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# A BEST SEQUENTIAL TEST FOR SYMMETRY WHEN THE PROBABILITY OF TERMINATION IS NOT ONE[1]

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A sequential test of a statistical hypothesis $H_0$ versus $H_1$ is said to be a test of Robbins type if there is a positive probability that the test will not stop if $H_0$ is true. Tests of this nature were introduced for testing the Bernoulli case by Darling and Robbins [1]; an earlier paper of Farrell [2] deals implicitly with the asymptotic expected sample size of such tests for testing the hypothesis $\theta = 0$ in the parametrized family of generalized density functions $h(\theta)e^{\theta x}\,d\mu$.

Let $\langle X_i \rangle_{i=1}^{\infty}$ be independent replicas of a random variable $X$ occurring as the by-product of a process. As long as the distribution function of $X$ satisfies a restriction, it is desired to have the process continue. $H_0$ would assert that the distribution function of $X$ satisfies the restriction, while $H_1$ would contain all undesirable possibilities for the distribution function of $X$. A test of Robbins type $T$ which stops with small probability under $H_0$ but stops the process under $H_1$ with probability one would be of value in regulating the process. In this paper a test of symmetry of Robbins type is constructed which has power one against any non-symmetric alternative. A measure of deviation from symmetry is introduced and the order of magnitude of the expected sample size of the constructed test for small deviations from symmetry is computed. A theorem of Roger Farrell's is cited to demonstrate that the order of magnitude of the expected sample size of the constructed test cannot be improved.

**1. The Smirnov statistic.** Let $X_1, X_2, \cdots, X_n$ be $n$ independent replicas of a random variable $X$ with the continuous distribution function $F$. To test $X$ for symmetry, that is to test the null hypothesis $H_0\, F(-x) = 1 - F(x)$ for all $0 \leqq x$, Smirnov suggested in 1947 [4] that the following statistic be used:

Rank the $n$ replicas $X_1, X_2, \cdots, X_n$ in order of increasing absolute value. Let $N_n{}^+(x)$ denote the number of positive $X_1, X_2, \cdots, X_n$ of absolute value less than or equal to $x$. Let $N^-(x)$ denote the number of negative $X_1, X_2, \cdots, X_n$ of absolute value less than or equal to $x$. The Smirnov statistic is $\sup_x |N^+(x) - N^-(x)|$. For the purposes of this paper advantage will be taken of the identical roles played by $N^+(x)$ and $N^-(x)$ when $X$ is symmetric and calculations will be made on the statistic $T_n = \sup_x [N^+(x) - N^-(x)]$.

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THEOREM 1. (Smirnov [4]). *Let $N = 2n$. Then*

$$P(T_N \geqq 2v) = \sum_{j=v}^{n} \frac{v}{j} \binom{2j}{j+v} 2^{-2j} .$$

**2. An analytical formula for $P(T_N \geqq \gamma N^{\frac{1}{2}})$.** It is necessary to determine how fast the Smirnov statistic grows as successive observations are made on a symmetric random variable. The first step in this direction is to determine an analytical approximation for the distribution function of $T_N$.

The following theorem, a special case of a problem of Uspensky ([5], page 135), which is the local limit theorem for the binomial distribution with $p = \frac{1}{2}$ together with an error term due to Uspensky, is required.

THEOREM 2. *The probability of exactly m successes in n independent trials with constant probability $\frac{1}{2}$ is $(2/\pi n)^{\frac{1}{2}} e^{-t^2/2} + \Delta$, where t is determined by the equation $m = (n + tn^{\frac{1}{2}})/2$ and $|\Delta| < 1.2n^{-\frac{3}{2}} + e^{-\frac{3}{2}n}$ providing $n \geqq 100$.*

Armed with Uspensky's result, the main theorem of this section follows.

THEOREM 3.

$$P(T_N \geqq \gamma N^{\frac{1}{2}}) = (2/\pi)^{\frac{1}{2}} \int_\gamma^\infty e^{-\frac{1}{2}u^2}\, du + R(\gamma)$$

*where R is less in absolute value than a term asymptotic to $(0.4/\gamma^{\frac{1}{2}})N^{-\frac{1}{4}}$ as $N \to \infty$.*

PROOF. Set $N = 2n$, then by Theorem 1:

$$P(T_N \geqq 2v) = \sum_{j=v}^{n} \frac{v}{j} \binom{2j}{j+v} \frac{1}{2^{2j}} .$$

If $2v \geqq 100$, as will be assumed, $P(T_N \geqq 2v) = v \sum_{j=v}^{n} (1/j)((1/(\pi j)^{\frac{1}{2}}) e^{-t_j^2/2} + \Delta_j)$ where $j + v = j + t_j(j/2)^{\frac{1}{2}}$, or $t_j = v(2/j)^{\frac{1}{2}}$, and

$$|\Delta_j| < \frac{(0.15)2^{\frac{3}{2}}}{j^{\frac{3}{2}}} + \exp[-\tfrac{3}{2}(j/2)^{\frac{1}{2}}]$$

by Uspensky's result. Let $f(x) = x^{-\frac{3}{2}} e^{-v^2/x}$ and let $R_n$ satisfy the equation

$$(v/\pi^{\frac{1}{2}}) \sum_{j=v}^{n} f(j) = (v/\pi^{\frac{1}{2}})[\int_0^n f(x)\, dx + R_n] .$$

Then

$$|R_n| \leqq \int_0^v f(x)\, dx + \sum_{j=v}^{n} \int_j^{j+1} |f(x) - f(j)|\, dx .$$

The function $f(x)$ has its maximum at $x = \frac{2}{3}v^2$. If $\frac{2}{3}v^2 < n + 1$ then

$$\sum_{j=v}^{n} \int_j^{j+1} |f(x) - f(j)|\, dx \leqq (f(\tfrac{2}{3}v^2) - f(v)) + (f(\tfrac{2}{3}v^2) - f(n+1)) \leqq 2f(\tfrac{2}{3}v^2) .$$

If $n + 1 \leqq \frac{2}{3}v^2$, then the value of the sum is less than $f(n + 1) - f(v)$ which is less than $f(\frac{2}{3}v^2)$. In the integral, let $y = v/x^{\frac{1}{2}}$ and conclude:

$$|R_n| \leqq (2/v) \int_{v^{\frac{1}{2}}}^\infty e^{-y^2}\, dy + 2(\tfrac{3}{2})^{\frac{3}{2}} v^{-3} e^{-\frac{3}{2}} .$$

The change of variable $y = 2^{\frac{1}{2}} v x^{-\frac{1}{2}}$ yields $(v/\pi^{\frac{1}{2}}) \int_0^n f(x)\, dx = (2/\pi)^{\frac{1}{2}} \int_{(2/n)^{\frac{1}{2}} v}^\infty e^{-y^2/2}\, dy$. Taking $v = \gamma(N^{\frac{1}{2}}/2)$ where $2n = N$ results in

$$P(T_N \geqq \gamma N^{\frac{1}{2}}) = (2/\pi)^{\frac{1}{2}} \int_\gamma^\infty e^{-y^2/2}\, dy + R(\gamma) ,$$

$$\text{where } \quad R(\gamma) \leqq |R_n| + v \sum_{j=v}^{n} |\Delta_{j/j}| .$$

An easy calculation shows that $\lim_{n\to\infty} n^{\frac{1}{2}}|R_n| = 0$ and $\lim_{n\to\infty} N^{\frac{1}{2}} v \sum_{j=v}^{n} |\Delta_{j/j}| = 0.4/\gamma$.

**3. The law of the iterated logarithm for successive Smirnov tests.** In this section it will be shown that the fluctuations to be expected in calculating successive Smirnov statistics when the random variable is symmetric obey the law of the iterated logarithm: $\limsup_{N\to\infty} T_N/(2N \ln \ln N)^{\frac{1}{2}} = 1$. As $T_N \geqq \sum_{i=1}^{N} \text{signum}\,(X_i)$, the ordinary law of the iterated logarithm implies that

$$\limsup_{N\to\infty} T_N/(2N \ln \ln N)^{\frac{1}{2}} \geqq 1 \;.$$

The proof presented here that $\limsup_{N\to\infty} T_N/(2N \ln \ln N)^{\frac{1}{2}} \leqq 1$ parallels closely the proof for the law of the iterated logarithm as presented in Feller [3]. To carry out the proof, the following lemma, which plays the role of a similar lemma in Feller's presentation, is needed.

LEMMA 1. $P\{\max_{1\leqq i\leqq n} T_i \geqq x\} \leqq 2P(T_n \geqq x)$.

PROOF. If $\max_{1\leqq i\leqq n} T_i \geqq x$, since the Smirnov statistic increases by at most one from one trial to the next, there must be a first trial for which the Smirnov statistic equals $x$. Therefore

$$P(\max_{1\leqq i\leqq n} T_i \geqq x) = \textstyle\sum_{i=1}^{n} P(T_i = x \ \text{ and } \ T_j < x, j < i)\,.$$

Now

$$\begin{aligned} P(T_n \geqq x) = \textstyle\sum_{i=1}^{n} & P(T_i = x \ \text{ and } \ T_j < x, j < i) \\ & \times P\{T_n \geqq x \mid T_i = x \ \text{ and } \ T_j < x, j < i\}\,. \end{aligned}$$

If $T_i = x$ as the first $i$ independent observations of $X$ are ranked in increasing order of absolute value then let $k$ designate the place in the ranking at which there are $x$ more positive than negative signs recorded. When the remaining $n - i$ observations are taken, an unknown number $r$ will be inserted into the block of those observations with absolute value less than or equal to the absolute value of that observation holding the $k$th place in rank at the $i$th trial. If the sum of the signums of these $r$ observations is nonnegative, the Smirnov statistic at the $n$th observation will clearly be greater than or equal to $x$. For any $r$, the probability that the sum of the signums of the $r$ observations is nonnegative is the probability that the sum of $r$ independent random variables with probability $\frac{1}{2}$ of being $+1$ and probability $\frac{1}{2}$ of being $-1$ is nonnegative. This probability is always at least one half. Therefore $P(T_n \geqq x \mid T_i = x \text{ and } T_{j<x, j<i}) \geqq \frac{1}{2}$. The equation for $P(T_n \geqq x)$ becomes the inequality:

$$P(T_n \geqq x) \geqq \tfrac{1}{2} \textstyle\sum_{i=1}^{n} P(T_i = x \ \text{ and } \ T_j < x, j < i)\,.$$

The sum, however, is just $P(\max_{1\leqq i\leqq n} T_i \geqq x)$. Therefore $2P(T_n \geqq x) \geqq P(\max_{1\leqq i\leqq n} T_i \geqq x)$. The result that $\limsup_{N\to\infty} T_N/(2N \ln \ln N)^{\frac{1}{2}} \leqq 1$ is now equivalent to the following theorem.

THEOREM 4. *With probability one only finitely many of the events* $T_N > (2\lambda N \ln \ln N)^{\frac{1}{2}}$ *for any* $\lambda > 1$ *are realized.*

PROOF. Let $\gamma_N = (2\lambda \ln \ln N)^{\frac{1}{2}}$ and let $\mu$ be a real number greater than one but less than $\lambda$. Let $n_r$ be the least integer greater than or equal to $\mu^r$. The event, for infinitely many $N$, $T_N \geqq \gamma_N N^{\frac{1}{2}}$, implies the event, for infinitely many $r$, $\max_{1 \leqq i \leqq n_r} T_i \geqq \gamma_{n_{r-1}}(n_{r-1})^{\frac{1}{2}}$. Using Lemma 1 and the Borel-Cantelli Lemma, to prove Theorem 4 it suffices to show $\sum_{r=1}^{\infty} 2P(T_{n_r} \geqq \gamma_{n_{r-1}}(n_{r-1})^{\frac{1}{2}})$ is finite. Using the estimate of $P(T_n \geqq \gamma N^{\frac{1}{2}})$ in Theorem 3 a routine calculation shows the series to be convergent.

**4. The construction of a level $\alpha$ test of Robbins type for symmetry.** Using the law of the iterated logarithm for successive Smirnov tests, it is possible to construct a test of Robbins type at a specified level of significance $\alpha$. If $\lambda > 1$, it is a consequence of the continuity of measure that as $C \to \infty$ the probability $T_N < (2\lambda N \ln \ln CN)^{\frac{1}{2}}$ for all $N \geqq 1$ tends to one. To design a test a $\lambda > 1$ is chosen. Then a number $C$ is chosen so that the probability that $T_N < (2\lambda N \ln \ln CN)^{\frac{1}{2}} - 1$ for all $N$ is at least $1 - \alpha/2$. The Smirnov statistic is calculated at each observation and testing stops, with the rejection of the symmetric hypothesis, the first stage $N$ that it exceeds $(2\lambda N \ln \ln CN)^{\frac{1}{2}} - 1$. With the above choice of $C$ and $\lambda$, the probability of stopping, if the random variable is symmetric, is less than $\alpha$. To show that the power of the test is one—that is, that testing stops with probability one if the random variable is not symmetric—it will be shown that the test has finite expected sample size against any non-symmetric alternative. This will be done using a technique suggested by the referee: the first step is to define a measure of asymmetry.

$$\tau = \sup_{x \geqq 0} |F(x) + F(-x) - 2F(0)| .$$

Define the stopping random variable $N_T(X)$ for a test $T$ of the random variable $X$ to be the positive integer valued random variable which assumes the value $n$ if $T$ stops at the $n$th stage of sampling. Let $T$ be the test of Robbins type described in this section and let $\tau$ be the measure of asymmetry just described. Let $S_n$ denote the Smirnov statistic at stage $n$.

THEOREM 5.

$$\frac{EN_T}{\ln \ln C(EN_T)} \leqq \frac{2\lambda}{\tau^2} .$$

PROOF. Define $\psi(x, t)$ as equal to one if $0 \leqq t \leqq x$, minus one if $-x \leqq t \leqq 0$ and as zero otherwise. It is easily verified that $\sup_x |E\psi(x, X)| = \tau$. As $|\sum_{j=1}^{n} \psi(x, X_j)| \leqq S_n$, $\tau \neq 0$ together with the weak law of large numbers implies that the constructed test stops with probability one. Using Wald's identities, one obtains: $(EN_T)(E\psi(x, X)) \leqq E|\sum_{j=1}^{N_T} \psi(x, X_j)|$. Thus $(EN_T)(E\psi(x, X)) \leqq ES_{N_T}$. Taking the supremum over $x$ on the left-hand side and using $S_{N_T} \leqq (2\lambda N_T \ln \ln CN_T)^{\frac{1}{2}}$ one obtains $(EN_T)(\tau) \leqq E(2\lambda N_T \ln \ln CN_T)^{\frac{1}{2}}$. As $(x \ln \ln Cx)^{\frac{1}{2}}$ is a concave function, Jensen's inequality may be applied yielding: $(EN_T)(\tau) \leqq [2\lambda(EN_T) \ln \ln C(EN_T)]^{\frac{1}{2}}$. From this, Theorem 5 follows. Theorem 5 shows that the order of growth of the expected sample size of the constructed test as $\tau \downarrow 0$

is less than $(2\lambda/\tau^2) \ln \ln 1/\tau$. The constant $\lambda$ may be chosen as close to one as desired, but the constant $C$ tends to infinity as $\lambda \downarrow 1$. Thus there is a trade off between good order properties of the expected sample size for small $\tau$ and quick detection of large departures from symmetry.

**5. The best order properties of the constructed sequential test.** Let $g(x) = P(X > 0 \mid |X| = x)$. Testing $X$ for symmetry is equivalent to testing the assertion $g(x) = \frac{1}{2}$ a.e. under the measure induced on the real line by the distribution function of $|X|$. With each observed value of $|X_i|$ there is associated a Bernoulli random variable $Y_i$ which takes the value 1 if $X_i \geqq 0$ and the value 0 if $X_i < 0$. Consider the testing problem in which $g_\theta(x) = \frac{1}{2} + \frac{1}{2}\theta h(x)$ where $|h(x)| \leqq 1$ and the distribution of $|X|$ does not depend on $\theta$. In this problem the asymptotic order of the expected sample size as $\theta \downarrow 0$ for a test of Robbins type must be at least the minimum order of the expected sample size for a randomized test of Robbins type testing $H_0 p = \frac{1}{2}$ versus $H_A p \neq \frac{1}{2}$ using a sequence of independent Bernoulli random variables each with probability $p(\theta) = \frac{1}{2} + \theta/2$ of equalling one. This statement follows as the statistician can simulate the distribution of $|X|$; if $|X| = x$ he randomly decides with probability $|h(x)|$ whether to observe the Bernoulli variable with unknown bias $\theta$, otherwise he simulates a Bernoulli variable with $p = \frac{1}{2}$. If $h(x) \geqq 0$, the statistician uses the Bernoulli variable specified at the last step; otherwise he uses one minus it.

Theorem 1 in R. H. Farrell's paper [2] proves that the order of the expected sample size must be at least $(\ln \ln 1/\theta)/\theta^{-2}$. For the case in which $h(x) = 1$, $\theta = \tau$ and this shows that the order properties of the expected sample size of the constructed test cannot be improved.

The author had obtained a different and independent proof of the order of magnitude of the asymptotic expected sample size. This proof depended only on the theorem of the mean and the fact that the harmonic series diverges. This elementary proof can be modified to obtain the same result Farrell obtained for the Bernoulli case; however it does not appear possible to derive Farrell's general theorem from purely elementary considerations.

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