The Annals of Statistics
1975, Vol. 3, No. 2, 457-460

RELATIONSHIPS BETWEEN THE UMVU ESTIMATORS OF THE MEAN AND MEDIAN OF A FUNCTION OF A NORMAL DISTRIBUTION[1]

BY FARHAD MEHRAN

Rutgers University

Let $\eta = f(\mu)$ and $\theta = Ef(Y)$, where f is a monotone function and $Y \sim N(\mu, \sigma^2)$. In this note we use the method of convolution transforms to show that the UMVU estimators of η and θ based on a pair of independent sufficient statistics $T \sim N(\mu, \alpha\sigma^2)$ and $S^2 \sim \sigma^2\chi^2_{(\nu)}$ are related to each other in a simple manner: the replacement α by $\alpha - 1$ in the expression of the UMVU estimator of η gives the corresponding expression of the UMVU estimator of θ. In addition, we show that a similar relationship also exists among the estimators of the variances.

1. Introduction. Let θ and η respectively denote the mean and median of a random variable $f(Y)$, where f is a monotone measurable function and Y is a normally distributed random variable with unknown mean and variance μ and σ^2. We assume that both θ and η are finite and write $\theta = Ef(Y)$, where E is the expected value operator, $Y \sim N(\mu, \sigma^2)$, and by monotonicity of f, $\eta = f(\mu)$.

Consider also a pair of statistics T and S^2 mutually independent and jointly sufficient (and boundedly complete) for μ and σ^2; T has a normal distribution $N(\mu, \alpha\sigma^2)$, where α is a known positive quantity and S^2 when divided by σ^2 has a chi-square distribution $\chi^2_{(\nu)}$ with certain number of degrees of freedom ν.

The uniform minimum variance unbiased (UMVU) estimators of θ and η, whenever they exist, are solutions of the integral equations,

$$EK_{\alpha,\nu}(T, S^2) = Ef(Y)\,, \tag{1}$$

$$EH_{\alpha,\nu}(T, S^2) = f(\mu)\,, \tag{2}$$

where $\hat{\theta} = K_{\alpha,\nu}(T, S^2)$ and $\hat{\eta} = H_{\alpha,\nu}(T, S^2)$ are the UMVU estimators of θ and η, respectively.

A number of writers (Washio, Morimoto and Ikeda (1956), Neyman and Scott (1960), Schmetterer (1960), Ghurye and Olkin (1969), and Gray, Watkins and Schucany (1973)), using various approaches, have obtained explicit solutions to (1) and (2), or more general results that may be applied to find these solutions. In a commentary to the paper by Gray, Watkins and Schucany (1973), the present author has indicated yet a different approach and applied it to solve (2) in the situation when the statistics T and S^2 are, respectively, the sample mean

Received March 1973; revised March 1974.

[1] Based on part of the author's dissertation written at Harvard University under the guidance of Professor F. Mosteller with partial support of grant GS-32327x of the National Science Foundation.

AMS 1970 *subject classifications.* Primary 62B05, 62F10; Secondary 62J99.

Key words and phrases. UMVU estimation, function of normal distribution.

and the corresponding sum of squares of deviations of a random sample from a $N(\mu, \sigma^2)$ distribution.

In this note we use the method of convolution transforms to show in Section 2 that the solutions to (1) and (2), whenever they exist, are related to each other by the relation $K_{\alpha,\nu}(T, S^2) \equiv H_{\alpha-1,\nu}(T, S^2)$ and obtain an expression for $H_{\alpha,\nu}(T, S^2)$. Thus if $\hat{\eta} = H_{\alpha,\nu}(T, S^2)$ is the UMVU estimator of η, then replacing α by $\alpha - 1$ in its expression gives the corresponding expression for $\hat{\theta} = K_{\alpha,\nu}(T, S^2)$, the UMVU estimator of θ.

In Section 3, we show that the same relationship also holds between the UMVU estimators of $V(\hat{\eta})$ and the difference $V(\hat{\theta}) - \phi^2$, where $V(\hat{\eta})$, $V(\hat{\theta})$ and ϕ^2 are the variances of $\hat{\eta}$, $\hat{\theta}$ and $f(Y)$, respectively. Since expressions for the UMVU estimators of $V(\hat{\theta})$ and ϕ^2 have been derived by Hoyle (1968) the result of Section 3 can be used to obtain the corresponding estimator of $V(\hat{\eta})$ without additional derivations.

2. Relationship between $\hat{\eta}$ and $\hat{\theta}$. We begin by recalling a definition and stating a known result which we shall use in subsequent arguments: (a) the convolution of two integrable functions $h(t)$ and $p(t)$ is defined by

$$\int_{-\infty}^{\infty} h(t)p(x-t)\,dt = h * p ,$$

and (b) its inversion formula is given by

$$h = G(D)[h * p] ,$$

where the inversion operator is defined by $G(D) = \sum_{k=0}^{\infty} (1/k!)G^{(k)}D^k$, where D is the differentiation operator, and $G^{(k)}$ is the kth derivative of $G(x) = 1/\int_{-\infty}^{\infty} \exp(-xt)p(t)\,dt$ evaluated at the origin. A proof and a discussion of the validity of (b) (with slightly different notations) can be found in Widder (1971), Chapter 7.

THEOREM 1. *Assume that $\hat{\eta} = H_{\alpha,\nu}(T, S^2)$ and $\hat{\theta} = K_{\alpha,\nu}(T, S^2)$, the* UMVU *estimators of η and θ based on statistics T and S^2, both exist. Then*

$$K_{\alpha,\nu}(T, S^2) \equiv H_{\alpha-1,\nu}(T, S^2) .$$

PROOF. Note that the relationship stated in Theorem 1 concerns only α. Since the distribution of S^2, the sufficient statistic for σ^2, is independent of T and does not involve α, it suffices to prove the theorem in the case when σ^2 is known. Equivalently, considering the conditional expectation $h_\alpha(T) = E\{H_{\alpha,\nu}(T, S^2) \mid T\}$ it is sufficient to show that $h_{\alpha-1}(T)$ is unbiased for θ.

Since $h_\alpha(T)$ is unbiased for η and $\eta = f(\mu)$, we have $\int_{-\infty}^{\infty} h_\alpha(t)p_\alpha(\mu - t)\,dt = f(\mu)$ where $p_\alpha(t) = (2\pi\alpha\sigma^2)^{-\frac{1}{2}} \exp(-t^2/2\alpha\sigma^2)$. The integral equation can be expressed as a convolution,

$$h_\alpha * p_\alpha = f . \tag{3}$$

Further, taking the convolution of each side of (3) with the density $p_1(t)$, obtained by letting $\alpha = 1$ in $p_\alpha(t)$, we derive

$$(4) \qquad \begin{aligned} (h_\alpha * p_\alpha) * p_1 &= f * p_1 , \\ h_\alpha * (p_\alpha * p_1) &= f * p_1 , \\ h_\alpha * p_{\alpha+1} &= f * p_1 . \end{aligned}$$

Finally, provided $h_{\alpha-1}$ exists, we may replace α by $\alpha - 1$ in (4) to obtain

$$\begin{aligned} h_{\alpha-1} * p_\alpha &= f * p_1 , \\ \int_{-\infty}^{\infty} h_{\alpha-1}(t) * p_\alpha(\mu - t)\, dt &= \int_{-\infty}^{\infty} f(t) p_1(\mu - t)\, dt , \\ E(h_{\alpha-1}(T)) &= Ef(Y) = \theta . \end{aligned} \qquad \square$$

Now we apply the inversion formula (b) to derive an expression for $H_{\alpha,\nu}(T, S^2)$. This result together with Theorem 1 gives the corresponding expression for $K_{\alpha,\nu}(T, S^2)$.

By the inversion formula (b), we have from (3)

$$(5) \qquad \begin{aligned} h_\alpha &= G(D)[h_\alpha * p_\alpha] \\ &= G(D)[f] \\ &= \exp(-\tfrac{1}{2}\alpha\sigma^2 D^2)[f] \end{aligned}$$

where we have utilized the fact that $G(x) = 1/\int_{-\infty}^{\infty} \exp(-xt) p_\alpha(t)\, dt = \exp(-\frac{1}{2}\alpha\sigma^2 x^2)$.

Since the sufficient statistic S^2 is distributed independently of T in order to derive $H_{\alpha,\nu}(T, S^2)$ from $h_\alpha(T)$ it suffices to find an unbiased estimator of $\exp(-\frac{1}{2}\alpha\sigma^2 x^2)$ based on S^2 (for a given x) and combine the result with (2.3). But, for any real number $a \neq 0$ an unbiased estimator of $\exp(\frac{1}{2}a\sigma^2)$ is given by

$$G_\nu(aS^2) = \sum_{k=0}^{\infty} \frac{1}{k!} \left\{ \Gamma\left(\frac{\nu}{2}\right) \Big/ \Gamma\left(\frac{\nu}{2} + k\right) \right\} (\tfrac{1}{4}aS^2)^k ,$$

where Γ denotes the gamma function, (see, e.g., Neyman and Scott (1960), page 652). Consequently,

$$(6) \qquad \begin{aligned} H_{\alpha,\nu}(T, S^2) &= G_\nu(-\alpha S^2 D^2)[f(T)] \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \left\{ \Gamma\left(\frac{\nu}{2}\right) \Big/ \Gamma\left(\frac{\nu}{2} + k\right) \right\} (-\tfrac{1}{4}\alpha S^2)^k f^{(2k)}(T) \end{aligned}$$

where $f^{(2k)}(T)$ denotes the $2k$th derivative of f evaluated at T. It should be noted that a sufficient condition for the series in (6) to converge uniformly is that f be an entire function of second order.

3. Relationship among the estimators of the variances. Let $V(\hat{\theta})$ and $V(\hat{\eta})$ denote the variances of the estimators $\hat{\theta}$ and $\hat{\eta}$, respectively, and let $\hat{V}(\hat{\theta})$ and $\hat{V}(\hat{\eta})$ denote their UMVU estimators based on the sufficient statistics T and S^2. Moreover, let ϕ^2 be the variance of the random variable $f(Y)$, and let $\hat{\phi}^2$ denote its corresponding UMVU estimator. We prove below that the difference $\hat{V}(\hat{\theta}) - \hat{\phi}^2$ is related to $\hat{V}(\hat{\eta})$ in the same manner as $\hat{\theta}$ is related to $\hat{\eta}$.

THEOREM 2. *If the* UMVU *estimators* $\hat{V}(\hat{\theta})$, $\hat{V}(\hat{\eta})$, *and* $\hat{\phi}^2$ *exist, replacing* α *by* $\alpha - 1$ *in the expression of* $\hat{V}(\hat{\eta})$ *yields the difference* $\hat{V}(\hat{\theta}) - \hat{\phi}^2$.

PROOF. For convenience, let us introduce the notation $e_1 \mathrm{R} e_2$ to indicate that expression e_2 may be deduced from e_1 by replacing α by $\alpha - 1$ in e_1. Thus by Theorem 1 we have $\hat{\eta} \mathrm{R} \hat{\theta}$, and by squaring each side of the relationship, $\hat{\eta}^2 \mathrm{R} \hat{\theta}^2$.

Now, let $\gamma = E[f^2(Y)]$ and apply Theorem 1 to the function f^2. It follows that $\widehat{\eta^2} \mathrm{R} \hat{\gamma}$, where $\widehat{\eta^2}$ and $\hat{\gamma}$ denote the UMVU estimators of η^2 and γ, respectively. Finally, combining the two preceding relationships we derive

$$(\hat{\eta}^2 - \widehat{\eta^2}) \mathrm{R} (\hat{\theta}^2 - \hat{\gamma})$$

$$(\hat{\eta}^2 - \widehat{\eta^2}) \mathrm{R} [(\hat{\theta}^2 - \widehat{\theta^2}) - (\hat{\gamma} - \widehat{\theta^2})]$$

$$\hat{V}(\hat{\eta}) \mathrm{R} [\hat{V}(\hat{\theta}) - \hat{\phi}^2] . \qquad \square$$

REFERENCES

GHURYE, S. G. and OLKIN, I. (1969). Unbiased estimation of some probability densities and related functions. *Ann. Math. Statist.* **40** 1261–1271.

GRAY, H. L., WATKINS, T. A. and SCHUCANY, W. R. (1973). On the jackknife statistic and its relation to UMVU estimators in the normal case. *Communications in Statistics* **2** 285–308. (Comments and rejoinder, 308–320.)

HOYLE, M. H. (1968). The estimation of variances after using a Gaussianating transformation. *Ann. Math. Statist.* **39** 1125–1143.

NEYMAN, J., and SCOTT, E. L. (1960). Correction for bias introduced by a transformation of variables. *Ann. Math. Statist.* **31** 643–655.

SCHMETTERER, L. (1960). On a problem of J. Neyman and E. Scott. *Ann. Math. Statist.* **31** 656–661.

WASHIO, Y., MORIMOTO, H. and IKEDA, N. (1956). Unbiased estimation based on sufficient statistics. *Bull. Math. Statist.* **6** 69–93.

WIDDER, D. V. (1971). *An Introduction to Transform Theory*. Academic Press, New York.

C/O VAHABZADEH
11 AVE. EUGENE PITTARD
1206 GENEVA, SWITZERLAND