The Annals of Statistics
1975, Vol. 3, No. 3, 581–598

TERMINATION, MOMENTS AND EXPONENTIAL BOUNDEDNESS OF THE STOPPING RULE FOR CERTAIN INVARIANT SEQUENTIAL PROBABILITY RATIO TESTS[1]

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It is well known that Wald's SPRT terminates with probability one and in fact the stopping time is exponentially bounded for every distribution P except in the trivial case where the log likelihood ratio vanishes with probability one. The results in the literature for invariant SPRT's, however, have been considerably less complete. In all the parametric problems studied, moment conditions of certain random variables have been assumed to prove termination, and the finiteness of their moment generating function has also been assumed to prove the exponential boundedness of the stopping rule. In this paper, we try to remove or weaken these conditions for certain invariant SPRT's. In particular, we show that like the Wald SPRT, the sequential t- and F-tests always terminate with probability one for any distribution P except in trivial cases. However, the stopping rules may fail to be exponentially bounded, and obstructive distributions are also exhibited. Sufficient conditions for exponential boundedness and finiteness of moments of the stopping rule are studied, and asymptotic expressions for the moments of the stopping rule are also given.

1. Introduction. Let $Z_1, Z_2, \cdots$ be independent, identically distributed (i.i.d.) random variables with a common distribution P. The joint distribution of the Z's will also be denoted by P. To test sequentially the simple hypothesis $H_0\colon P = P_0$ versus the simple alternative $H_1\colon P = P_1$, Wald's sequential probability ratio test (SPRT) stops sampling as soon as $R_n \notin (A, B)$, where R_n denotes the likelihood ratio $\prod_{i=1}^n p_1(Z_i)/p_0(Z_i)$, p_i being the density of P_i with respect to some common dominating measure μ $(i = 0, 1)$. If we put $Y_i = \log[p_1(Z_i)/p_0(Z_i)]$ and $L_n = \sum_1^n Y_i$, then the stopping rule for the SPRT is simply the first time N when the random walk $\{L_n, n \geqq 1\}$ exits from the interval (a, b), where $a = \log A$ and $b = \log B$. In [20], Wald proved that $P[N < \infty] = 1$ for every P except if $P[Y_1 = 0] = 1$, and under the same assumption $P[Y_1 = 0] < 1$, Stein [17] showed that

$$(1) \qquad \exists\, c > 0 \text{ and } 0 < \rho < 1 \text{ for which } P[N > n] < c\rho^n, \quad n = 1, 2, \cdots.$$

The property $P[N < \infty] = 1$ is commonly referred to as "termination with probability one", while the property (1) of a stopping rule N is referred to as "exponential boundedness" by Berk [1] and Wijsman [21], [22].

Received June 1973; revised September 1974.

[1] Research supported by the Public Health Service under Grant GM-16895-03.

AMS 1970 *subject classifications.* Primary 6245; Secondary 6225, 6030.

Key words and phrases. Invariant sequential probability ratio tests, termination with probability one, exponentially bounded stopping rule, obstructive distributions, invariantly sufficient sequence, likelihood ratio, asymptotic behavior of moments.

If the hypotheses to be tested are composite, it may be possible to reduce the the composite hypotheses to simple ones by using the principle of invariance. The resulting test is called an invariant SPRT (cf. [5]). Here we again stop as soon as R_n exits an interval (A, B), where R_n is the likelihood ratio of the maximal invariant (with respect to a given group of invariance transformations) at stage n. If we let $L_n = \log R_n$ as before, then unlike the case of Wald's SPRT, the sequence $(L_n, n \geqq 1)$ does not have to be a random walk since we take R_n to be the likelihood ratio of the maximal invariant as stage n instead of the likelihood ratio for the original data sequence $Z_1, \cdots, Z_n$. The questions of termination with probability one and of exponential boundedness of the stopping rule N for an invariant SPRT turn out to be much harder to answer than in the case of Wald's SPRT. In fact, Stein's result for the Wald SPRT does not always carry over to an invariant SPRT. Relative to a given invariant SPRT, a distribution P is called *obstructive* if (1) is not satisfied. This terminology is due to Wijsman who exhibited certain obstructive distributions for several invariant SPRT's, including the sequential t-test (cf. [22], [23], [24]).

Wijsman's proof of the termination and exponential boundedness of the stopping rules for a wide class of invariant SPRT's relies on the demonstration of the following approximation for the log likelihood ratio L_n of the maximal invariant at stage n:

$$\text{For each } \xi \in R^k,\ \exists c > 0 \text{ together with a neighborhood } V \text{ of } \xi \text{ and a continuous function } \Phi \text{ on } V \text{ such that } |L_n - n\Phi(\bar{X}_n)| < c \text{ if } \bar{X}_n \in V, \quad n = 1, 2, \cdots. \tag{2}$$

The random variable $\bar{X}_n$ in (2) denotes $n^{-1} \sum_{i=1}^n X_i$, where X_i is some vector-valued function of Z_i, i.e., for some function s from the range of Z_1 into R^k, we have

$$X_i = s(Z_i), \qquad i = 1, 2, \cdots. \tag{3}$$

The property (3) implies that $X_1, X_2, \cdots$ are i.i.d. random vectors and so under the assumption that $EX_1 = \xi$ exists and is finite, the strong law of large numbers yields that $P[\lim_{n\to\infty} \bar{X}_n = \xi] = 1$.

For the sequential t-test, Wijsman's results (cf. [23]) show that $P[N < \infty] = 1$ if $\infty > EZ_1^2 > 0$ and if P is not one of the family of two-point distributions defined by:

$$\begin{aligned} P[Z_1 = (\sigma^2 + \zeta^2)^{\frac{1}{2}}\zeta^{-1}\{(\sigma^2 + \zeta^2)^{\frac{1}{2}} \pm \sigma\}] \\ = \tfrac{1}{2}[1 \mp \sigma(\sigma^2 + \zeta^2)^{-\frac{1}{2}}], \qquad \sigma > 0, \zeta \neq 0. \end{aligned} \tag{4}$$

Furthermore, N is exponentially bounded if $E\exp(tZ_1^2) < \infty$ for all small t and if P does not satisfy (4). As the function s in (3) reduces to $X_i = s(Z_i) = (Z_1, Z_i^2)'$ in the present case, it is conceivable why Wijsman's argument requires the assumption of the finiteness of EZ_1^2 for termination with probability one and the assumption of finite mgf of Z_1^2 for the exponential boundedness of the stopping rule. Berk [1], by a different method, obtained essentially the same conclusions, and Wijsman in [24] has also shown that the two-point distributions defined by (4) are indeed obstructive (see Section 3 below). The question of whether $P[N < \infty]$ is always one if $P[Z_1 = 0] < 1$ still remains open, and so does the question of whether there are other obstructive distributions than those defined by (4). In this paper, we shall study both of these problems.

As mentioned above, Wijsman's approach is to approximate the log likelihood ratio L_n by $n\Phi(\bar{X}_n)$ in (3), so that we still have cumulative averages of i.i.d. random vectors $X_1, X_2, \cdots$ to work with. The approach we shall take is that instead of viewing $\Phi(\bar{X}_n)$ as a function of the cumulative average $\bar{X}_n$, we shall re-write $\Phi(\bar{X}_n)$ as a function of T_n, say $\Phi(\bar{X}_n) = \Psi(T_n)$, where T_n is a suitably chosen invariantly sufficient sequence. To prove termination with probability one, we need only show that

$$1 = P[\limsup_{n\to\infty} |n\Psi(T_n)| = \infty] = P[\limsup_{n\to\infty} |L_n| = \infty] . \tag{5}$$

We shall show how this can be done for the sequential t-test in Section 2. In this case, we can take $T_n = n^{-1}\sum_1^n Z_i/(n^{-1}\sum_1^n Z_i^2)^{\frac{1}{2}}$. Instead of handling the two random walks $\sum_1^n Z_i$, $\sum_1^n Z_i^2$ separately, we here work directly with the sequence T_n which, though not a random walk, has some nice limiting behavior. For example, it will be shown in Section 2 that if $P[Z_1 = 0] < 1$, then

$$P\left[\limsup_{n\to\infty} |\textstyle\sum_1^n Z_i| \Big/ \left(\frac{1}{n}\sum_1^n Z_i^2\right)^{\frac{1}{2}} = \infty\right] = 1 . \tag{6}$$

Other interesting properties of the limiting behavior of the sequence T_n are also given in Section 2.

In Section 3, again by directly analyzing the sequence T_n, we shall exhibit other obstructive distributions for the sequential t-test. Conditions for the exponential boundedness of the stopping rule N are also studied. In Section 4, conditions for the finiteness of moments of the stopping rule are given, and we also find asymptotic expressions for these moments and generalize a theorem of Sacks [14]. Though we specialize for definiteness our discussion in Sections 2, 3, 4 to the sequential t-test, the ideas used in the analysis are of a fairly general nature and can be used to handle other invariant SPRT's. In Section 5, we extend our results to other classical invariant SPRT's such as the sequential F and the sequential T^2 tests.

2. Limiting behavior of the invariantly sufficient sequence T_n and termination with probability one for the sequential t-test. Let $Z_1, Z_2, \cdots$ be i.i.d. random variables with a common distribution P. We want to test the null hypothesis H_0 that P is $N(\zeta, \sigma^2)$ with $\zeta/\sigma = \gamma_0$ versus the alternative H_1 that P is $N(\zeta, \sigma^2)$ with $\zeta/\sigma = \gamma_1$, where γ_0 and γ_1 are two distinct real numbers. The problem is invariant under the transformations $Z_i \to cZ_i$, $c > 0$, and the group G of invariance transformations is isomorphic to the multiplicative group of positive reals. At stage n, a maximal invariant under the group G is $(x_1/|x_n|, \cdots, x_n/|x_n|)$

and the likelihood ratio R_n of the maximal invariant can be written as $R_n = U_n(\gamma_1)/U_n(\gamma_0)$, where

$$U_n(\gamma) = \int_0^\infty u^{-1} \exp[nf(u, T_n; \gamma)]\, du\,, \tag{7}$$

in which $T_n = (n^{-1} \sum_{i=1}^n Z_i)/\{n^{-1} \sum_{i=1}^n Z_i^2\}^{\frac{1}{2}}$ is an invariantly sufficient sequence (we define T_n to be 0 when the denominator vanishes), and

$$f(u, y; \gamma) = -\tfrac{1}{2}u^2 + \gamma y u + \log u - \tfrac{1}{2}\gamma^2 \tag{8}$$

(cf. [23], page 1866). The sequential t-test stops sampling at stage $N = \inf\{n : R_n \notin (A, B)\}$ and accepts H_0 or H_1 according as $R_n \leqq A$ or $R_n \geqq B$, with $0 < A < 1 < B$. David and Kruskal [4] have shown that $P[N < \infty] = 1$ for P in the normal model. As mentioned in Section 1, Berk and Wijsman have shown $P[N < \infty] = 1$ for P outside the parametric model such that $0 < EZ_1^2 < \infty$ and P is not one of the family of two-point distributions defined by (4). In this section, we shall prove that $P[N < \infty] = 1$ for any distribution P such that $P[Z_1 = 0] < 1$. To do this, we shall show that (5) holds. We first study the limiting behavior of the sequence T_n in the following lemmas.

LEMMA 1. *Suppose $Z_1, Z_2, \cdots$ are* i.i.d. *random variables such that $P[Z_1 = 0] < 1$. Let $S_n = Z_1 + \cdots + Z_n$, $v_n^2 = n^{-1} \sum_{i=1}^n Z_i^2$. Then*

$$\limsup_{n\to\infty} |S_n|/v_n = \infty \quad \text{a.s.} \tag{9}$$

PROOF. If $EZ_1^2 < \infty$, then by the strong law of large numbers, $v_n^2 \to EZ_1^2$ a.s. Furthermore, since $P[Z_1 = 0] < 1$, it is well known that $\limsup_{n\to\infty} |S_n| = \infty$ a.s., and so (9) holds. Now assume that $E|Z_1| < \infty$ and $EZ_1^2 = \infty$. Then by the converse to the law of the iterated logarithm (cf. [19]),

$$\limsup_{n\to\infty} |S_n|/(n \log\log n)^{\frac{1}{2}} = \infty \quad \text{a.s.}$$

Also it follows from Lemma 2 below together with Kronecker's lemma that $\lim_{n\to\infty} n^{-2} \sum_{i=1}^n Z_i^2 = 0$ a.s., and so $\lim_{n\to\infty} n^{-\frac{1}{2}} v_n = 0$ a.s. Therefore (9) holds. Now consider the case $E|Z_1| = \infty$. By a theorem of Kesten [9], we have

$$\limsup_{n\to\infty} |Z_n|/(|Z_1| + \cdots + |Z_{n-1}|) = \infty \quad \text{a.s.} \tag{10}$$

From (10), it follows that

$$\limsup_{n\to\infty} |S_n|/\{\textstyle\sum_{i=1}^n Z_i^2\}^{\frac{1}{2}} \geqq 1 \quad \text{a.s.}, \tag{11}$$

and so (9) holds.

LEMMA 2. *Suppose $Z_1, Z_2, \cdots$ are* i.i.d. *random variables such that $E|Z_1| < \infty$. Then the series $\sum_{n=1}^\infty n^{-\alpha}|Z_n|^\alpha$ converges* a.s. *for any $\alpha > 1$.*

PROOF. Obviously $\sum P[n^{-\alpha}|Z_n|^\alpha \geqq 1] = \sum P[|Z_n| \geqq n] < \infty$. Also for any $\beta > 1$,

$$\begin{aligned} \sum_{n=1}^\infty n^{-\beta} E|Z_n|^\beta I_{[|Z_n| \leqq n]} &\leqq \sum_{k=1}^\infty k^\beta P[k-1 < |Z_1| \leqq k] \sum_{n=k}^\infty n^{-\beta} \\ &\leqq 1 + (\beta - 1)^{-1} \sum_{k=1}^\infty k P[k-1 < |Z_1| \leqq k]\,. \end{aligned} \tag{12}$$

Hence $\sum E(n^{-\alpha}|Z_n|^\alpha I_{[n^{-\alpha}|Z_n|^\alpha \leq 1]}) < \infty$ and $\sum \operatorname{Var}(n^{-\alpha}|Z_n|^\alpha I_{[n^{-\alpha}|Z_n|^\alpha \leq 1]}) < \infty$. The desired conclusion then follows from Kolmogorov's three-series theorem (cf. [12], page 237).

LEMMA. 3. *Suppose $Z_1, Z_2, \cdots$ are* i.i.d. *random variables with $E|Z_1| = \infty$. Let $\bar{Z}_n = n^{-1}\sum_{i=1}^n Z_i$, $v_n^2 = n^{-1}\sum_{i=1}^n Z_i^2$. Then*

$$\liminf_{n\to\infty} |\bar{Z}_n|/v_n = 0 \quad \text{a.s.} \tag{13}$$

PROOF. If follows from (10) that with probability one, 1 is an accumulation point of the sequence $(|S_n|/\{\sum_{i=1}^n Z_i^2\}^{\frac{1}{2}}, n \geqq 1)$. Therefore

$$\liminf_{n\to\infty} |\bar{Z}_n|/v_n = \liminf_{n\to\infty} n^{-\frac{1}{2}}|S_n|/\{\textstyle\sum_{i=1}^n Z_i^2\}^{\frac{1}{2}} = 0 \quad \text{a.s.}$$

LEMMA 4. *With the same assumptions and notation as in Lemma* 1, *for any real number λ,*

$$\limsup_{n\to\infty} |S_n - \lambda n v_n| = \infty \quad \text{a.s.} \tag{14}$$

except in the case where Z_1 is degenerate and $\lambda = \operatorname{sgn}(EZ_1)$.

PROOF. Since (14) is trivial in the case $\lambda = 0$, we shall assume below that $\lambda \neq 0$. If $E|Z_1| = \infty$, it is easy to see from Lemma 3 that (14) holds. If $E|Z_1| < \infty$ and $EZ_1^2 = \infty$, then $v_n \to \infty$ a.s. while $S_n/n \to EZ_1$ a.s., and so (14) is obvious.

We now assume $EZ_1^2 < \infty$. Let $EZ_1 = \zeta$, $EZ_1^2 = \tau^2$, $\operatorname{Var} Z_1 = \sigma^2 = \tau^2 - \zeta^2$. Since $S_n/n \to \zeta$ a.s. and $v_n \to \tau$ a.s., (14) is obvious in the case $\zeta \neq \lambda\tau$. Now consider the case $\zeta = \lambda\tau$. If $EZ_1^4 = \infty$, then by the law of the iterated logarithm and its converse,

$$\limsup_{n\to\infty} |S_n - n\zeta|/(2n \log\log n)^{\frac{1}{2}} = \sigma \quad \text{a.s.} \tag{15}$$

$$\limsup_{n\to\infty} |\textstyle\sum_1^n Z_i^2 - n\tau^2|/(n \log\log n)^{\frac{1}{2}} = \infty \quad \text{a.s.} \tag{16}$$

Since $\lambda \neq 0$ and $\zeta = \lambda\tau$, it is easy to see from (15) and (16) that (14) holds.

We now consider the case $\zeta = \lambda\tau$ and $EZ_1^4 < \infty$. By Taylor's theorem, we can write $x^{\frac{1}{2}} - \tau = (x - \tau^2)((1/2\tau) + u(x))$, $x > 0$, where $\lim_{x\to\tau^2} u(x) = 0$. Therefore

$$v_n = \left(\frac{1}{n}\textstyle\sum_{i=1}^n Z_i^2\right)^{\frac{1}{2}} = \left(\frac{1}{n}\textstyle\sum_{i=1}^n Z_i^2\right)\Big/2\tau + \tau/2 + \left(\frac{1}{n}\textstyle\sum_1^n Z_i^2 - \tau^2\right)u_n\,,$$

where $u_n = u(n^{-1}\sum_1^n Z_i^2) \to 0$ a.s. as $n \to \infty$. Hence

$$\begin{aligned} S_n - \lambda n v_n &= S_n - (\lambda \textstyle\sum_{i=1}^n Z_i^2/2\tau) - n\zeta/2 - (\sum_{i=1}^n Z_i^2 - n\tau^2)\lambda u_n \\ &= (\textstyle\sum_{i=1}^n X_i - nEX_1) - (\sum_{i=1}^n Z_i^2 - n\tau^2)\lambda u_n\,, \end{aligned} \tag{17}$$

where $X_i = Z_i - (\lambda/2\tau)Z_i^2$ and so $EX_1 = \zeta/2$. In the case $P[X_1 = EX_1] < 1$, we have $\operatorname{Var} X_1 > 0$ and it follows from the law of the iterated logarithm that

$$\limsup_{n\to\infty} |\textstyle\sum_{i=1}^n X_i - nEX_1|/(2n\log\log n)^{\frac{1}{2}} = (\operatorname{Var} X_1)^{\frac{1}{2}} \quad \text{a.s.} \tag{18}$$

On the other hand, since $EZ_1^4 < \infty$ and $\lim_{n\to\infty} u_n = 0$ a.s., we have

$$\lim_{n\to\infty} |\textstyle\sum_{i=1}^n Z_i^2 - n\tau^2||u_n|/(n \log\log n)^{\frac{1}{2}} = 0 \quad \text{a.s.} \tag{19}$$

From (17), (18) and (19), we obtain (14).

It remains to consider the case $P[X_1 = EX_1] = 1$. First assume that Z_1 is nondegenerate so that $\sigma > 0$. Since $2EX_1 = \zeta = \lambda\tau$ and $\tau^2 = \zeta^2 + \sigma^2$, this case is equivalent to the situation where Z_1 has the two-point distribution defined by (4). To prove (14) holds in this case, we can without loss of generality assume that $\tau^2 = \zeta^2 + \sigma^2 = 1$, so that (4) reduces to

$$P[Z_1 = \zeta^{-1}(1 \pm \sigma)] = \tfrac{1}{2}(1 \mp \sigma) .$$

Let $p = \frac{1}{2}(1 + \sigma)$, $q = \frac{1}{2}(1 - \sigma)$, $p_n = n^{-1} \sum_{i=1}^n I_{[Z_i = \zeta^{-1}(1-\sigma)]}$, $q_n = n^{-1} \sum_{i=1}^n I_{[Z_i=\zeta^{-1}(1+\sigma)]}$. For $n > 3$, define ε_n by $p_n = p + \varepsilon_n\{(2pq \log\log n)/n\}^{\frac{1}{2}}$. Obviously, this implies that $q_n = q - \varepsilon_n\{(2pq \log\log n)/n\}^{\frac{1}{2}}$. We note that

$$\begin{aligned} S_n &= np_n\zeta^{-1}(1 - \sigma) + nq_n\zeta^{-1}(1 + \sigma) \\ &= n\zeta - \sigma\varepsilon_n(2n \log\log n)^{\frac{1}{2}} . \end{aligned} \tag{20}$$

The last relation in (20) follows from the fact that $1 - \sigma^2 = \zeta^2$ and $pq = \frac{1}{4}(1 - \sigma^2) = \frac{1}{4}\zeta^2$. We also note that

$$\begin{aligned} \lambda n v_n &= \zeta n v_n = \zeta n^{\frac{1}{2}}(\textstyle\sum_1^n Z_i^2)^{\frac{1}{2}} \\ &= \zeta n^{\frac{1}{2}}\{np_n\zeta^{-2}(1 - \sigma)^2 + nq_n\zeta^{-2}(1 + \sigma)^2\}^{\frac{1}{2}} \\ &= \zeta n\{1 - 4\zeta^{-1}\sigma\varepsilon_n(\log\log n/2n)^{\frac{1}{2}}\}^{\frac{1}{2}} . \end{aligned} \tag{21}$$

Since $\limsup_{n\to\infty} |\varepsilon_n| = 1$ a.s., it is easy to see from (20) and (21) that

$$\limsup_{n\to\infty} |S_n - \lambda n v_n|/(2\zeta^{-1}\sigma^2 \log\log n) = 1 \quad \text{a.s.}$$

The only remaining case now is when Z_1 is degenerate, say $Z_1 = c\ (\neq 0)$ a.s. In this case, $v_n = |c|$ a.s. and since $\lambda \neq \operatorname{sgn} c$, (14) obviously holds.

THEOREM 1. *The sequential t-test (as described in the beginning of the present section) terminates with probability one under any distribution P for which $P[Z_1 = 0] < 1$.*

PROOF. We first observe that $|T_n| \leqq 1$. Using Laplace's asymptotic formula (cf. [23]), for $\gamma_0 \neq \gamma_1$,

$$\begin{aligned} &\log\{\textstyle\int_0^\infty u^{-1} \exp[nf(u, y; \gamma_1)]\, du\} - \log\{\textstyle\int_0^\infty u^{-1} \exp[nf(u, y; \gamma_0)]\, du\} \\ &\qquad = n\{\beta(\gamma_1 y) - \beta(\gamma_0 y) - \tfrac{1}{2}\gamma_1^2 + \tfrac{1}{2}\gamma_0^2\} + O(1) \qquad \text{as } n \to \infty , \end{aligned} \tag{22}$$

with the $O(1)$ term being uniform for $|y| \leqq 1$, where the function f is as defined in (8) and the function β is defined by

$$\beta(u) = \tfrac{1}{2}u\alpha(u) + \log\alpha(u) , \quad \text{setting} \quad \alpha(u) = \tfrac{1}{2}[u + (u^2 + 4)^{\frac{1}{2}}] \tag{23}$$

(cf. [23], page 1867). It then follows from (22) that there exists a positive constant c for which

$$|\log R_n - n\Psi(T_n)| \leqq c\,, \qquad n = 1, 2, \cdots, \tag{24}$$

where we define

$$\Psi(y) = \beta(\gamma_1 y) - \beta(\gamma_0 y) - \tfrac{1}{2}\gamma_1^2 + \tfrac{1}{2}\gamma_0^2\,. \tag{25}$$

In view of (24), we need only show that $P[\limsup |n\Psi(T_n)| = \infty] = 1$.

The function Ψ is continuously differentiable, and since $\beta'(u) = \alpha(u)$, we have $\Psi'(y) = \gamma_1\alpha(\gamma_1 y) - \gamma_0\alpha(\gamma_0 y)$. Since the function $u\alpha(u)$ is strictly increasing, it follows that $\Psi'(y) \neq 0$ for all y. In the case $\Psi(0) = 0$, or equivalently $\gamma_1 = -\gamma_0$, this implies that there exists $c > 0$ such that $|\Psi(x)| \geqq c|x|$ for $|x| \leqq 1$. Then by Lemma 1,

$$\limsup_{n\to\infty} |n\Psi(T_n)| \geqq \limsup_{n\to\infty} c|nT_n| = c \limsup_{n\to\infty} |S_n/v_n| = \infty \quad \text{a.s.}$$

Now consider the case $\Psi(0) \neq 0$. If $E|Z_1| = \infty$, then by Lemma 3, $\liminf_{n\to\infty} |T_n| = 0$ a.s. If $E|Z_1| < \infty$ and $EZ_1^2 = \infty$, then $\lim_{n\to\infty} T_n = 0$ a.s. Hence in either case, $\limsup_{n\to\infty} |n\Psi(T_n)| = \infty$ a.s.

The only remaining case is $\Psi(0) \neq 0$ and $EZ_1^2 < \infty$. Define $h(\gamma, y) = \beta(\gamma y) - \frac{1}{2}\gamma^2$ and note that $(d/d\gamma)h(\gamma, 1) = \alpha(\gamma) - \gamma > 0$ and $(d/d\gamma)h(\gamma, -1) = -\alpha(-\gamma) - \gamma < 0$ for all real γ. Now $\Psi(y) = h(\gamma_1, y) - h(\gamma_0, y)$ and $\gamma_1 \neq \gamma_0$, and so it follows that

$$\begin{gathered}\Psi(1) \neq 0\,, \qquad \Psi(-1) \neq 0\,, \\ \Psi(1) \quad \text{and} \quad \Psi(-1) \quad \text{are of opposite signs.}\end{gathered} \tag{26}$$

Since $EZ_1^2 < \infty$, $\lim_{n\to\infty} T_n = \lambda$ a.s., where $\lambda = EZ_1/(EZ_1^2)^{\frac{1}{2}}$. If $\Psi(\lambda) \neq 0$, then obviously $\limsup_{n\to\infty} |n\Psi(T_n)| = \infty$ a.s. Now suppose $\Psi(\lambda) = 0$. Then $\lambda \notin \{-1, 0, 1\}$. Since Ψ' vanishes nowhere, we can find a constant $c > 0$ such that $|\Psi(y)| = |\Psi(y) - \Psi(\lambda)| \geqq c|y - \lambda|$ for $|y| \leqq 1$. Using Lemma 4 (with $\lambda \notin \{-1, 0, 1\}$), we then obtain that

$$\limsup_{n\to\infty} |n\Psi(T_n)| \geqq \limsup_{n\to\infty} (c/v_n)|S_n - n\lambda v_n| = \infty \quad \text{a.s.}$$

3. Obstructive distributions and exponential boundedness of the stopping rule for the sequential t-test. In the case $\gamma_0^2 \neq \gamma_1^2$, Wijsman [24] has shown that the two-point distributions defined by (4) are obstructive for the sequential t-test if the stopping bounds are such that B is sufficiently large while A is sufficiently small. These two-point distributions are not obstructive in the case $\gamma_0 = -\gamma_1$ (see Theorem 2 below), and it is interesting to ask whether there are indeed any obstructive distributions in this case. The answer turns out to be affirmative. Since $\Psi(0) = 0$ in this case and since Ψ is continuously differentiable, we can find $d > 0$ such that $|\Psi(x)| \leqq d|x|$ for $|x| \leqq 1$. Therefore in view of (24), if B and A^{-1} are sufficiently large, then there exists $a > 1$ such that $N \geqq M$, where $N = \inf\{n \geqq 1 : R_n \notin (A, B)\}$ and

$$M = \inf\{n \geqq 1 : n|T_n| \geqq a\} = \inf\{n \geqq 1 : |S_n| \geqq av_n\}\,. \tag{27}$$

In the following lemma, we shall construct distributions P under which all the higher moments of M (and therefore of N) are infinite. Hence such distributions are obstructive.

LEMMA 5. *Let* $\{z_i : i = 1, 2, \cdots\}$ *be a set of positive numbers with* $\lim_{i\to\infty} z_i = \infty$. *Suppose* $Z_1, Z_2, \cdots$ *are* i.i.d. *symmetric random variables taking values in the set* $\{\cdots, -z_2, -z_1, z_1, z_2, \cdots\}$. *Let* $p_i = P[Z_1 = z_i]$, $S_n = Z_1 + \cdots + Z_n$, $v_n^2 = n^{-1}\sum_{i=1}^n Z_i^2$, *and define* M *by* (27) *with* $a > 1$. *If* $\sum_1^\infty p_i z_i^\delta < \infty$ *for some* $0 < \delta < 2$ *and* $\sum_1^\infty p_i^2 z_i^\beta = \infty$ *for some* $\beta > \delta$, *then* $EM^{\beta(2+\delta)/2\delta} = \infty$.

PROOF. By Lemma 1, $P[M < \infty] = 1$. Since $\sum_1^\infty p_i z_i^\delta < \infty$ implies that $E|Z_1|^\delta < \infty$, it follows from the Marcinkiewicz–Zygmund strong law of large numbers ([12], page 243) that $\lim_{n\to\infty} n^{-1/\delta}\sum_{i=3}^n Z_i = 0$ a.s. Hence letting $T = \sup\{n \geqq 3 : |\sum_{i=3}^n Z_i| \geqq n^{1/\delta}\}$ (putting $T = 3$ when the above set is empty), we have $P[T < \infty] = 1$. We note that in the event $[Z_1 + Z_2 = 0, M > T]$,

$$a(\textstyle\sum_1^M Z_i^2/M)^{\frac{1}{2}} \leqq |\sum_1^M Z_i| = |\sum_3^M Z_i| < M^{1/\delta}\,,$$

and so $a^2\sum_1^M Z_i^2 \leqq M^{(2+\delta)/\delta}$. Therefore

$$a^\beta|Z_1|^\beta I_{[Z_1+Z_2=0, M>T]} \leqq M^{\beta(2+\delta)/2\delta}\,. \tag{28}$$

For $x \geqq 0$, define $L(x) = \inf\{n \geqq 3 : |\sum_3^n Z_i| \geqq a[n^{-1}(x + \sum_3^n Z_i^2)]^{\frac{1}{2}}\}$. Since $a > 1$, we have $M \geqq 2$ and

$$[Z_1 + Z_2 = 0, M > T] = [Z_1 + Z_2 = 0, L(Z_1^2 + Z_2^2) > T]\,.$$

It then follows from (28) that

$$a^\beta E|Z_1|^\beta I_{[Z_1+Z_2=0, L(2Z_1^2)>T]} \leqq EM^{\beta(2+\delta)/2\delta}\,. \tag{29}$$

We note that

$$E(|Z_1|^\beta I_{[Z_1+Z_2=0, L(2Z_1^2)>T]} \mid Z_1 = z) = |z|^\beta P[Z_2 = -z]P[L(2z^2) > T]\,.$$

Therefore using (29), we obtain that

$$EM^{\beta(2+\delta)/2\delta} \geqq 2a^\beta \textstyle\sum_1^\infty p_i^2 z_i^\beta P[L(2z_i^2) > T] = \infty\,,$$

since $\sum p_i^2 z_i^\beta = \infty$ and $P[L(x) > T] \uparrow P[T < \infty] = 1$ as $x \uparrow \infty$.

The above example shows that if the distribution of Z_1 does not have a sufficiently small tail, then N may fail to be exponentially bounded, and in fact it may not even have finite moments.

The following theorem gives sufficient conditions for the exponential boundedness of N.

THEOREM 2. *In the case* $\gamma_0 = -\gamma_1$ *the stopping rule* N *of the sequential t-test is exponentially bounded if any one of the following conditions is satisfied.*

(i) $0 < EZ_1 \leqq \infty$ *and* $E\exp(tZ_1^-) < \infty$ *for some* $t > 0$.
(ii) $0 > EZ_1 \geqq -\infty$ *and* $E\exp(tZ_1^+) < \infty$ *for some* $t > 0$.
(iii) $EZ_1 = 0$, $E|Z_1| > 0$ *and* $E\exp(tZ_1^2) < \infty$ *for some* $t > 0$.

PROOF. Since $\gamma_0 = -\gamma_1$ is equivalent to the case where $\Psi(0) = 0$, and since we have shown that there exists $c > 0$ for which $|\Psi(x)| \geqq c|x|$ whenever $|x| \leqq 1$, it follows from (24) that we need only show that M is exponentially bounded for any $a > 0$, where M is defined by (27).

We first prove the theorem under condition (i). Since $EZ_1 > 0$, we can choose $b > 0$ such that $EZ_1' > 0$, where $Z_i' = Z_i I_{[Z_i \leqq b]}$. Let $Z_i'' = Z_i - Z_i'$, $S_n' = \sum_1^n Z_i'$, $S_n'' = \sum_1^n Z_i''$. Define $T_1 = \sup\{n \geqq 1 : S_n' \leqq nEZ_1'/2\}$ ($\sup \varnothing = 0$). Since $Ee^{t|Z_1'|} < \infty$ for some $t > 0$ and $EZ_1' > 0$, it follows from a theorem of Chernoff [2] (see also Lemma 3.3 of [21]) that T_1 is exponentially bounded. Noting that $Z_n'' \geqq 0$, we obtain that $S_n''^2 \geqq a^2(\sum_1^n (Z_i'')^2/n)$ if $n \geqq a^2$. Since for $n \geqq a^2$, $S_n'' \geqq a[n^{-1} \sum_1^n (Z_i'')^2]^{\frac{1}{2}}$ and $\sum_1^n Z_i^2 = \sum_1^n (Z_i')^2 + \sum_1^n (Z_i'')^2$, we obtain that

$$M \leqq \inf\{n \geqq \max(T_1 + 1, a^2) : nEZ_1'/2 \geqq a[n^{-1} \sum_1^n (Z_i')^2]^{\frac{1}{2}}\}. \tag{30}$$

Noting that $[\sum_1^n (Z_i')^2]^{\frac{1}{2}} \leqq \sum_1^n |Z_i'|$, (30) implies that

$$\begin{aligned} M &\leqq T_1 + a^2 + 2 + \sup\{n \geqq 1 : \textstyle\sum_1^n |Z_i'| > n^{\frac{3}{2}} EZ_i'/(2a)\} \\ &= T_1 + a^2 + 2 + L, \quad \text{say.} \end{aligned}$$

Obviously, by Chernoff's theorem, L is exponentially bounded. Hence M is exponentially bounded. Replacing Z_i by $-Z_i$ in the above argument, we obtain the theorem also under condition (ii). The situation under condition (iii) is an immediate consequence of the results of Wijsman [23].

Theorem 3 below considers the situation $\gamma_0 \neq -\gamma_1$. In this case, $\Psi(0) \neq 0$. By (26) and the continuity and monotonicity of Ψ, there exists unique $\lambda \in (-1, 1)$ such that $\Psi(\lambda) = 0$. We shall use this fact in Theorem 3.

THEOREM 3. *Let $\gamma_0^2 \neq \gamma_1^2$, and let λ be the unique number in $(-1, 1)$ such that $\Psi(\lambda) = 0$. Then $\lambda \neq 0$, and the stopping time N of the sequential t-test is exponentially bounded if any one of the following conditions is satisfied*:

(i) $\lambda > 0$, $P[Z_1 < 0] > 0$ *and* $E\exp(tZ_1^+) < \infty$ *for some* $t > 0$.
(ii) $\lambda < 0$, $P[Z_1 > 0] > 0$ *and* $E\exp(tZ_1^-) < \infty$ *for some* $t > 0$.
(iii) $\lambda > 0$, $E\exp(tZ_1^+) < \infty$ *for some* $t > 0$ *and* $-\infty \leqq EZ_1 < \lambda(EZ_1^2)^{\frac{1}{2}} \leqq \infty$.
(iv) $\lambda < 0$, $E\exp(tZ_1^-) < \infty$ *for some* $t > 0$ *and* $\infty \geqq EZ_1 > \lambda(EZ_1^2)^{\frac{1}{2}} \geqq -\infty$.
(v) $E\exp(tZ_1^2) < \infty$ *for some* $t > 0$ *and the distribution of* Z_1 *is not a member of the family of two-point distributions defined by* (4).

PROOF. Since there exists $d > 0$ such that $|\Psi(x)| \geqq d|x - \lambda|$ for $|x| \leqq 1$, it follows from (24) that we need only show that M_a is exponentially bounded for any $a > 0$, where

$$M_a = \inf\{n \geqq 1 : n|T_n - \lambda| \geqq a\} = \inf\{n \geqq 1 : S_n \notin ((n\lambda - a)v_n, (n\lambda + a)v_n)\}.$$

First assume condition (i). There exist $\delta > 0$, $c > 0$ and $0 < \rho < 1$ such that $P[S_n > \delta n] \leqq c\rho^n$, $n = 1, 2, \cdots$. This can be proved by truncating Z_i from below and then applying Chernoff's theorem. Take $\alpha > 4\ a\ \delta/\lambda$ and choose a positive integer r such that $P[Z_1 + \cdots + Z_r < -\alpha] = p > 0$. We observe that

$$P[M_a > (r+1)n] \leqq PC_n + \sum_{i=0}^n P[S_{n+ir} > \delta(n+ir)] \leqq PC_n + c\sum_{i=0}^n \rho^{n+ir}, \tag{31}$$

where we define the event C_n by

$$C_n = [\forall i = 0, 1, \cdots, n,\ S_{n+ir} \leqq \delta(n+ir) \text{ and } (\lambda(n+ir) - a)v_{n+ir} < S_{n+ir} < (\lambda(n+ir)+a)v_{n+ir}].$$

For $n \geqq n_0$, we have $\lambda(n+ir) - a > \lambda(n+ir)/2$, $i = 0, 1, \cdots, n$ and so $v_{n+ir} < 2\delta/\lambda$ on C_n. Since jv_j is nondecreasing in j, we have for $n \geqq n_0$ and $i = 0, 1, \cdots, n-1$,

$$\{\lambda(n+ir)+a\}v_{n+ir} - \{\lambda(n+(i+1)r) - a\}v_{n+(i+1)r} \leqq av_{n+ir} + av_{n+(i+1)r} < 4a\delta/\lambda \quad \text{on} \quad C_n. \tag{32}$$

Therefore for $n \geqq n_0$ and $i = 0, 1, \cdots, n-1$, we have on $C_n \cap [X_i^{(n)} < -\alpha]$, where we set $X_i^{(n)} = Z_{n+ir+1} + \cdots + Z_{n+(i+1)r}$,

$$\begin{aligned} S_{n+(i+1)r} = S_{n+ir} + X_i^{(n)} &< (\lambda(n+ir)+a)v_{n+ir} - \alpha \\ &< (\lambda(n+ir)+a)v_{n+ir} - (4a\delta/\lambda) \\ &< \{\lambda(n+(i+1)r) - a\}v_{n+(i+1)r}. \end{aligned} \tag{33}$$

From (33), we see that $C_n \cap [X_i^{(n)} < -\alpha] = \emptyset$, and so for $n \geqq n_0$,

$$C_n \subset [X_i^{(n)} \geqq -\alpha \text{ for } i = 0, 1, \cdots, n-1].$$

This implies that $PC_n \leqq (1-p)^n$ and in view of (31), M_a is exponentially bounded.

Now assume condition (iii). Choose $\zeta > 0$ and $0 < \gamma < \lambda$ such that $\gamma(EZ_1^2)^{\frac{1}{2}} > \xi > EZ_1$. Define $T = \sup\{n \geqq 1 : S_n \geqq \xi n\}$. By a suitable truncation and Chernoff's theorem, it can be shown that T is exponentially bounded. For $n \geqq n_1$, we have $n\lambda - a \geqq \gamma n$, and so

$$\begin{aligned} M_a &\leqq \inf\{n \geqq \max(T+1, n_1) : (n\lambda - a)v_n \geqq S_n\} \\ &\leqq \inf\{n \geqq \max(T+1, n_1) : \gamma n v_n \geqq \xi n\} \\ &\leqq T + n_1 + 2 + \sup\{n \geqq 1 : \gamma^2 \textstyle\sum_1^n Z_i^2 < \xi^2 n\}. \end{aligned} \tag{34}$$

By truncating Z_i^2 suitably from above and applying Chernoff's theorem, it is easy to see from (34) that M_a is exponentially bounded. The situation under condition (v) has been considered by Wijsman [23].

4. Moments of the stopping rule for the sequential t-test. In Section 3, we have given examples to show that moments of the stopping rule N for the sequential t-test may be infinite. Theorems 4 and 5 below give some conditions for the finiteness of moments of N.

THEOREM 4. *Let $\beta > 0$. In the case $\gamma_0 = -\gamma_1$, $EN^\beta < \infty$ if any one of the following conditions is satisfied:*

(i) $0 < EZ_1 \leqq \infty$ *and* $E(Z_1^-)^{\beta+1} < \infty$.
(ii) $0 > EZ_1 \geqq -\infty$ *and* $E(Z_1^+)^{\beta+1} < \infty$.
(iii) $EZ_1 = 0$ *and* $0 < E|Z_1|^{2(\beta+1)} < \infty$.

THEOREM 5. *Let* $\gamma_0^2 \neq \gamma_1^2$, *and let* λ *be the unique solution of* $\Psi(\lambda) = 0$. *Then for any* $\beta > 0$, $EN^\beta < \infty$ *if any one of the following conditions is satisfied*:

(i) $\lambda > 0$, $P[Z_1 < 0] > 0$ *and* $E(Z_1^+)^{\beta+1} < \infty$.
(ii) $\lambda < 0$, $P[Z_1 > 0] > 0$ *and* $E(Z_1^-)^{\beta+1} < \infty$.
(iii) $\lambda > 0$, $E(Z_1^+)^{\beta+1} < \infty$ *and* $-\infty \leqq EZ_1 < \lambda(EZ_1^2)^{\frac{1}{2}} \leqq \infty$.
(iv) $\lambda < 0$, $E(Z_1^-)^{\beta+1} < \infty$ *and* $\infty \geqq EZ_1 > \lambda(EZ_1^2)^{\frac{1}{2}} \geqq -\infty$.
(v) $E|Z_1|^{2(\beta+1)} < \infty$ *and the distribution of* Z_1 *is not a member of the family of two-point distributions defined by* (4).

The above two theorems can be proved by using similar ideas as the proof of Theorems 2 and 3 and replacing Chernoff's theorem by the following lemma (cf. [3] and [10]).

LEMMA 6. *Suppose* $Z_1, Z_2, \cdots$ *are* i.i.d. *random variables and* δ *is a real number. Let* $S_n = Z_1 + \cdots + Z_n$, $\beta > 0$.

(i) *If* $\delta > EZ_1 \geqq -\infty$ *and* $E(Z_1^+)^{\beta+1} < \infty$, *then* $\sum n^{\beta-1}P[\sup_{k\geqq n} k^{-1}S_k > \delta] < \infty$.
(ii) *If* $\delta < EZ_1 \leqq \infty$ *and* $E(Z_1^-)^{\beta+1} < \infty$, *then* $\sum n^{\beta-1}P[\inf_{k\geqq n} k^{-1}S_k < \delta] < \infty$.

In [14], Sacks has proved that for the sequential t-test, if $\gamma_0 = 0$, $\gamma_1 = 1$ and the stopping bounds $A(< 1) < B$ are such that $A = B^{-1}$, then

$$EN \sim 2\log B/\log 2 \quad \text{as} \quad B \to \infty\,, \tag{35}$$

under the assumption that $Z_1, Z_2, \cdots$ are i.i.d. $N(1, 1)$ random variables. Here we give a simple proof of a general form of this theorem where our conclusion still holds outside the normal parametric model.

THEOREM 6. *Suppose* $Z_1, Z_2, \cdots$ *are* i.i.d. *random variables such that* $0 < E|Z_1|^{2(\beta+1)} < \infty$ *for some* $\beta > 0$. *Let* $N = \inf\{n \geqq 1 : R_n \notin (A, B)\}$, *where* $R_n = U_n(\gamma_1)/U_n(\gamma_0)$ *with* $\gamma_1 \neq \gamma_0$ *and* $U_n(\gamma)$ *defined by* (7). *Let* $a = \log A$, $b = \log B$ *and* $\lambda = EZ_1/(EZ_1^2)^{\frac{1}{2}}$. *Defining* Ψ *by* (25), *we have as* $\min(-a, b) \to \infty$,

$$\begin{aligned} EN^\beta &\sim (b/\Psi(\lambda))^\beta \qquad \text{if} \quad \Psi(\lambda) > 0\,; \\ EN^\beta &\sim (a/\Psi(\lambda))^\beta \qquad \text{if} \quad \Psi(\lambda) < 0\,. \end{aligned} \tag{36}$$

PROOF. In view of (24), it suffices to show that the asymptotic relation (36) as $\min(-a, b) \to \infty$ is satisfied by the stopping time M defined by

$$M = \inf\{n \geqq 1 : n\Psi(T_n) \notin (a, b)\}\,.$$

Since $\lim_{n\to\infty} T_n = \lambda$ a.s., it is easy to see that with probability one, $M\Psi(\lambda)/b \to 1$

if $\Psi(\lambda) > 0$ and $M\Psi(\lambda)/a \to 1$ if $\Psi(\lambda) < 0$ as $\min(-a, b) \to \infty$. Hence by Fatou's lemma,

$$(37) \qquad \begin{aligned} &\liminf_{\min(-a,b)\to\infty} (\Psi(\lambda)/b)^\beta EM^\beta \geqq 1 \qquad \text{if} \quad \Psi(\lambda) > 0\,, \\ &\liminf_{\min(-a,b)\to\infty} (\Psi(\lambda)/a)^\beta EM^\beta \geqq 1 \qquad \text{if} \quad \Psi(\lambda) < 0\,. \end{aligned}$$

For definiteness, consider the case $\Psi(\lambda) > 0$. Then $M \leqq M(b) = \inf\{n \geqq 1: n\Psi(T_n) \geqq b\}$. Given any small $\varepsilon > 0$, let

$$L = \sup\left\{n \geqq 1: \left|\frac{1}{n}\sum_1^n Z_i - EZ_1\right| > \varepsilon \text{ or } \left|\frac{1}{n}\sum_1^n Z_i^2 - EZ_1^2\right| > \varepsilon\right\}.$$

Since $E(Z_1^2)^{\beta+1} < \infty$, it follows from Lemma 6 that $EL^\beta < \infty$. Let $\rho(\varepsilon) = \min\{\Psi(u/v): |u - EZ_1| \leqq \varepsilon, |v^2 - EZ_1^2| \leqq \varepsilon\}$. Then $\rho(\varepsilon) > 0$ if ε is sufficiently small. We note that

$$(38) \qquad \begin{aligned} M(b) &\leqq (L+1)I_{[L+1\geqq M(b)]} + M(b)I_{[M(b)>L+1]} \\ &\leqq L + (b/\rho(\varepsilon)) + 2\,. \end{aligned}$$

The second inequality in (38) follows from the fact that on $[M(b) > L + 1]$, we have

$$(39) \qquad (M(b) - 1)\rho(\varepsilon) \leqq (M(b) - 1)\Psi(T_{M(b)-1}) < b\,.$$

Hence if $0 < \beta \leqq 1$, then it follows from (38) that

$$(40) \qquad EM^\beta(b) \leqq EL^\beta + (b/\rho(\varepsilon))^\beta + 2\,.$$

Therefore $\limsup_{b\to\infty} EM^\beta(b)/b^\beta \leqq 1/(\rho(\varepsilon))^\beta$, and since ε is arbitrary, we have established (36) in the case $0 < \beta \leqq 1$.

Next consider the case where β is a positive integer. Then it follows from (38) that

$$M^\beta(b) \leqq ((L+2) + (b/\rho(\varepsilon)))^\beta\,.$$

Applying the binomial expansion to the right-hand side of the above inequality, we then obtain using induction that $\limsup_{b\to\infty} EM^\beta(b)/b^\beta \leqq 1/(\rho(\varepsilon))^\beta$, and so (36) is also established in this case.

Finally for any real number $\beta > 1$, we write $\beta = k + \delta$, where k is a positive integer and $0 \leqq \delta < 1$. It then follows from (38) that

$$(41) \qquad M^\beta(b) \leqq ((L+2) + (b/\rho(\varepsilon)))^k(L+2)^\delta + ((L+2) + (b/\rho(\varepsilon)))^k(b/\rho(\varepsilon))^\delta\,.$$

It is easy to see from (41) that $\limsup_{b\to\infty} EM^\beta(b)/b^\beta \leqq 1/(\rho(\varepsilon))^\beta$, and the desired conclusion then follows upon letting $\varepsilon \downarrow 0$. The case $\Psi(\lambda) < 0$ can be handled similarly.

In the case $\gamma_0 = 0$, $\gamma_1 = 1$, we have $\Psi(1/2^{\frac{1}{2}}) = \frac{1}{2}\log 2$. Thus if $EZ_1 = 1$, $\operatorname{Var} Z_1 = 1$ and $EZ_1^4 < \infty$, then $EZ_1^2 = 2$ and (36) with $\beta = 1$ reduces to the result (35) obtained by Sacks.

5. Other invariant sequential probability ratio tests. In this section, we shall extend our method to study other invariant SPRT's. First we consider the sequential F-test studied by Ifram [6]. Here Z_n are i.i.d. k-dimensional random vectors with $Z_n = (Z_{n1}, \cdots, Z_{nk})$, where $\{Z_{n1}: n = 1, 2, \cdots; i = 1, \cdots, k\}$ is a family of independent random variables. The parametric model assumes that Z_{ni} is $N(\zeta_i, \sigma^2)$ and that for some given $s \leqq k$, $\zeta_i = 0$ if $i > s$. For some given $1 \leqq q \leqq s$, letting $\theta = \sum_{i=1}^{q} \zeta_i^2/k\sigma^2$, we want to test $H_0: \theta = \theta_0$ against the alternative $H_1: \theta = \theta_1$ with $\theta_0 \neq \theta_1$, $\theta_0 \geqq 0$, $\theta_1 \geqq 0$. By a procedure similar to that on page 267 in [11], sufficiency and invariance under some group of transformations reduce the data to the sequence (W_n), where

$$W_n = \left(\sum_{i=1}^{q} \bar{Z}_{ni}^2\right)\Big/\left(\frac{1}{n} \sum_{j=1}^{n}\left\{\sum_{i=1}^{s}(Z_{ji} - \bar{Z}_{ni})^2 + \sum_{i=s+1}^{k} Z_{ji}^2\right\}\right),$$
$$\bar{Z}_{ni} = \sum_{j=1}^{n} Z_{ji}/n .$$

We can write the likelihood ratio $R_n = p_{\theta_1,n}(W_n)/p_{\theta_0,n}(W_n)$, where

$$p_{\theta,n}(x) = (1/B(c, \lambda - a - c))x^{c-1}(1 + x)^{-(\lambda-a)} \times e^{-\lambda\theta}\,{}_1F_1(\lambda - a, c; \lambda\theta x/(1 + x)) \tag{42}$$

with $2a = s - q$, $2c = q$ and $2\lambda = kn$. The function ${}_1F_1(\alpha, \beta; z)$ in (42) denotes Kummer's confluent hypergeometric function (cf. [7], [16]). Letting $Y_n = W_n/(1 + W_n)$, we have

$$\log R_n = -\tfrac{1}{2}kn(\theta_1 - \theta_0) + \log {}_1F_1(-a + \tfrac{1}{2}kn, c; \tfrac{1}{2}kn\theta_1 Y_n) - \log {}_1F_1(-a + \tfrac{1}{2}kn, c; \tfrac{1}{2}kn\theta_0 Y_n) . \tag{43}$$

The sequential F-test stops sampling at stage $N = \inf\{n: R_n \notin (A, B)\}$ and accepts H_0 or H_1 according as $R_N \leqq A$ or $R_N \geqq B$. The particular case $k = q = 1$ corresponds to the sequential t^2-test (or the two-sided sequential t-test) studied by Rushton [13] and Schneiderman and Armitage [15]. We now consider the problem of termination with probability one under any distribution P. First we generalize Kesten's theorem to the case of several populations in the following lemma. The proof of Lemma 7 makes use of similar ideas as Kesten's proof and is omitted here.

LEMMA 7. *Suppose $\{Z_{ni}: n = 1, 2, \cdots, i = 1, \cdots, k\}$ is a family of independent random variables such that for each fixed i, $Z_{1i}, Z_{2i}, \cdots$ have the same distribution. Let $S_{ni} = Z_{1i} + \cdots + Z_{ni}$. If $\max_{1\leqq i\leqq k} E|Z_{1i}| = \infty$, then*

$$\limsup_{n\to\infty} \max_{1\leqq i\leqq k} |Z_{ni}|/\max_{1\leqq i\leqq k} |S_{n-1,i}| = \infty \quad \text{a.s.}$$

LEMMA 8 (cf. [18]). *With the same notation as in Lemma 7, assume that $EZ_{1i} = 0$, $EZ_{1i}^2 = \sigma_i^2 < \infty$ for $i = 1, \cdots, k$.*

(i) *If $\sigma^2 = \sum_{i=1}^{k} \sigma_i^2 > 0$, then*

$$\limsup_{n\to\infty} \left|\sum_{i=1}^{k} S_{ni}\right|/(2n \log\log n)^{\frac{1}{2}} = \sigma \quad \text{a.s.}$$

(ii) *If $\sigma_i^2 > 0$ for $i = 1, \cdots, k$, then with probability one, the set of limit points of the sequence of random vectors $(S_{n1}/(2\sigma_1 n \log\log n)^{\frac{1}{2}}, \cdots, S_{nk}/(2\sigma_k n \log\log n)^{\frac{1}{2}})$, $n \geqq 3$, coincides with the unit ball $\{(x_1, \cdots, x_k): \sum_{i=1}^{k} x_i^2 \leqq 1\}$ in R^k.*

We now define a function G as follows:

$$G(y) = k\{H(y, \theta_1) - H(y, \theta_0)\} \tag{44}$$
$$H(y, \theta) = -\tfrac{1}{2}\theta + \tfrac{1}{4}[\theta y + (\theta y)^{\frac{1}{2}}(\theta y + 4)^{\frac{1}{2}}] + \log((\theta y)^{\frac{1}{2}} + (\theta y + 4)^{\frac{1}{2}}) .$$

This function plays the same role for the sequential F-test as that played by the function Ψ defined by (25) for the sequential t-test. We note that G is continuous on $[0, \infty)$ and its derivative is continuous and is nowhere zero on $(0, \infty)$. In fact

$$G'(y) = \frac{k}{4}\{\theta_1 + (\theta_1^2 + 4\theta_1 y^{-1})^{\frac{1}{2}} - \theta_0 - (\theta_0^2 + 4\theta_0 y^{-1})^{\frac{1}{2}}\} , \qquad y > 0 .$$

Suppose $\theta_1 > \theta_0$. Then $G(0) = -(k/2)(\theta_1 - \theta_0) < 0$. Also for all $\theta > 0$,

$$\frac{d}{d\theta} H(1, \theta) = \tfrac{1}{4}\{(1 + 4\theta^{-1})^{\frac{1}{2}} - 1\} > 0$$

and so $G(1) = k\{H(1, \theta_1) - H(1, \theta_0)\} > 0$. In the case $\theta_1 < \theta_0$, then $G(0) > 0$ and $G(1) < 0$. Hence in either case there exists a unique $\beta \in (0, 1)$ such that $G(\beta) = 0$.

THEOREM 7. *Let $\beta \in (0, 1)$ be the unique solution of $G(\beta) = 0$. If $q = 1$ or if θ_0, θ_1 are both nonzero, then the sequential F-test terminates with probability one under any distribution P for which the following statement does not hold*:

$$\begin{aligned} &P[Z_{1i} = \mu_i \ \textit{for} \ i = 1, \cdots, q;\ Z_{1i}^2 = \tau_i^2 \ \textit{for} \ i = s+1, \cdots, k] = 1 \\ &\qquad \textit{with} \quad \textstyle\sum_1^q \mu_i^2 = \beta(\sum_{i=1}^q \mu_i^2 + \sum_{i=s+1}^k \tau_i^2) , \quad \textit{and} \\ &\qquad Z_{1i} \quad \textit{is degenerate for} \quad i = q+1, \cdots, s . \end{aligned} \tag{45}$$

In the case where $q \geqq 2$ and one of θ_0, θ_1 vanishes, the sequential F-test terminates with probability one under any distribution P for which the following statement does not hold:

$$P[Z_{1i} = 0 \ \textit{for} \ i = 1, \cdots, k] = 1 . \tag{46}$$

PROOF. We first note that

$$Y_n = \textstyle\sum_{i=1}^q \bar{Z}_{ni}^2 \Big/ \Big\{\frac{1}{n} \sum_{j=1}^n \sum_{i=1}^q Z_{ji}^2 + \frac{1}{n} \sum_{j=1}^n \sum_{i=q+1}^s (Z_{ji} - \bar{Z}_{ni})^2$$
$$+ \frac{1}{n} \textstyle\sum_{j=1}^n \sum_{i=s+1}^k Z_{ji}^2\Big\} .$$

(We take Y_n to be 0 if the denominator in the above expression vanishes.) If $E|Z_{1i}| = \infty$ for some $i = 1, \cdots, k$, then it is easy to see from Lemma 7 that $\liminf_{n\to\infty} Y_n = 0$ a.s. Now assume that $\max_{1\leqq i\leqq k} E|Z_{1i}| < \infty$ but $\max_{1\leqq i\leqq k} EZ_{1i}^2 = \infty$. Then the strong law of large numbers implies that $\lim_{n\to\infty} Y_n = 0$ a.s. In the case $\max_{1\leqq i\leqq k} EZ_{1i}^2 < \infty$ and $EZ_{1i} = 0$ for all $i = 1, \cdots, q$, we also have $\lim_{n\to\infty} Y_n = 0$ a.s. But using the limiting behavior of Kummer's function (cf. [7], [16]), if $\lim_{n\to\infty} Y_n = 0$ a.s., then it follows from (43) that $\lim_{n\to\infty} n^{-1}|\log R_n| = (k/2)|\theta_1 - \theta_0|$ a.s., while if $\liminf_{n\to\infty} Y_n = 0$ a.s., then $\limsup_{n\to\infty} n^{-1}|\log R_n| \geqq (k/2)|\theta_1 - \theta_0|$ a.s. Hence in all the above cases, $\limsup_{n\to\infty} |\log R_n| = \infty$ a.s.

We shall now assume that $\max_{1 \le i \le k} EZ_{1i}^2 < \infty$ and $EZ_{1i} \neq 0$ for some $i = 1, \cdots, q$. In this case, setting

$$(47)\qquad \mu_i = EZ_{1i}\,, \qquad \tau_i^2 = EZ_{1i}^2\,, \qquad \sigma_i^2 = \operatorname{Var} Z_{1i}\,,$$
$$\gamma = (\textstyle\sum_{i=1}^q \mu_i^2)/\{\sum_{i=1}^q \tau_i^2 + \sum_{i=q+1}^s \sigma_i^2 + \sum_{i=s+1}^k \tau_i^2\}\,,$$

we obtain from the strong law of large numbers that $\lim_{n\to\infty} Y_n = \gamma\ (> 0)$ a.s. From the uniform asymptotic expansion of Kummer's function (cf. [7], [16], [25]), we can choose $\delta > 0$ such that $\gamma - \delta > 0$ and $\rho > 0$ such that for $\gamma - \delta \leqq y \leqq \gamma + \delta$ and $n = 1, 2, \cdots$,

$$(48)\qquad |\{-\tfrac{1}{2}kn(\theta_1 - \theta_0) + \log {}_1F_1(-a + \tfrac{1}{2}kn, c; \tfrac{1}{2}kn\theta_1 y)$$
$$- \log {}_1F_1(-a + \tfrac{1}{2}kn, c; \tfrac{1}{2}kn\theta_0 y)\} - \{nG(y) + \nu \log n\}| \leqq \rho$$

where $\nu = 0$ if θ_0 and θ_1 are both positive, while in the case when one of θ_0, θ_1 vanishes, we have $\nu = \frac{1}{2} - c$ if $\theta_1 \neq 0$ and $\nu = c - \frac{1}{2}$ if $\theta_0 \neq 0$. (This is due to the fact that ${}_1F_1(\lambda, \zeta; 0) = 1$. Also note that when $q = 1$, we always have $\nu = 0$.) Hence if $G(\gamma) \neq 0$, then (43) and (48) imply that $\limsup_{n\to\infty} |\log R_n| = \infty$ a.s.

Now assume that $G(\gamma) = 0$, i.e., $\gamma = \beta$. Then there exist positive constants $c_1 > c_2$ such that

$$(49)\qquad c_1|y - \beta| \geqq |G(y)| \geqq c_2|y - \beta|\,, \qquad 0 \leqq y \leqq 1\,.$$

If $\max_{1 \le i \le k} EZ_{1i}^4 = \infty$, then as in the proof of Lemma 4, we can show that $\limsup_{n\to\infty} |\log R_n| = \infty$ a.s. Now suppose that $\max_{1 \le i \le k} EZ_{1i}^4 < \infty$. For $i = 1, \cdots, s$, we can write $x^2 - \mu_i^2 = (x - \mu_i)(2\mu_i + v_i(x))$ with $\lim_{x\to\mu_i} v_i(x) = 0$. Therefore

$$n \textstyle\sum_{i=1}^q \bar{Z}_{ni}^2 - \beta\{\sum_{i=1}^q \sum_{j=1}^n Z_{ji}^2 + \sum_{i=q+1}^s \sum_{j=1}^n (Z_{ji} - \bar{Z}_{ni})^2$$
$$+ \textstyle\sum_{i=s+1}^k \sum_{j=1}^n Z_{ji}^2\}$$
$$(50)\qquad = \textstyle\sum_{i=1}^q (\sum_{j=1}^n X_{ji} - nEX_{1i}) - \beta \sum_{i=q+1}^s \{\sum_{j=1}^n (Z_{ji} - \mu_i)^2 - n\sigma_i^2\}$$
$$- \beta \textstyle\sum_{i=s+1}^k (\sum_{j=1}^n Z_{ji}^2 - n\tau_i^2)$$
$$+ \textstyle\sum_{i=1}^q (\sum_{j=1}^n Z_{ji} - n\mu_i)v_{ni} + n\beta \sum_{i=q+1}^s (\bar{Z}_{ni} - \mu_i)^2$$

where $X_{ji} = 2\mu_i Z_{ji} - \beta Z_{ji}^2$ and $v_{ni} = v_i(\bar{Z}_{ni})$, $i = 1, \cdots, q$. In the case where $\sum_{i=1}^q \operatorname{Var}(X_{1i}) + \sum_{i=q+1}^s \operatorname{Var}((Z_{1i} - \mu_i)^2) + \sum_{i=s+1}^k \operatorname{Var}(Z_{1i}^2) > 0$, we can use (43), (48), (49), (50) and Lemma 8 to show that $\limsup_{n\to\infty} |\log R_n| = \infty$ a.s. by a similar argument as in the proof of Lemma 4.

It remains only to consider the case $\gamma = \beta$ and $\sum_{i=1}^q \operatorname{Var}(X_{1i}) + \sum_{i=q+1}^s \operatorname{Var}((Z_{1i} - \mu_i)^2) + \sum_{i=s+1}^k \operatorname{Var}(Z_{1i}^2) = 0$. Then for $i = s + 1, \cdots, k$, Z_{1i}^2 is degenerate, while for $1 = 1, \cdots, s$, the distribution of Z_{1i} is either degenerate or puts its mass only to two points. For $i = 1, \cdots, q$, let $P[Z_{1i} = a_i] = p_i$ and $P[Z_{1i} = b_i] = q_i$ with $p_i + q_i = 1$ (a_i and b_i need not be distinct). Set

$$d_i = (p_i q_i)^{\frac{1}{2}}\{(2a_i^2 p_i + 2a_i b_i q_i + \beta b_i^2) - (2b_i^2 q_i + 2a_i b_i p_i + \beta a_i^2)\}\,;$$
$$p_{ni} = \frac{1}{n} \textstyle\sum_{j=1}^n I_{[Z_{ji} = a_i]} = p_i + \varepsilon_{ni}\{(2p_i q_i \log\log n)/n\}^{\frac{1}{2}}\,;$$
$$q_{ni} = 1 - p_{ni} = q_i - \varepsilon_{ni}\{(2p_i q_i \log\log n)/n\}^{\frac{1}{2}}\,.$$

Then it can be shown that

$$(51)\qquad n\bar{Z}_{ni}^2 - \beta \sum_{j=1}^n Z_{ji}^2 = n\mu_i^2 - n\beta\tau_i^2 + d_i\varepsilon_{ni}(2n\log\log n)^{\frac{1}{2}} + 2p_iq_i\varepsilon_{ni}^2(a_i - b_i)^2\log\log n\,, \qquad i = 1, \cdots, q\,.$$

For $i = q+1, \cdots, s$, since $\operatorname{Var}((Z_{1i} - \mu_i)^2) = 0$, we have $P[Z_{1i} = \mu_i + \sigma_i] = P[Z_{1i} = \mu_i - \sigma_i]$ and this probability is 1 or $\frac{1}{2}$ according as σ_i is zero or not. In the case $\sigma_i = 0$, obviously $\sum_{j=1}^n (Z_{ji} - \bar{Z}_{ni})^2 = 0$ a.s. In the case $\sigma_i \neq 0$, letting $n^{-1}\sum_{j=1}^n I_{[Z_{ji}=\mu_i+\sigma_i]} = \frac{1}{2} + \varepsilon_{ni}(\log\log n/2n)^{\frac{1}{2}}$, we have

$$(52)\qquad \sum_{j=1}^n (Z_{ji} - \bar{Z}_{ni})^2 = n\sigma_i^2 - 2\sigma_i^2\varepsilon_{ni}\log\log n\,.$$

Therefore if $\max_{1\leq i\leq q}|d_i| > 0$, then (51), (52) together with Lemma 8 imply

$$(53)\qquad \limsup_{n\to\infty} |n\sum_{i=1}^q \bar{Z}_{ni}^2 - \beta\sum_{j=1}^n \{\sum_{i=1}^q Z_{ji}^2 + \sum_{i=q+1}^s (Z_{ji} - \bar{Z}_{ni})^2 + \sum_{i=s+1}^k Z_{ji}^2\}|/(n\log\log n)^{\frac{1}{2}} = \zeta \quad \text{a.s.}$$

for some finite positive constant ζ, and so it follows from (43), (48) and (53) that $\limsup_{n\to\infty}|\log R_n| = \infty$ a.s. In the case where $\max_{1\leq i\leq q}|d_i| = 0$, if θ_0 and θ_1 are both positive or if $q = 1$, then (45) cannot hold and there must exist some $i = 1, \cdots, s$ such that Z_{1i} is nondegenerate, and so we have from (51), (52) and Lemma 8 that

$$(54)\qquad \limsup_{n\to\infty} |n\sum_{i=1}^q \bar{Z}_{ni}^2 - \beta\sum_{j=1}^n \{\sum_{i=1}^q Z_{ji}^2 + \sum_{i=q+1}^s (Z_{ji} - \bar{Z}_{ni})^2 + \sum_{i=s+1}^k Z_{ji}^2\}|/(\log\log n) = \xi \quad \text{a.s.}$$

for some positive number ξ. It is then obvious from (43), (48) and (54) that $\limsup_{n\to\infty}|\log R_n| = \infty$. The same result obviously still holds when $\xi = 0$ in (54) and $\nu \neq 0$, and that is why we do not have to exclude the case (45) when $q \geqq 2$ and one of θ_0, θ_1 vanishes, provided that (46) does not hold.

It is clear from the above proof that in the case where (45) is satisfied and θ_0, θ_1 are both nonzero, then the sequential F-test fails to have the property of termination with probability one when the stopping bounds A, B are such that B is sufficiently large and A is sufficiently small. The function G defined by (44) which plays a central role in the termination problem also gives us the asymptotic behavior of moments of the stopping rule for the sequential F-test. Using (43), (48) and a similar argument as in the proof of Theorem 6, we can prove the following theorem.

THEOREM 8. *Suppose* $\{Z_{ni}: n = 1, 2, \cdots; i = 1, \cdots, k\}$ *is a family of independent random variables such that for each fixed i,* $Z_{1i}, Z_{2i}, \cdots$ *have the same distribution. Let* $N = \inf\{n \geqq 1 : \log R_n \notin (a, b)\}$, *where* $\log R_n$ *is defined by* (43) *with* $\theta_1 \neq \theta_0$, $\theta_1 \geqq 0$, $\theta_0 \geqq 0$. *Assume that for some* $\eta > 0$, $E|Z_{1i}|^{2(\eta+1)} < \infty$ *for* $i = 1, \cdots, k$. *Let* $\gamma = (\sum_{i=1}^q \mu_i^2)/\{\sum_{i=1}^q \tau_i^2 + \sum_{i=q+1}^s \sigma_i^2 + \sum_{i=s+1}^k \tau_i^2\}$, *where* $\mu_i = EZ_{1i}$, $\tau_i^2 = EZ_{1i}^2$, $\sigma_i^2 = \operatorname{Var} Z_{1i}$; *and define G by* (44). *Suppose that* $\gamma > 0$ *and* $G(\gamma) \neq 0$. *Then* $EN^\eta < \infty$ *and as* $\min(-a, b) \to \infty$,

$$EN^\eta \sim (b/G(\gamma))^\eta \qquad \text{if} \quad G(\gamma) > 0\,;$$
$$EN^\eta \sim (a/G(\gamma))^\eta \qquad \text{if} \quad G(\gamma) < 0\,.$$

The function $H(y, \theta)$ defined by (44) also gives us the asymptotic behavior of moments of the stopping rule for the sequential T^2-test. Suppose Z_n are i.i.d. k-dimensional random vectors with $Z_n = (Z_{n1}, \cdots, Z_{nk})$. The parametric model assumes that Z_n is normal with mean vector ζ and positive definite covariance matrix Σ. We want to test the null hypothesis $H_0: \zeta'\Sigma^{-1}\zeta = \theta_0$ versus the alternative $H_1: \zeta'\Sigma^{-1}\zeta = \theta_1$ with $\theta_0 \neq \theta_1$, $\theta_0 \geqq 0$, $\theta_1 \geqq 0$. Let $\bar{Z}_n$ and S_n be the sample mean vector and the sample covariance matrix (at stage n) respectively, and let $Q_n = \bar{Z}_n' S_n^{-1} \bar{Z}_n$. The sequential T^2-test stops sampling at stage $N = \inf\{n \geqq 2: R_n \notin (A, B)\}$, and accepts H_0 or H_1 according as $R_N \leqq A$ or $R_N \geqq B$, where the likelihood ratio R_n in the present case is given by:

$$\begin{aligned}(55) \qquad \log R_n = & -\tfrac{1}{2}n(\theta_1 - \theta_0) + \log {}_1F_1\left(\frac{n}{2}, \frac{k}{2}; n\theta_1 Q_n/2(1 + Q_n)\right) \\ & - \log {}_1F_1\left(\frac{n}{2}, \frac{k}{2}; n\theta_0 Q_n/2(1 + Q_n)\right)\end{aligned}$$

(cf. [5], [8]). The following theorem gives the asymptotic behavior of the moments of the stopping rule N.

THEOREM 9. *Let $\eta > 0$ and suppose $Z_n = (Z_{n1}, \cdots, Z_{nk})$ are* i.i.d. *k-dimensional random vectors such that $E|Z_{1i}|^{2(\eta+1)} < \infty$ for $i = 1, \cdots, k$. Suppose Z_1 has a positive definite covariance matrix Σ and mean vector ζ. Let $N = \inf\{n \geqq 2:$ $\log R_n \notin (a, b)\}$ where $\log R_n$ is defined by (55) and $\theta_1 \neq \theta_0$, $\theta_1 \geqq 0$, $\theta_0 \geqq 0$. Define $H(y, \theta)$ as in (44) and let $G_1(y) = H(y, \theta_1) - H(y, \theta_0)$. Suppose $\lambda > 0$ and $G_1(\lambda) \neq 0$, where $\lambda = (\zeta'\Sigma^{-1}\zeta)/(1 + \zeta'\Sigma^{-1}\zeta)$. Then $EN^\eta < \infty$ and as* $\min(-a, b) \to \infty$,

$$\begin{aligned} EN^\eta &\sim (b/G_1(\lambda))^\eta \qquad \text{if} \quad G_1(\lambda) > 0\,; \\ EN^\eta &\sim (a/G_1(\lambda))^\eta \qquad \text{if} \quad G_1(\lambda) < 0\,. \end{aligned}$$

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