG (t, q) *such that* $\dim(W \cap V_{\mu_1} \cap V_{\mu_2}) = n$, $\dim(W \cap V_{\mu_i}) = m_i$ $(i = 1, 2)$ *and* $\dim(W \cap T(V_{\mu_1}, V_{\mu_2})) = (m_1 + m_2 - n) + l$ *is equal to*

$$\begin{aligned} &\Gamma_{l,s}(\mu; \mu_1, \mu_2, m, m_1, m_2, n, t, q) \\ (3.11)\qquad &= q^c\phi(m, n, q)\phi(\mu_1 - m - 1, m_1 - n - 1, q) \\ &\times \phi(\mu_2 - m - 1, m_2 - n - 1, q) \\ &\times \phi(t - \mu_1 - \mu_2 + m - 1, s - 1, q) \\ &\times \chi(\mu_1 - m - m_1 + n, \mu_2 - m - m_2 + n, l; q) \end{aligned}$$

where l is an integer satisfying the condition (3.8), $s = \mu - (m_1 + m_2 - n + l)$ *and* $c = (m - n)(l + m_1 + m_2 - 2n) + s(\mu_1 + \mu_2 + n - m - m_1 - m_2 - l)$.

PROOF. From Lemma 3.5, it follows that for any flats V_{m_1}, V_{m_2} and V_n such that $V_{m_1} \subset V_{\mu_1}$, $V_{m_2} \subset V_{\mu_2}$ and $V_n \subset V_{\mu_1} \cap V_{\mu_2}$, the number of $(l + m_1 + m_2 - n)$-flats W_0 in $T(V_{\mu_1}, V_{\mu_2})$ such that $W_0 \cap V_{\mu_i} = V_{m_i}$ $(i = 1, 2)$ and $W_0 \cap V_{\mu_1} \cap V_{\mu_2} = V_n$ is equal to $q^{(m-n)l}\chi(\mu_1 - m - m_1 + n, \mu_2 - m - m_2 + n, l; q)$. Since the number of n-flats V_n in the m-flat $V_{\mu_1} \cap V_{\mu_2}$ is equal to $\phi(m, n, q)$, it follows from Lemma 3.4 that the number of $(l + m_1 + m_2 - n)$-flats W_0 such that $\dim(W_0 \cap V_{\mu_1}) = m_1$, $\dim(W_0 \cap V_{\mu_2}) = m_2$ and $\dim(W_0 \cap V_{\mu_1} \cap V_{\mu_2}) = n$ is equal to $\phi(m, n, q)\eta(m_1; m, n, \mu_1, q)\eta(m_2; m, n, \mu_2, q)q^{(m-n)l}\chi(\mu_1 - m - m_1 + n, \mu_2 - m - m_2 + n, l; q)$. Since the number of $(s + l + m_1 + m_2 - n)$-flats W in PG (t, q) such that $W \cap T(V_{\mu_1}, V_{\mu_2}) = W_0$ is equal to $\eta(s + l + m_1 + m_2 - n; \mu_1 + \mu_2 - m, l + m_1 + m_2 - n, t, q)$ for any flat W_0 in $T(V_{\mu_1}, V_{\mu_2})$, we have the required result.

LEMMA 3.7. *Let V be a μ-flat in $\mathscr{B}(t, \mu, q)$. Then, the number of μ-flats W in $\mathscr{B}(t, \mu, q)$ such that $W \cap V$ is an m-flat and $W \cap V \cap U$ is an $(m - \varepsilon)$-flat is equal to*

$$\begin{aligned} (3.12)\qquad N_0(\mu; m, t, q) &= q^{(\mu-m-1)(\mu-m)}\phi(\mu - 1, m, q) \\ &\times \phi(t - \mu - 1, \mu - m - 2, q)(q^\theta - 1) \end{aligned}$$

or

$$N_1(\mu; m, t, q) = q^{(\mu-m)(\mu-m+1)}\phi(\mu-1, m-1, q)\phi(t-\mu-1, \mu-m-1, q) \tag{3.13}$$

according as $\varepsilon = 0$ *or* 1 *where* $\theta = t - 2\mu + m + 1$.

PROOF. Since $T(V, U)$ is the t-flat in PG (t, q), the number of μ-flats W in $\mathscr{B}(t, \mu, q)$ such that $\dim(W \cap V) = m$ and $\dim(W \cap V \cap U) = m - \varepsilon$ is equal to the number of μ-flats W^* in $T(V, U)$ such that $\dim(W^* \cap V) = m$, $\dim(W^* \cap U) = \mu - 1$, $\dim(W^* \cap V \cap U) = m - \varepsilon$ and $\dim(W^* \cap T(V, U)) = \mu$ for the μ-flat V and the $(t-1)$-flat U. It follows, therefore, from Lemma 3.6 that the number of μ-flats W in $\mathscr{B}(t, \mu, q)$ such that $\dim(W \cap V) = m$ and $\dim(W \cap V \cap U) = m - \varepsilon$ is equal to $\Gamma_{1-\varepsilon,0}(\mu; \mu, t-1, \mu-1, m, \mu-1, m-\varepsilon, t, q)$. Hence, we have the required result from (3.11).

LEMMA 3.8. *Let* V_1 *and* V_2 *be any* μ*-flats in* $\mathscr{B}(t, \mu, q)$ *such that* $\dim(V_1 \cap V_2) = m$ *and* $\dim(V_1 \cap V_2 \cap U) = m - \varepsilon$. *Then, the number of* μ*-flats* W *in* $\mathscr{B}(t, \mu, q)$ *such that*

$$\dim(W \cap V_i) = m_i \quad \text{and} \quad \dim(W \cap V_i \cap U) = m_i - \varepsilon_i \tag{3.14}$$

for $i = 1, 2$ *is equal to*

$$\begin{aligned} \Phi^{(\varepsilon)}_{(\varepsilon_1,\varepsilon_2)}&(\mu; m, m_1, m_2, t, q) \\ &= \textstyle\sum_{r=0}^{1-\varepsilon} \sum_{n=u^*}^{w^*} \sum_{l=0}^{z^*} q^{c^*(r,n,l;\varepsilon)}\phi(m-\varepsilon, n, q) \\ &\quad \times \phi(\mu - m + \varepsilon - 2, m_1 - \varepsilon_1 - n - 1, q) \\ &\quad \times \phi(\mu - m + \varepsilon - 2, m_2 - \varepsilon_2 - n - 1, q) \\ &\quad \times \phi(t - 2\mu + m - 1, \mu + n - m_1 + \varepsilon_1 - m_2 \\ &\quad + \varepsilon_2 - r - l - 2, q) \\ &\quad \times \chi(\mu + n - m + \varepsilon - m_1 + \varepsilon_1 - 1, \mu + n \\ &\quad - m + \varepsilon - m_2 + \varepsilon_2 - 1, l; q) \\ &\quad \times \{(1-\varepsilon_1)(1-\varepsilon_2)q^{t-\mu} - (-1)^{\varepsilon_1}(1-\varepsilon_2)q^{\mu-m_1+\varepsilon_1-1} \\ &\quad - (-1)^{\varepsilon_2}(1-\varepsilon_1)q^{\mu-m_2+\varepsilon_2-1} + (-1)^{\varepsilon_1+\varepsilon_2}(r+\varepsilon)q^{m+l-\varepsilon-n}\} \end{aligned} \tag{3.15}$$

for $\varepsilon, \varepsilon_1, \varepsilon_2 = 0, 1$ *where*

$$\begin{aligned} u^* &= \max\{-1, m - \varepsilon + m_1 - \varepsilon_1 - \mu + 1, m - \varepsilon + m_2 - \varepsilon_2 \\ &\qquad - \mu + 1, m_1 - \varepsilon_1 + m_2 - \varepsilon_2 - \mu + 1\}, \\ w^* &= \min\{m - \varepsilon, m_1 - \varepsilon_1, m_2 - \varepsilon_2\}, \\ z^* &= \min\{\mu + n + \varepsilon + \varepsilon_1 - m - m_1 - 1, \mu + n + \varepsilon + \varepsilon_2 \\ &\qquad - m - m_2 - 1, \mu + n + \varepsilon_1 + \varepsilon_2 - m_1 - m_2 - r - 1\}, \\ c^*(r, n, l; \varepsilon) &= (n + \mu + \varepsilon_1 + \varepsilon_2 - m_1 - m_2 - l - 1) \\ &\qquad \times (n + 2\mu + \varepsilon_1 + \varepsilon_2 - r - m - m_1 - m_2 - l - 1) \\ &\qquad + (m - n - \varepsilon)(l + m_1 + m_2 - 2n - \varepsilon_1 - \varepsilon_2). \end{aligned} \tag{3.16}$$

Note that if $u^* > w^*$ or $z^* < 0$, $\Phi^{(\varepsilon)}_{(\varepsilon_1,\varepsilon_2)}(\mu; m, m_1, m_2, t, q) = 0$.

PROOF. Let W_0 be a $(\mu - 1)$-flat in the $(t - 1)$-flat U such that

$$(3.17) \quad \begin{aligned} &\dim (W_0 \cap V_i) = m_i - \varepsilon_i \quad (i = 1, 2), \qquad \dim (W_0 \cap V_1 \cap V_2) = n, \\ &\dim (W_0 \cap T(V_1, V_2)) = (m_1 - \varepsilon_1 + m_2 - \varepsilon_2 - n) + k \end{aligned}$$

for some integers n and k such that $u^* \leqq u \leqq w^*$ and $0 \leqq k \leqq \mu - 1 - (m_1 - \varepsilon_1 + m_2 - \varepsilon_2 - n)$ and let W be the μ-flat generated by the defining points of W_0 and a point (δ) in PG (t, q) not contained in U. Then, it follows from Lemma 3.1 that a necessary and sufficient condition for the point (δ) that W is a μ-flat satisfying the condition (3.14) is that (δ) is a point $(\notin U)$ in PG (t, q) satisfying the condition: (i) $(\delta) \notin \{T(W_0, V_1) \cup T(W_0, V_2)\}$, (ii) $(\delta) \in \{T(W_0, V_1) - T(W_0, V_2)\}$ or (iii) $(\delta) \in \{T(W_0, V_1) \cap T(W_0, V_2)\}$ according as $(\varepsilon_1, \varepsilon_2) = (0, 0), (1, 0)$ or $(1, 1)$. Since $T(W_0, V_1) \cap U = T(W_0, V_1 \cap U)$ for any flat W_0 in U, we have

$$(3.18) \quad T(W_0, V_1) \cap T(W_0, V_2) \cap U = T(W_0, V_1 \cap U) \cap T(W_0, V_2 \cap U).$$

Since both $V_1 \cap U$ and $V_2 \cap U$ are $(\mu - 1)$-flats, it follows from Lemma 3.3 and (3.18) that

$$(3.19) \quad \begin{aligned} &\dim \{T(W_0, V_1) \cap T(W_0, V_2)\} \\ &\qquad = (m_1 - \varepsilon_1 + m_2 - \varepsilon_2 + m - 2n) + 2k + s \\ &\qquad = \mu - 1 + m - n + k \end{aligned}$$

and

$$(3.20) \quad \begin{aligned} &\dim \{T(W_0, V_1) \cap T(W_0, V_2) \cap U\} \\ &\qquad = (m_1 - \varepsilon_1 + m_2 - \varepsilon_2 + m - \varepsilon - 2n) + 2(k - r) + r + s \\ &\qquad = \mu - 1 + m - \varepsilon - n + k - r \end{aligned}$$

where $s = \mu - 1 - (m_1 - \varepsilon_1 + m_2 - \varepsilon_2 - n + k)$ and r is a nonnegative integer such that

$$(3.21) \quad \dim \{W_0 \cap T(V_1 \cap U, V_2 \cap U)\} = (k - r) + (m_1 - \varepsilon_1 + m_2 - \varepsilon_2 - n).$$

Since U is a $(t - 1)$-flat, $\dim \{T(W_0, V_1) \cap T(W_0, V_2)\} - \dim \{T(W_0, V_1) \cap T(W_0, V_2) \cap U\} \leqq 1$. Therefore, we have $0 \leqq r \leqq 1 - \varepsilon$ (i.e., $r = 0$ in the case $\varepsilon = 1$ and $r = 0$ or 1 in the case $\varepsilon = 0$) from (3.19) and (3.20). Since $\dim T(W_0, V_i) = 2\mu - 1 - (m_i - \varepsilon_i)$ for $i = 1, 2$, in the case $(\varepsilon_1, \varepsilon_2) = (0, 0)$, the number of points (δ) $(\notin U)$ satisfying the condition (i) is equal to $\{\phi(t, 0, q) - \phi(t - 1, 0, q)\} - \{\phi(2\mu - m_1 - 1, 0, q) - \phi(2\mu - m_1 - 2, 0, q)\} - \{\phi(2\mu - m_2 - 1, 0, q) - \phi(2\mu - m_2 - 2, 0, q)\} + \{\phi(\mu - 1 + m - n + l + r, 0, q) - \phi(\mu - 1 + m - \varepsilon - n + l, 0, q)\} = q^t - q^{2\mu - m_1 - 1} - q^{2\mu - m_2 - 1} + q^{\mu + l + m - n - \varepsilon}(q^{\varepsilon + r} - 1)/(q - 1)$ where $l = k - r$. Similarly, in the case $(\varepsilon_1, \varepsilon_2) = (1, 0)$, the number of points (δ) $(\notin U)$ satisfying the condition (ii) is equal to $q^{2\mu - m_1} - q^{\mu + l + m - n - \varepsilon}(q^{\varepsilon + r} - 1)/(q - 1)$ and in the case $(\varepsilon_1, \varepsilon_2) = (1, 1)$, the number of points (δ) $(\notin U)$ satisfying the condition (iii) is equal to $q^{\mu + l + m - n - \varepsilon}(q^{\varepsilon + r} - 1)/(q - 1)$.

On the other hand, each μ-flat W satisfying the condition (3.14) can be generated by the defining points of the $(\mu - 1)$-flat W_0 and any one point (δ) of q^μ points in $W - W_0$. Hence, the number of μ-flats W such that $W \cap U = W_0$ is equal to $\{(1-\varepsilon_1)(1-\varepsilon_2)q^{t-\mu}-(-1)^{\varepsilon_1}(1-\varepsilon_2)q^{\mu-m_1+\varepsilon_1-1}-(-1)^{\varepsilon_2}(1-\varepsilon_1)q^{\mu-m_2+\varepsilon_2-1}+(-1)^{\varepsilon_1+\varepsilon_2}(\varepsilon+r)q^{m+l-\varepsilon-n}\}$ for any $(\mu-1)$-flat W_0 satisfying the conditions (3.17) and (3.21) because $(q^\zeta-1)/(q-1)=\zeta$ for $\zeta=0$ or 1. Therefore, it is sufficient to obtain the number of $(\mu-1)$-flats W_0 in U satisfying the conditions (3.17) and (3.21) in order to obtain the number of μ-flats W satisfying the condion (3.14).

Let $V_i^*=U\cap V_i$ for $i=1,2$. Then, the conditions (3.17) and (3.21) can be also expressed as follows:

$$(3.22)\quad \begin{aligned}&\dim(W_0\cap V_i^*)=m_i-\varepsilon_i\quad(i=1,2)\,,\\&\dim(W_0\cap V_1^*\cap V_2^*)=n\,,\\&\dim(W_0\cap T(V_1^*,V_2^*))=(m_1-\varepsilon_1+m_2-\varepsilon_2-n)+(k-r)\,,\\&\dim(W_0\cap T(V_1,V_2)\cap U)=(m_1-\varepsilon_1+m_2-\varepsilon_2-n)+k\end{aligned}$$

where r is an integer such that $0\leqq r\leqq 1-\varepsilon$. Note that $T(V_1^*,V_2^*)\subset T(V_1,V_2)\cap U$ and $\dim(T(V_1,V_2)\cap U)=\dim T(V_1^*,V_2^*)+(1-\varepsilon)$ because $\dim(T(V_1,V_2)\cap U)=2\mu-m-1$ and $\dim T(V_1^*,V_2^*)=2(\mu-1)-(m-\varepsilon)$.

(i) In the case $\varepsilon=1$, $T(V_1,V_2)\cap U=T(V_1^*,V_2^*)$ and $r=0$. Therefore, it follows from Lemma 3.4 and Lemma 3.6 that the number of $(\mu-1)$-flats W_0 in U satisfying the condition (3.22) is equal to $\Gamma_{l,0}(l+m_1-\varepsilon_1+m_2-\varepsilon_2-n;\mu-1,\mu-1,m-1,m_1-\varepsilon_1,m_2-\varepsilon_2,n,2\mu-m-1,q)\eta(\mu-1;2\mu-m-1,l+m_1-\varepsilon_1+m_2-\varepsilon_2-n,t-1,q)$ where $l=k$.

(ii) In the case $\varepsilon=0$, $\dim(T(V_1,V_2)\cap U)=\dim T(V_1^*,V_2^*)+1$ and $r=0$ or 1. Therefore, it follows from Lemma 3.4 and Lemma 3.6 that the number of $(\mu-1)$-flats W_0 in U satisfying the condition (3.22) is equal to $\Gamma_{l,r}(r+l+m_1-\varepsilon_1+m_2-\varepsilon_2-n;\mu-1,\mu-1,m,m_1-\varepsilon_1,m_2'-\varepsilon_2,n,2\mu-m-1,q)\eta(\mu-1;2\mu-m-1,l+m_1-\varepsilon_1+m_2-\varepsilon_2-n,t-1,q)$ for $r=0,1$ where $l=k-r$. Since those results hold for any integers n and l such that $u^*\leqq n\leqq w^*$ and $0\leqq l\leqq z^*$, the number of μ-flats W satisfying the condition (3.14) is equal to

$$\begin{aligned}&\sum_{r=0}^{1-\varepsilon}\sum_{n=u^*}^{w^*}\sum_{l=0}^{z^*}\Gamma_{l,r}(r+l+m_1-\varepsilon_1+m_2-\varepsilon_2-n;\mu-1,\mu-1,\\&\qquad m-\varepsilon,m_1-\varepsilon_1,m_2-\varepsilon_2,n,2\mu-m-1,q)\\&\qquad\quad\times\eta(\mu-1;2\mu-m-1,l+m_1-\varepsilon_1+m_2-\varepsilon_2-n,t-1,q)\\&\qquad\quad\times\{(1-\varepsilon_1)(1-\varepsilon_2)q^{t-\mu}-(-1)^{\varepsilon_1}(1-\varepsilon_2)q^{\mu-m_1+\varepsilon_1-1}\\&\qquad\quad-(-1)^{\varepsilon_2}(1-\varepsilon_1)q^{\mu-m_2+\varepsilon_2-1}+(-1)^{\varepsilon_1+\varepsilon_2}(\varepsilon+r)q^{m+l-\varepsilon-n}\}\,.\end{aligned}$$

Therefore, we have the required result from (3.9) and (3.11).

Proof of Theorem 3.1. From Definition 3.1, Lemma 3.7 and Lemma 3.8, it is easy to see that (i) $n_{(i,\varepsilon)}=N_\varepsilon(\mu;\mu-i,t,q)$ and $p^{(i,\varepsilon)}_{(j,\zeta)(k,\xi)}=\Phi^{(\varepsilon)}_{(\zeta,\xi)}(\mu;\mu-i,\mu-j,\mu-k,t,q)$ for $\varepsilon,\zeta,\xi=0,1$, $i=1,2,\cdots,\gamma_\varepsilon$, $j=1,2,\cdots,\gamma_\zeta$ and $k=1,2,\cdots,\gamma_\xi$ where $\gamma_0=\min\{\mu+1,t-\mu\}$ and $\gamma_1=\min\{\mu,t-\mu\}$ and (ii) $n_{(i,0)}=N_0(\mu;\mu-i,t,q)>0$ and $n_{(i,1)}=N_1(\mu;\mu-i,t,q)>0$ for any integer i such that $1\leqq i\leqq\min\{\mu,t-\mu\}$. Since $n_{(\mu+1,0)}=N_0(\mu;-1,t,q)=q^{\mu(\mu+1)}\phi(t-\mu-1,\mu-1,q)(q^{t-2\mu}-1)$ in the special case $i=\mu+1$ and $\varepsilon=0$, $n_{(\mu+1,0)}$ is a positive integer or zero according as $t > 2\mu$ or not. Hence, the association defined by Definition 3.1 is an association scheme with $m = \min\{2\mu + 1, 2(t - \mu)\}$ associate classes and parameters $n_{(i,\varepsilon)}$ and $p^{(i,\varepsilon)}_{(j,\zeta)(k,\xi)}$ given by (3.1) and (3.2), respectively. This completes the proof.

4. The dual of the BIB design AG $(t, q): \mu$. It is well known [1] that by identifying the points of AG (t, q) with the v^* treatments and identifying the μ-flats $(0 < \mu < t)$ of AG (t, q) with the b^* blocks, a BIB design, denoted by AG (t, q): μ, with parameters

$$(4.1) \qquad \begin{aligned} &v^* = q^t, \qquad b^* = \phi(t, \mu, q) - \phi(t - 1, \mu, q), \qquad k^* = q^\mu, \\ &r^* = \phi(t - 1, \mu - 1, q) \qquad \text{and} \qquad \lambda^* = \phi(t - 2, \mu - 2, q) \end{aligned}$$

is obtained from AG (t, q) where $\phi(t, \mu, q)$ is given by (2.1).

THEOREM 4.1. *The dual of a* BIB *design* AG $(t, q): \mu$ *is an affine geometrical type* PBIB *design with* $m = \min\{2\mu + 1, 2(t - \mu)\}$ *associate classes and parameters*

$$(4.2) \qquad \begin{aligned} &v = \phi(t, \mu, q) - \phi(t - 1, \mu, q), \qquad b = q^t, \qquad r = q^\mu, \\ &k = \phi(t - 1, \mu - 1, q), \qquad \lambda_{(i,1)} = q^{\mu - i} \quad (i = 1, 2, \cdots, \gamma_1), \\ &\lambda_{(1,0)} = \lambda_{(2,0)} = \cdots = \lambda_{(\gamma_0,0)} = 0 \end{aligned}$$

and $n_{(i,\varepsilon)}$ *and* $p^{(i,\varepsilon)}_{(j,\zeta)(k,\xi)}$ *given by* (3.1) *and* (3.2), *respectively, where* $\gamma_0 = \min\{\mu + 1, t - \mu\}$ *and* $\gamma_1 = \min\{\mu, t - \mu\}$.

PROOF. It is obvious that parameters v, b, r and k are given by (4.2). Let V_α^* and V_β^* be any two μ-flats in AG (t, q) which are (i, ε)th associates. Then, the number, $\lambda_{(i,\varepsilon)}$, of points in AG (t, q) contained in $V_\alpha^* \cap V_\beta^*$ is equal to $\phi(\mu - i, 0, q) - \phi(\mu - i - \varepsilon, 0, q) = \varepsilon q^{\mu - i}$ for $\varepsilon = 0, 1$ and $i = 1, 2, \cdots, \gamma_\varepsilon$. Therefore, we have the required result from Definition 3.1 and Theorem 3.1.

In the special case $\mu = t - 1$, we have the

COROLLARY 4.1. *The dual of a* BIB *design* AG $(t, q): t - 1$ *is a (semi-regular) group divisible type* PBIB *design with parameters*

$$(4.3) \qquad \begin{aligned} &v = (q^{t+1} - q)/(q - 1), \qquad b = q^t, \qquad r = q^{t-1}, \\ &k = (q^t - 1)/(q - 1), \qquad \lambda_{(1,1)} = q^{t-2}, \qquad \lambda_{(1,0)} = 0, \\ &n_{(1,1)} = (q^{t+1} - q^2)/(q - 1), \qquad n_{(1,0)} = q - 1, \qquad p^{(1,1)}_{(1,1)(1,0)} = q - 1 \\ &\textit{and} \qquad p^{(1,0)}_{(1,1)(1,0)} = 0. \end{aligned}$$

In the special case $\mu = 1$, we have the

COROLLARY 4.2. *For any integer* $t \geqq 3$, *the dual of a* BIB *design* AG $(t, q): 1$ *is an affine geometrical type* PBIB *design with three associate classes and parameters*

$$(4.4) \qquad \begin{aligned} &v = q^{t-1}(q^t - 1)/(q - 1), \qquad b = q^t, \qquad r = q, \\ &k = (q^t - 1)/(q - 1), \qquad \lambda_1 = 1, \qquad \lambda_2 = 0, \qquad \lambda_3 = 0, \\ &n_1 = (q^{t+1} - q^2)/(q - 1), \qquad n_2 = q^{t-1} - 1, \\ &n_3 = (q^{t+1} - q^2)(q^{t-2} - 1)/(q - 1), \end{aligned}$$

$$\|p^1_{ij}\| = \begin{bmatrix} \frac{(q^t+q^3-3q^2+q)}{(q-1)} & (q-1) & (q^t-q^2) \\ & 0 & (q^{t-1}-q) \\ \text{(sym.)} & & \frac{(q^{t-1}-q)(q^t-q^2-q+1)}{(q-1)} \end{bmatrix},$$

$$(4.5)\quad \|p^2_{ij}\| = \begin{bmatrix} q^2 & 0 & (q^{t+1}-q^3)/(q-1) \\ & (q^{t-1}-2) & 0 \\ \text{(sym.)} & & (q^t-q^2)(q^{t-1}-q-1)/(q-1) \end{bmatrix},$$

$$\|p^3_{ij}\| = \begin{bmatrix} q^2 & q & (q^{t+1}-q^3-q^2+q)/(q-1) \\ & 0 & (q^{t-1}-q-1) \\ \text{(sym.)} & & \frac{(q^{2t-1}-2q^{t+1}-2q^t+q^{t-1}+q^3+3q^2-2q)}{(q-1)} \end{bmatrix}$$

where the numbers 1, 2 *and* 3 *represent* (1, 1), (1, 0) *and* (2, 0), *respectively.*

In Section 5, it will be shown that the number of the associate classes of this dual design can be reduced from three to two.

5. Reduction of the number of the associate classes. Since $\lambda_{(1,0)} = \lambda_{(2,0)} = \cdots = \lambda_{(\gamma_0,0)} = 0$, it seems that the number of the associate classes of this dual design can be reduced to associate classes less than m where $m = \min\{2\mu+1, 2(t-\mu)\}$. In this section, we shall show that in the case $\mu = 1$ and $t \geqq 3$, the number of the associate classes of this dual design can be reduced from three to two but it is not reducible except for the above case.

Among $v = \phi(t, 1, q) - \phi(t-1, 0, q)$ 1-flats V_α^* ($\alpha = 1, 2, \cdots, v$) in AG (t, q), we define a relation of association, called a redµced affine geometrical (RAG) association scheme, as follows:

DEFINITION 5.1. Two 1-flats V_α^* and V_β^* ($\alpha \neq \beta$) in AG (t, q) are said to be 1st associates or 2nd associates according as $V_\alpha^* \cap V_\beta^*$ is a 0-flat or a (-1)-flat.

Note that two 1-flats V_α^* and V_β^* are 1st associates if V_α^* and V_β^* are (1, 1)th associates by Definition 3.1 but two 1-flats V_α^* and V_β^* are 2nd associates if V_α^* and V_β^* are (1, 0)th or (2, 0)th associates by Definition 3.1.

THEOREM 5.1. *The association defined above is an association scheme with two associate classes and parameters*

$$(5.1)\quad \tilde{n}_1 = (q^{t+1}-q^2)/(q-1), \qquad \tilde{n}_2 = (q^{t-1}-1)(q^t-q^2+q-1)/(q-1),$$

$$(5.2)\quad \|\tilde{p}^1_{ij}\| = \begin{bmatrix} \frac{(q^t+q^3-3q^2+q)}{(q-1)} & (q^t-q^2+q-1) \\ \text{(sym.)} & \frac{(q^{2t-1}-2q^{t+1}+q^t-q^{t-1}+q^3-q^2+q)}{(q-1)} \end{bmatrix},$$

$$\|\tilde{p}^2_{ij}\| = \begin{bmatrix} q^2 & (q^{t+1}-q^3)/(q-1) \\ \text{(sym.)} & (q^{2t-1}-2q^{t+1}-q^{t-1}+q^3+q^2-2q+2)/(q-1) \end{bmatrix}$$

and the dual of the BIB *design* AG $(t, q): 1$ *is a* RAG *tyqe* PBIB *design with two associate classes and parameters*

$$(5.3) \qquad \begin{aligned} &v = \phi(t, 1, q) - \phi(t-1, 1, q)\,, \qquad b = q^t\,, \qquad r = q\,, \\ &k = (q^t - 1)/(q-1)\,, \qquad \tilde{\lambda}_1 = 1\,, \qquad \tilde{\lambda}_2 = 0 \end{aligned}$$

and $\tilde{n}_i$, $\tilde{p}^i_{jk}$ $(i, j, k = 1, 2)$ *given by* (5.1) *and* (5.2), *respectively.*

PROOF. From (4.5) in Corollary 4.2, it is easy to see that

$$p^2_{11} = p^3_{11}\,, \qquad \sum_{j=2}^3 p^2_{1j} = \sum_{j=2}^3 p^3_{1j} \qquad \text{and}$$
$$\sum_{i=2}^3 \sum_{j=2}^3 p^2_{ij} = \sum_{i=2}^3 \sum_{j=2}^3 p^3_{ij}\,.$$

This implies that in the case $\mu = 1$ and $t \geqq 3$, the AG association scheme with three associate classes can be reduced to the RAG association scheme with two associate classes by combining 2nd associate with 3rd associate. Since $\lambda_2 = \lambda_3$, we have the required results from Corollary 4.2,

REMARK. This result coincides with Shrikhande's result [9]. Because the dual of the BIB design AG $(t, q): 1$ with parameter $\lambda = 1$ is a PBIB design with two associate classes from Shrikhande's result.

THEOREM 5.2. *If* $\mu \geqq 2$, *the number of the associate classes of a* AG *type* PBIB *design,* $D^*(t, \mu, q)$, *with* $m = \min\{2\mu + 1, 2(t-\mu)\}$ *associate classes and parameters* $v, b, r, k, \lambda_i, n_{(i,\varepsilon)}, p^{(i,\varepsilon)}_{(j,\zeta)(k,\xi)}$ *given by* (4.2), (3.1) *and* (3.2) *cannot be reduced to a number less than* m.

PROOF. Since $\lambda_{(1,1)} > \lambda_{(2,1)} > \cdots > \lambda_{(\gamma_1,1)}$, we cannot combine $(i, 1)$th associate with $(j, 1)$th associate for any distinct integers i and j such that $1 \leqq i, j \leqq \gamma_1$ where $\gamma_1 = \min\{\mu, t-\mu\}$. Similarly, we cannot combine $(i, 1)$th associate with $(k, 0)$th associate for any integers i and k such that $1 \leqq i \leqq \gamma_1$ and $1 \leqq k \leqq \gamma_0$, where $\gamma_0 = \min\{\mu+1, t-\mu\}$, because $\lambda_{(i,1)} \neq \lambda_{(k,0)}$. Hence, if the number of the associate classes of the design $D^*(t, \mu, q)$ can be reduced, there must exist at least one pair $((i, 0), (j, 0))$ $(i \neq j)$ such that we can combine $(i, 0)$th associate with $(j, 0)$th associate for some integers i and j. In order to prove Theorem 5.2, it is, therefore, sufficient to show that we cannot combine $(i, 0)$th associate with $(j, 0)$th associate, i.e., there exists at least one integer l such that $p^{(i,0)}_{(1,1)(l,1)} \neq p^{(j,0)}_{(1,1)(l,1)}$, for any integers i and j such that $1 \leqq i < j \leqq \gamma_0$.

From (3.2) and (3.3), it is easy to see that $p^{(i,0)}_{(1,1)(l,1)} = 0$ for $l = 2, 3, \cdots, \gamma_1$ and $i = 1, 2, \cdots, l-1$ because $u > w$. On the other hand, $p^{(l,0)}_{(1,1)(l,1)} = q^{2l}\phi(\mu - l, \mu - l - 1, q)\phi(t - \mu - l - 1, -1, q) > 0$ for $l = 2, 3, \cdots, \gamma_1$. Hence, there exists at least one integer l $(l = j)$ such that $p^{(i,0)}_{(1,1)(l,1)} \neq p^{(j,0)}_{(1,1)(l,1)}$ for any integers i and j such that $1 \leqq i < j \leqq \gamma_1$. If $\mu < t \leqq 2\mu$, $\gamma_1 = \gamma_0 = t - \mu$ and if $t > 2\mu$, $\gamma_1 = \mu$ and $\gamma_0 = \mu + 1$. It is, therefore, sufficient to show that in the case $t > 2\mu$, there exists at least one integer l $(2 \leqq l \leqq \mu)$ such that $p^{(i,0)}_{(1,1)(l,1)} \neq p^{(\mu+1,0)}_{(1,1)(l,1)}$ for any integer i such that $1 \leqq i \leqq \mu$. Since $p^{(\mu,0)}_{(1,1)(\mu,1)} = q^{2\mu}$, $p^{(\mu+1,0)}_{(1,1)(\mu,1)} = q^{\mu+1}(q^\mu - 1)/(q-1)$ and $p^{(i,0)}_{(1,1)(\mu,1)} = 0$ for $i = 1, 2, \cdots, \mu - 1$, we have the required result.

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