The Annals of Statistics
1973, Vol. 1, No. 5, 940-943

A NOTE ON TWO-SIDED CONFIDENCE INTERVALS FOR LINEAR FUNCTIONS OF THE NORMAL MEAN AND VARIANCE[1]

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The confidence sets for linear functions $\mu + \lambda\sigma^2$ of the mean μ and variance σ^2 of a normal distribution, defined in terms of the uniformly most powerful unbiased level α tests of hypotheses of form $H_0(\lambda, m)$: $\mu + \lambda\sigma^2 = m$ against the two-sided alternative $H_1(\lambda, m)$: $\mu + \lambda\sigma^2 \neq m$ for $-\infty < m < \infty$, for fixed α and λ, are shown to be intervals if the number of degrees of freedom for estimating σ^2 is $\geqq 2$.

1. Introduction. Uniformly most accurate unbiased level $1 - \alpha$ confidence procedures for linear functions of the mean μ and variance σ^2 of a normal distribution have been defined in terms of the uniformly most powerful unbiased (UMPU) level α tests of null hypotheses of form H_0: $\mu + \lambda\sigma^2 = m$ against the usual one- and two-sided alternatives ([1]). The procedures defined in terms of tests against one-sided alternatives were shown to give confidence sets that are one-sided intervals, provided that ν, the number of degrees of freedom for estimating σ^2, is at least two. The conjecture that the tests against two-sided alternatives define confidence intervals when $\nu \geqq 2$ was proved only for the case $\nu = 2$. This note completes the proof, for the case $\nu > 2$.

2. The problem. Let Y be normally distributed with mean μ and variance σ^2/γ, where γ is known, and let S^2/σ^2 be independently distributed as chi-square with ν degrees of freedom. For arbitrary λ, the acceptance region of the UMPU level α test of H_0: $\mu + \lambda\sigma^2 = m$ vs. H_1: $\mu + \lambda\sigma^2 \neq m$ is the set of data points (y, s^2) such that $t_1(\nu, -\lambda z, \alpha) \leqq t \leqq t_2(\nu, -\lambda z, \alpha)$, where

$$t = (\gamma\nu)^{\frac{1}{2}}(y - m)/s\,, \qquad z = \gamma^{\frac{1}{2}}[s^2 + \gamma(y-m)^2]^{\frac{1}{2}}/(\nu+1)\,. \tag{1}$$

The critical values $t_1 = t_1(\nu, -\lambda z, \alpha)$ and $t_2 = t_2(\nu, -\lambda z, \alpha)$ are determined by the two equations

$$\int_{t_1}^{t_2} f_\nu(t\,|-\lambda z)\,dt = (1-\alpha)\int_{-\infty}^{\infty} f_\nu(t\,|-\lambda z)\,dt\,, \tag{2}$$

$$\int_{t_1}^{t_2} t(\nu + t^2)^{-\frac{1}{2}} f_\nu(t\,|-\lambda z)\,dt = (1-\alpha)\int_{-\infty}^{\infty} t(\nu+t^2)^{-\frac{1}{2}} f_\nu(t\,|-\lambda z)\,dt\,, \tag{3}$$

Received July 1972; revised January 1973.

[1] Work supported in part by the National Research Council of Canada under Grant Number A8199.

[2] The results in this paper were originally submitted independently by the first two and third authors, respectively.

AMS 1970 *subject classifications.* Primary 62F25; Secondary 62F05.

Key words and phrases. Confidence intervals, normal distribution, linear functions of mean and variance, lognormal distribution.

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where

$$f_\nu(t \mid -\lambda z) = (\nu + t^2)^{-\frac{1}{2}(\nu+1)} \exp\{(\nu + 1)(-\lambda z)t(\nu + t^2)^{-\frac{1}{2}}\}$$

for $-\infty < t < \infty$.

We will show that for any (y, s^2) the set of values m such that $t_1(\nu, -\lambda z, \alpha) \leqq t \leqq t_2(\nu, -\lambda z, \alpha)$ is an interval, for $-\infty < \lambda < \infty$, $0 < \alpha < 1$, and $\nu > 2$. That is, we will show that if $t \leqq t_i$ for $m = m'$ then $t < t_i$ for $m > m'$ $(i = 1, 2)$. This result was proved for the case $\nu = 2$ in [1].

3. The solution. For each $\nu > 0$, the monotone transformation $q_\nu(t) = t/(\nu + t^2)^{\frac{1}{2}}$ maps the real line onto the interval $(-1, 1)$. By applying this transformation to the critical values t_1 and t_2, we define $p_i = zq_\nu(t_i)$ for $i = 1, 2$. These transformed critical values may be shown to satisfy, by (2) and (3), the two equations

$$\int_{p_1}^{p_2} g_k(u)\, du = (1 - \alpha) \int_{-z}^{z} g_k(u)\, du\,, \tag{4}$$

$$\int_{p_1}^{p_2} u g_k(u)\, du = (1 - \alpha) \int_{-z}^{z} u g_k(u)\, du\,, \tag{5}$$

where for $k = \frac{1}{2}\nu - 1$, $-z < u < z$, and $\beta = -\lambda(\nu + 1)$ we have

$$g_k(u) = (z^2 - u^2)^k \exp(\beta u)\,.$$

Since $y - m = (\gamma\nu)^{-\frac{1}{2}} st = [(\nu + 1)/\gamma]zt/(\nu + t^2)^{\frac{1}{2}} = [(\nu + 1)/\gamma]zq_\nu(t)$ the desired result can be obtained by proving that $[(\nu + 1)/\gamma]p_i - (y - m)$ is a monotone increasing function of m. Since p_i is differentiable, it is enough to show that the partial derivative $\partial p_i/\partial m$ is greater than $-\gamma/(\nu + 1)$, for $i = 1, 2$. The partial derivative $\partial z/\partial m$ is obtained from (1) as

$$\partial z/\partial m = -[\gamma/(\nu + 1)](y - m)/[s^2/\gamma + (y - m)^2]^{\frac{1}{2}}\,, \tag{6}$$

which is less than $\gamma/(\nu + 1)$ in absolute value. Therefore it is enough to show that p_i', which we define by $p_i' = \partial p_i/\partial z$, satisfies the inequality $|p_i'| \leqq 1$, for $i = 1, 2$.

For $\beta = 0$, (4) and (5) are satisfied for $p_1 = -p_2$, where

$$\int_{-z}^{p_2} g_k(u)\, du = (1 - \tfrac{1}{2}\alpha) \int_{-z}^{z} g_k(u)\, du\,.$$

Therefore the problem reduces to the one-sided case proved in [1]. For $\beta \neq 0$, integration by parts of (5), taking the anti-derivative of $\exp(\beta u)$, yields

$$\frac{1}{\beta} u g_k(u) \Big|_{p_1}^{p_2} - \frac{1}{\beta} \int_{p_1}^{p_2} [g_k(u) - 2ku^2 g_{k-1}(u)]\, du$$
$$= -(1 - \alpha) \frac{1}{\beta} \int_{-z}^{z} [g_k(u) - 2ku^2 g_{k-1}(u)]\, du\,.$$

After cancellation and rearrangement of terms, plus reduction by (4), we obtain

$$p_2 g_k(p_2) - p_1 g_k(p_1) = -2kz^2\{\int_{p_1}^{p_2} g_{k-1}(u)\, du - (1 - \alpha) \int_{-z}^{z} g_{k-1}(u)\, du\}\,. \tag{7}$$

If we differentiate both sides of (6) with respect to z we obtain

$$p_2' g_k(p_2) - p_1' g_k(p_1) = -2kz\{\int_{p_1}^{p_2} g_{k-1}(u)\, du - (1 - \alpha) \int_{-z}^{z} g_{k-1}(u)\, du\}\,,$$

which implies, by (7)

$$(8) \qquad (p_2' - p_2/z)g_k(p_2) = (p_1' - p_1/z)g_k(p_1) \,.$$

Differentiation of (5) gives

$$(9) \qquad p_2'p_2 g_k(p_2) - p_1'p_1 g_k(p_1) = -2kz[\textstyle\int_{p_1}^{p_2} u g_{k-1}(u)\,du - (1-\alpha)\int_{-z}^{z} u g_{k-1}(u)\,du] \,.$$

Integration by parts of the right-hand side of (9), this time taking the anti-derivative of $u(z^2 - u^2)^{k-1}$, reduces it to $z\{g_k(p_2) - g_k(p_1) - \beta[\int_{p_1}^{p_2} g_k(u)\,du - (1-\alpha)\int_{-z}^{z} g_k(u)\,du]\}$, which by (4) reduces the equation to

$$(10) \qquad (p_2'p_2/z - 1)g_k(p_2) = (p_1'p_1/z - 1)g_k(p_1) \,.$$

Equations (8) and (10) can be solved for p_1' and p_2' in terms of p_1, p_2 and z:

$$(11) \qquad p_1' = [p_1p_2 + (z^2 - p_2^2)g_k(p_2)/g_k(p_1) - z^2]/[z(p_2 - p_1)] \,,$$

$$(12) \qquad p_2' = [p_1p_2 + (z^2 - p_1^2)g_k(p_1)/g_k(p_2) - z^2]/[z(p_1 - p_2)] \,.$$

In particular, it follows from (11) and (12) that $|p_i'| \leqq 1$, $i = 1, 2$, if and only if

$$(13) \qquad (z + p_1)/(z + p_2) \leqq g_k(p_2)/g_k(p_1) \leqq (z - p_1)/(z - p_2) \,.$$

This is expression (7.13) in [1].

The symmetry relation $t_1(\nu, -\lambda z, \alpha) = -t_2(\nu, \lambda z, \alpha)$ ((2.12 in [1]) yields a similar relation for p_1 and p_2 expressed as functions of $\beta = -\lambda(\nu + 1)$,

$$(14) \qquad p_1(-\beta) = -p_2(\beta) \,,$$

where ν, α, and z are assumed to be fixed. If we write $g_k(u, \beta)$ to emphasize that g_k is a function of β as well as u, we see that $g_k(-u, -\beta) = g_k(u, \beta)$. Therefore $g_k(p_1(-\beta), -\beta) = g_k(p_2(\beta), \beta)$, by (14), and it also follows that the first inequality in (13) holds if and only if the second one holds.

If we multiply both sides of (4) by z and add (5) we obtain

$$(15) \qquad \textstyle\int_{p_1}^{p_2} (z + u)g_k(u)\,du = (1 - \alpha)\int_{-z}^{z} (z + u)g_k(u)\,du \,.$$

Now we can integrate both sides of (15) by parts as in the derivation of (7), and after reduction we have

$$(16) \qquad (z + p_2)g_k(p_2) - (z + p_1)g_k(p_1) = -2kz\{\textstyle\int_{p_1}^{p_2} (z + u)g_{k-1}(u)\,du - (1 - \alpha)\int_{-z}^{z} (z + u)g_{k-1}(u)\,du\} \,.$$

Clearly the first inequality in (13), and hence our result, holds if the expression in brackets on the right-hand side of (16) is non-positive.

For $\alpha = 0$, (6) and (7) give $p_1 = -z$, $p_2 = z$, and for $\alpha = 1$ we have $p_1 = p_2$. Therefore the result is true for $\alpha = 0$ or 1, since in each case the right-hand side of (16) is zero. For $0 < \alpha < 1$, we can divide this expression by $1 - \alpha$, and rewrite the integrand, so that the problem is to show the inequality

$$(17) \qquad \textstyle\int_{-z}^{z} (z - u)^{-1}g_k(u)\,du \geqq \dfrac{1}{1 - \alpha}\int_{p_1}^{p_2} (z - u)^{-1}g_k(u)\,du \,.$$

We complete the proof with the aid of the following lemma, whose proof is easy.

LEMMA. *If μ is a finite measure on the real line R, and I is an interval such that*

$$\int_I x \, d\mu(x) = \frac{\mu(I)}{\mu(R)} \int_R x \, d\mu(x) , \tag{18}$$

then

$$\int_I x^2 \, d\mu(x) \leqq \frac{\mu(I)}{\mu(R)} \int_R x^2 \, d\mu(x) . \tag{19}$$

(The sign of strict inequality holds in (19) if $\mu(I) < \mu(R)$.) That is, if the distribution we obtain after conditioning on an interval has the same mean as the unconditional distribution, the variance, and hence the second moment, of the conditional distribution is not increased over that of the unconditional distribution.

We make the transformation of variables $x = 1/(z - u)$ in (4), (5) and (17), so that x varies from $1/(2z)$ to ∞. Now we put

$$f(x) = (z - u)g_k(u) \, du/dx ,$$

where $x = 1/(z - u)$, the density vanishing for $x < 1/(2z)$. Setting $x_1 = 1/(z - p_1)$ and $x_2 = 1/(z - p_2)$, letting $I = (x_1, x_2)$, and defining $\mu(A) = \int_A f(x) \, dx$ for Borel-measurable sets A, it follows from (4) and (5) that (18) is satisfied for μ and I. But (19) is the same as (17), since $x^2 f(x) = (z - u)^{-1} g_k(u) \, du/dx$ for $x = 1/(z - u)$.

A note on the case $\nu = 1$ is in order. It can be shown that, as in the one-sided case treated in [1], there exist values of y, s^2, λ, γ, m, and α such that $\partial p_i/\partial m < -\gamma/(\nu + 1)$ for at least one of $i = 1, 2$. Therefore it does not follow, by the line of reasoning used in the present paper, that confidence intervals are always obtained when $\nu = 1$.

REFERENCE

[1] LAND, C. E. (1971). Confidence intervals for linear functions of the normal mean and variance. *Ann. Math. Statist.* **42** 1187–1205.

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