BAYESIAN VARIABLE SAMPLING PLANS FOR THE EXPONENTIAL DISTRIBUTION WITH TYPE I CENSORING

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In this article, a model of single variable sampling plan with type I censoring is studied. Under the assumption that the variable is exponentially distributed and the loss function is a polynomial, an explicit expression of the Bayes risk is derived. Then, a simple and finite algorithm to determine an optimal sampling plan for minimizing the Bayes risk is suggested. Furthermore, a discretization method is proposed so that one can easily obtain an approximately optimal sampling plan.

1. Introduction. In the realm of quality control, there are various schemes for choosing a single sampling plan [e.g., see Wetherill (1977)]. Among these schemes, the decision approach is probably more scientific and realistic because the sampling plan is determined by making an optimal decision on the basis of some economic considerations such as maximizing the return or minimizing the loss. The research work using this approach has generated an extensive literature; most statisticians working on this problem are confined to a linear loss function, however [see, e.g., Fertig and Mann (1974) and Wetherill and Köllerström (1979)]. Hald (1967, 1981) considered polynomial loss functions, but the optimal sample size derived is by and large not an integer.

Lam (1988a, b) developed a model for single sampling plans with polynomial loss function. In his papers, the quality of an item in a batch is assumed to be a random variable with normal distribution $N(\mu, \tau^2)$, where τ^2 is known and μ has a normal prior distribution; then an explicit expression for the Bayes risk is obtained. He also suggested a finite algorithm for numerical solution to the optimal sampling plan. Later on, Lam and Lau (1993) considered the same problem for the case when τ^2 is also unknown.

In practice, the normal distribution has often been used in modelling many random variables. However, if the quality of an item is measured by its lifetime, the normal distribution may be appropriate only if its negative tail is negligible. In this event, the exponential distribution will be a better alternative in modelling the lifetime data.

Suppose we are given a batch of N products for acceptance sampling. The quality of an item in the batch is measured by its lifetime X, with standard value μ_0. If $X \geq \mu_0$, we shall accept the item without any additional cost. Assume that X has an exponential distribution $\text{Exp}(\lambda)$ with density $f(x) = \lambda e^{-\lambda x}$, $x > 0$. Lam

(1990) showed that when the batch is accepted the cost function is given by

$$
(1.1) \quad N \int_0^{\mu_0} C(\mu_0 - x)\lambda \exp(-\lambda x)\, dx = NC\left\{\mu_0 - \frac{1 - \exp(-\lambda \mu_0)}{\lambda}\right\}
$$
$$
= \alpha_0 + \alpha_1 \lambda + \alpha_2 \lambda^2 + \cdots,
$$

where $C, \alpha_0, \alpha_1, \alpha_2, \ldots$ are constants and independent of λ.

The quality of an item may be measured by the reliability of the item at t_0. Suppose that X is the lifetime of an item. Then the reliability of the item at t_0 is given by $R(t_0) = P(X > t_0)$, with standard value p_0. If $R(t_0) \geq p_0$, we shall accept the item without any extra cost. Assume again that X has an exponential distribution $\text{Exp}(\lambda)$. Lam (1990) showed that the cost function of an accepted batch is given by

$$
(1.2) \quad NC'\big(p_0 - R(t_0)\big) = NC'_{\!\!*}\{p_0 - \exp(-\lambda t_0)\}
$$
$$
= \alpha'_0 + \alpha'_1 \lambda + \alpha'_2 \lambda^2 + \cdots,
$$

where $C', \alpha'_0, \alpha'_1, \alpha'_2, \ldots$ are constants and independent of λ.

In both cases, the exact cost functions are all power functions of λ. Hence, it is more befitting to consider a polynomial cost function rather than a linear cost function, because the former is more accurate in approximating cost functions (1.1) and (1.2).

Furthermore, lifetime data are often censored. For example, in clinical trials, a patient's lifetime may be censored in one of the following forms: loss to follow-up, dropping out or termination of the study. If the cost of inspection increases heavily with time, we may put n items on inspection and terminate it at a preassigned time t_0. This is *type I censoring*. If the items are unduly expensive, we may put n items on inspection and terminate it when a preassigned number of items, say $m\ (\leq n)$, have failed. This is *type II censoring*.

In the past twenty years or so, a great deal of research work has focussed on the acceptance sampling problems for the exponential distribution with type II censoring. Among those are Grubbs (1971), Pierce (1973), Guenther, Patil and Uppuluri (1976), Engelhardt and Bain (1978) and Kocherlakota and Balakrishnan (1986). The same problem was also investigated by Lam (1990). For a polynomial loss function, Lam (1990) derived an optimal single sampling plan for minimizing the Bayes risk. He also suggested a simple algorithm for determining an optimal sampling plan in a finite number of search steps.

In this paper, we deal with the problem of a single sampling plan with a polynomial loss function for the exponential distribution with type I censoring. In Section 2, the model is formulated. After considering the sampling plan (n, t, T), where n is the sample size, t is the preassigned censoring time and T is the minimum acceptance time, an explicit expression for the Bayes risk will be derived. In Section 3, a finite algorithm for determining an optimal sampling plan (n_0, t_0, T_0) is given. Furthermore, a discretization method for finding an approximately optimal sampling plan is proposed, followed by some numerical illustrations. Finally, in Section 4, we explain why the proposed sampling

plan can save some resources. The sensitivities of the parameters of the prior distribution and of the coefficients of the loss function are also discussed.

The same problem for the exponential distribution with random censoring has been studied by Lam and Choy (1993).

2. Model. Suppose that a batch of items is presented for acceptance sampling or further processing. The lifetime of an item is a random variable X which has an exponential distribution $\text{Exp}(\lambda)$ with density function

$$(2.1) \qquad f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

Moreover, λ has a conjugate Gamma prior distribution $\Gamma(a, b)$ with density

$$(2.2) \qquad g(\lambda) = \begin{cases} b^a \lambda^{a-1} e^{-b\lambda} / \Gamma(a), & \lambda > 0, \\ 0, & \lambda \leq 0, \end{cases}$$

where parameters a and b are known.

A random sample $\mathbf{X} = (X_1, \ldots, X_n)$ of size n is taken from the batch, giving an observation $\mathbf{x} = (x_1, \ldots, x_n)$.

As in Lam (1990), the polynomial loss function has the following form:

$$(2.3) \qquad l\{\lambda, \delta(\mathbf{x})\} = \begin{cases} nC_s + a_0 + a_1 \lambda + \cdots + a_k \lambda^k, & \delta(\mathbf{x}) = d_0, \\ nC_s + C_r, & \delta(\mathbf{x}) = d_1, \end{cases}$$

where coefficients $a_0, a_1, \ldots, a_k$, C_r and C_s are constants, C_s refers to the inspection cost per item and C_r is the cost of rejecting a batch, $\delta(\mathbf{x})$ is the decision function, d_0 denotes the decision of accepting the batch and d_1 the decision of rejecting the batch. In addition, we assume that

$$(2.4) \qquad a_0 + a_1 \lambda + \cdots + a_k \lambda^k \geq 0 \qquad \forall \lambda > 0.$$

Inequality (2.4) is a reasonable assumption because the left-hand side of (2.4) represents a part of loss due to accepting the batch and hence must be nonnegative.

Let $X_{(1)} \leq \cdots \leq X_{(n)}$ be the order statistics of $\mathbf{X} = (X_1, \ldots, X_n)$. Since the sample is subject to type I censoring, the true observations are as follows:

$$(2.5) \qquad Y_i = \begin{cases} X_{(i)}, & X_{(i)} \leq t, \\ t, & X_{(i)} > t, \end{cases} \qquad i = 1, 2, \ldots, n.$$

Let $M = \max\{i : X_{(i)} \leq t\}$ be the number of failures by time t. It is known that if $M > 0$, the maximum likelihood estimator (MLE) of the expected lifetime $\theta = 1/\lambda$ is given by

$$(2.6) \qquad \widehat{\theta}_M = \frac{\sum_{i=1}^{n} Y_i}{M} \frac{Y_i}{M} = \frac{\sum_{i=1}^{M} X_{(i)} + (n - M)t}{M};$$

if $M = 0$, then $\widehat{\theta}_M = nt$ [see, e.g., Sinha (1986)]. Obviously, M has a binomial distribution $B(n, p)$ with

$$(2.7) \qquad P(M = m) = \binom{n}{m} p^m (1-p)^{n-m}, \qquad m = 0, 1, \ldots, n,$$

where $p = 1 - e^{-\lambda t}$. Furthermore, for $\lambda > 0$ and $M = m$, the joint density function of $(X_{(1)}, \ldots, X_{(m)})$ is given by

$$(2.8) \qquad f(x_{(1)}, \ldots, x_{(m)}) = \begin{cases} \dfrac{n!}{(n-m)!} \lambda^m \exp\left\{ -\lambda \left(\displaystyle\sum_{i=1}^{m} x_{(i)} + (n-m)t \right) \right\}, \\ \qquad m = 1, \ldots, n, \quad 0 \le x_{(1)} \le \cdots \le x_{(m)} \le t, \\[2mm] 0, \quad \text{otherwise.} \end{cases}$$

Here, it is natural to use the one-sided decision function

$$(2.9) \qquad \delta(\mathbf{X}) = \begin{cases} d_0, & \widehat{\theta}_M \ge T, \\ d_1, & \widehat{\theta}_M < T. \end{cases}$$

As mentioned in Section 1, if the quality of an item is measured by its lifetime X, when $X \ge \mu_0$, we shall accept it without additional cost. In other words, we can accept the batch if the value of θ is large. It follows from (2.6) that we should use the one-sided decision function (2.9). If the quality of an item is measured by its reliability $R(t_0) = \exp(-\lambda t_0)$, when $R(t_0) \ge p_0$, we shall accept the item without extra cost. Since the MLE of $R(t_0)$ is $\exp(-t_0/\widehat{\theta}_M)$, we also have the one-sided decision function (2.9). Besides, if the M-th order statistic $X_{(M)} = t$, then $\widehat{\theta}_M$ is the best linear unbiased estimator of θ [see, e.g., Balakrishnan and Cohen (1991)]. This justifies again our choice of decision function (2.9).

Later on, we may drop the parentheses in the subscripts for convenience.

It follows from (2.6) that $0 \le \widehat{\theta}_M \le nt$. Then, on the basis of (2.9) we can assume that

$$(2.10) \qquad\qquad\qquad 0 < T \le nt.$$

In fact, if $T > nt$, then $\delta(\mathbf{X}) \equiv d_1$ and we shall always reject the batch; if $T = 0$, then $\delta(\mathbf{X}) \equiv d_0$, and we shall accept the batch at all times. However, the Bayes risks for both cases include the term nC_s; they are, respectively, greater than the Bayes risks corresponding to the case without sampling. Hence (2.10) is a natural assumption.

To illustrate the evaluation of the Bayes risk, we assume that the degree of the loss function in (2.3) is 2.

Now, the loss function is quadratic and the Bayes risk is given by

$$r(n, t, T) = E\left\{E\Big(l(\lambda, \delta(\mathbf{X})) \mid \lambda\Big)\right\}$$

$$= E\{(nC_s + a_0 + a_1\lambda + a_2\lambda^2)P(\widehat{\theta}_M \geq T) + (nC_s + C_r)P(\widehat{\theta}_M < T)\}$$

(2.11)
$$= E\left[nC_s + a_0 + a_1\lambda + a_2\lambda^2 + (C_r - a_0 - a_1\lambda - a_2\lambda^2) \right.$$

$$\times \left\{ I_{(nt < T)}P(M = 0) + \sum_{m=1}^{n} \int \cdots \int_{\Delta_m} \frac{n!}{(n-m)!} \right.$$

$$\left. \times \lambda^m \exp\left(-\lambda\left(\sum_{i=1}^{m} x_i + (n-m)t\right)\right) dx_1 \cdots dx_m \right\} \right],$$

where I_A is the indicator of A,

$$\Delta_m = \left\{ (x_1, \ldots, x_m) \,\middle|\, 0 < x_1 \leq \cdots \leq x_m \leq t, \ \sum_{i=1}^{m} x_i < b_m \right\}$$

and $b_m = mT - (n-m)t$.

From (2.2), (2.10) and (A.9), (2.11) becomes

$$r(n, t, T) = nC_s + a_0 + a_1\mu_1 + a_2\mu_2$$

(2.12)
$$+ \int_0^{\infty} (C_r - a_0 - a_1\lambda - a_2\lambda^2) \sum_{m=1}^{n} \frac{n!}{(n-m)!}$$

$$\times \exp\{-\lambda(n-m)t\}G_m(b_m, t)\frac{b^a \lambda^{a-1} e^{-b\lambda}}{\Gamma(a)}\, d\lambda,$$

where μ_1 and μ_2 are the first and second moments of λ about 0, respectively.

Let $[x]$ be the integer part of the real number x, and write

$$m_1 = \left[\frac{nt}{T+t}\right] + 1, \qquad m_2 = \left[\frac{nt}{T}\right] \quad \text{and} \quad m_3 = m_2 + 1.$$

Then, we shall assume tentatively that

(2.13) $$0 < m_1 \leq m_2 < m_3 \leq n.$$

Now, from (A.9), we have the following:

1. $m < m_1 \Longleftrightarrow b_m \leq 0$ and hence

(2.14) $$G_m(b_m, t) = 0;$$

2. $m_1 \leq m \leq m_2 \Longleftrightarrow 0 < b_m \leq mt$ and then

$$G_m(b_m, t)$$

(2.15)
$$= \frac{\lambda^m}{m!(m-1)!} \sum_{j=0}^{[b_m/t]} \int_0^{b_m - jt} \binom{m}{j}(-1)^j u^{m-1} \exp\left[-\lambda(u+jt)\right] du;$$

3. $m \geq m_3 \Longleftrightarrow b_m > mt$ and

$$(2.16) \qquad G_m(b_m, t) = \frac{1}{m!}(1 - e^{-\lambda t})^m.$$

Thus, the Bayes risk (2.12) can be partitioned into three terms:

$$r(n, t, T) = r_1 + r_2 + r_3,$$

where

$$r_1 = nC_s + a_0 + a_1\mu_1 + a_2\mu_2,$$

$$r_2 = \sum_{m=m_1}^{m_2} \int_0^\infty (C_r - a_0 - a_1\lambda - a_2\lambda^2)$$

$$\times \frac{n!}{(n-m)!} \exp\{ - \lambda(n-m)t\} G_m(b_m, t) \frac{b^a \lambda^{a-1} \exp(-b\lambda)}{\Gamma(a)} d\lambda$$

and

$$r_3 = \sum_{m=m_3}^{n} \int_0^\infty (C_r - a_0 - a_1\lambda - a_2\lambda^2)$$

$$\times \frac{n!}{(n-m)!} \exp\{ - \lambda(n-m)t\} G_m(b_m, t) \frac{b^a \lambda^{a-1} \exp(-b\lambda)}{\Gamma(a)} d\lambda.$$

Note that if (2.13) does not hold, then $r(n, t, T)$ will either be the sum of r_1 and r_2 or the sum of r_1 and r_3 only.

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It follows from (2.15) that

$$(2.17) \qquad \begin{aligned} r_2 = \sum_{m=m_1}^{m_2} \sum_{j=0}^{[b_m/t]} \binom{n}{m}\binom{m}{j}(-1)^j \frac{1}{(m-1)!} \frac{b^a}{\Gamma(a)} \\ \times \int_0^{b_m - jt} u^{m-1} \left\{ \frac{(C_r - a_0)\Gamma(m+a)}{(d+jt+u)^{(m+a)}} - \frac{a_1\Gamma(m+a+1)}{(d+jt+u)^{m+a+1}} \right. \\ \left. - \frac{a_2\Gamma(m+a+2)}{(d+jt+u)^{m+a+2}} \right\} du \end{aligned}$$

where $d = (n-m)t + b$. To simplify the expression further, let

$$\beta_y(u, v) = \int_0^y x^{u-1}(1-x)^{v-1} dx, \qquad 0 \leq y \leq 1,$$

be the incomplete Beta function. Then after taking a transformation $w = cx/(1-x)$, we have

$$(2.18) \qquad \int_0^z \frac{w^{u-1}}{(w+c)^{u+v}} dw = \frac{\beta_y(u, v)}{c^v},$$

where $y = z/(z + c)$. Note that $I_y(u, v) = \beta_y(u,v)/\beta(u,v)$ is the Beta distribution function, the values of $I_y(u,v)$ and Beta function $\beta(u,v)$ can be computed using numerical subroutines [e.g., IMSL (1991)]. By (2.18), equation (2.17) is reduced to

$$
r_2 = \sum_{m=m_1}^{m_2} \sum_{j=0}^{[b_m/t]} \binom{n}{m}\binom{m}{j}(-1)^j\left(1/(m-1)!\right)\left(b^a/\Gamma(a)\right)
$$

$$
(2.19) \qquad \times \left\{(C_r - a_0)\Gamma(m+a)(d+jt)^2\beta_\gamma(m,a)\right.
$$

$$
- a_1\Gamma(m+a+1)(d+jt)\beta_\gamma(m,a+1)
$$

$$
\left. - a_2\Gamma(m+a+2)\beta_\gamma(m,a+2)\right\}/(d+jt)^{a+2},
$$

where $\gamma = (b_m - jt)/(b_m + d)$. Moreover, it follows from (2.16) that

$$
r_3 = \sum_{m=m_3}^{n} \sum_{j=0}^{m} \binom{n}{m}\binom{m}{j}(-1)^j b^a
$$

$$
(2.20) \qquad \times \frac{\left\{(C_r - a_0)(d+jt)^2 - a_1 a(d+jt) - a_2 a(a+1)\right\}}{(d+jt)^{a+2}}.
$$

In general, for a loss function given in (2.3) of degree $k > 2$, the Bayes risk can be evaluated in a similar way.

3. A finite algorithm and the discretization method.

Based on the explicit expression of the Bayes risk, a simple algorithm to determine an optimal sampling plan can be implemented using the following steps:

1. Start with $n = 0$; minimize $r(0, t, T)$ with respect to t and T.
2. Move n to 1; minimize $r(1, t, T)$ with respect to t and T.
3. Move n to $n + 1$; repeat the above procedure and continue.
4. By comparison, choose the triple (n_0, t_0, T_0) corresponding to the smallest Bayes risk $r(n, t, T)$.

As a result of the algorithm, $r(n_0, t_0, T_0)$ is the minimum Bayes risk and (n_0, t_0, T_0) is the optimal sampling plan.

The following theorem implies that the algorithm is finite, that is, we can find an optimal sampling plan in a finite number of search steps.

THEOREM 1. *For $n \geq 1$, let $r(n, t_n, T_n) = \min_{t,T} r(n, t, T)$. Then the size n_0 of the optimal sampling plan satisfies*

$$
(3.1) \qquad n_0 \leq \min\left\{\frac{C_r}{C_s}, \frac{a_0 + a_1\mu_1 + a_2\mu_2}{C_s}, \frac{r(n, t_n, T_n)}{C_s}\right\}.
$$

PROOF. From (2.3) and (2.4), it is easy to see that for an optimal sampling plan (n_0, t_0, T_0) we have

$$
(3.2) \qquad\qquad r(n_0, t_0, T_0) \geq n_0 C_s.
$$

Let $(0, 0, \infty)$ be the plan of rejecting the batch without sampling, and let $(0, 0, 0)$ be the plan of accepting the batch without sampling. Then, obviously,

$$(3.3) \qquad r(n_0, t_0, T_0) \leq \min \big(r(0, 0, \infty), r(0, 0, 0), r(n, t_n, T_n) \big).$$

It is clear that $r(0, 0, \infty)) = C_r$ and $r(0, 0, 0) = a_0 + a_1\mu_1 + a_2\mu_2$.

The proof is completed by the combination of (3.2) and (3.3). □

Theorem 1 gives an adaptive upper bound for the size of the optimal sampling plan, which will save the computing time considerably.

The expression of $r(n, t, T)$ is very complicated because $G_m(x, t)$ is not even a continuous function. Many numerical optimization methods, such as Newton–Gauss, steepest descent and conjugate gradient methods, are not applicable in this situation. An alternative method is to consider a sequence of particular sampling plans (n, t, lt), where l takes the values from a discrete set of $(0, n]$. Thus, for fixed n and l, we can minimize $r(n, t, lt)$ with respect to t (it is a one-dimensional problem only !), then determine the minimum Bayes risk and the optimal sampling plan by comparison.

This method can be applied to the case where a lower bound and an upper bound of t can be preassigned. Although t is bounded below by 0, it is not clear in general whether t is also bounded above. However, an alternative way is to find an interval $[t_L, t_U]$ such that

$$P(t_L \leq X \leq t_U) = 1 - \nu,$$

where $0 < \nu < 1$. To do this, we can choose t_L and t_U such that $P(X < t_L) = \nu/2$ and $P(X > t_U) = \nu/2$. Because

$$P(X < t_L) = \int_0^\infty \int_0^{t_L} \lambda e^{-\lambda x} \frac{b^a}{\Gamma(a)} \lambda^{a-1} e^{-b\lambda} \, dx \, d\lambda$$
$$= 1 - \left(1 + \frac{t_L}{b}\right)^{-a} = \frac{\nu}{2},$$

hence

$$(3.4) \qquad t_L = b \left\{ \left(1 - \frac{\nu}{2}\right)^{-1/a} - 1 \right\}.$$

Similarly

$$(3.5) \qquad t_U = b \left\{ \left(\frac{\nu}{2}\right)^{-1/a} - 1 \right\}.$$

Such t_L and t_U can be called a $\nu/2$ lower bound and $\nu/2$ upper bound, respectively.

Although the optimal sampling plan obtained by searching a sequence of particular sampling plans (n, t, lt) is in general not a globally optimal sampling

TABLE 1

The minimum Bayes risks and optimal sampling plans as a and (or) b vary (varies)

(1)		(2)	(3)	(4)	(5)	(6) Risk for true	(7)
a	b	$r(n_0, t_0, T_0)$	n_0	t_0	T_0	parameters	Efficiency
0.2	0.2	12.1499	4	0.0270	0.1080	31.0779	0.8024
2.5	0.4	29.7506	1	0.7978	0.7978	27.3042	0.9133
2.5	0.6	27.7834	3	0.8537	0.4268	25.1406	0.9919
1.5	0.8	16.6233	3	0.5262	0.2631	25.4831	0.9786
2.0	0.8	21.2153	3	0.6051	0.3026	25.0864	0.9940
2.5*	0.8*	24.9367	3	0.7077	0.3539	24.9367*	1.0000
3.0	0.8	27.6136	3	0.8170	0.4085	25.0589	0.9951
3.5	0.8	29.2789	2	1.0037	0.5019	25.7683	0.9677
2.5	1.0	21.7640	3	0.5483	0.2742	25.3408	0.9841
2.5	1.2	18.6097	3	0.4158	0.2079	26.6736	0.9349
10.0	3.0	29.5166	2	0.7928	0.3964	25.2993	0.9857

TABLE 2

The minimum Bayes risks and optimal sampling plans as a_0 varies

a_0	$r(n_0, t_0, T_0)$	n_0	t_0	T_0	Risk for true coefficient	Efficiency
0.1	23.9743	4	0.6539	0.3269	24.9904	0.9979
0.5	24.1874	3	0.6808	0.3404	24.9451	0.9997
1.5	24.6925	3	0.6808	0.3404	24.9451	0.9997
2.0*	24.9367	3	0.7077	0.3539	24.9367*	1.0000
3.0	25.4172	3	0.7346	0.3673	24.9457	0.9996
5.0	26.3287	3	0.7884	0.3942	25.0071	0.9972
10.0	28.2745	2	1.0037	0.5018	25.7682	0.9677

plan, it is at least an approximate one. This approximation is called the *discretization method*.

To illustrate the model, the algorithm and the discretization method developed in Sections 2 and 3, some numerical examples are studied and tabulated in Tables 1–6. The values $a = 2.5, b = 0.8, a_0 = 2, a_1 = 2, a_2 = 2, C_s = 0.5$ and $C_r = 30$ are used for comparison. In each table only two parameters or one coefficient can vary and the others are fixed. As before, $r(n_0, t_0, T_0)$ is used to denote the minimum Bayes risk, while (n_0, t_0, T_0) is the optimal sampling plan.

In these examples, the interval $[t_L, t_U]$ is determined such that $P(t_L \leq X \leq t_U) = 0.95$. For each n, a sequence of sampling plans (n, t, lt) is chosen in the following way:

$$t = t_L + \frac{i(t_U - t_L)}{100}, \qquad i = 0, 1, \ldots, 100,$$

TABLE 3

The minimum Bayes risks and optimal sampling plans as a_1 varies

a_1	$r(n_0, t_0, T_0)$	n_0	t_0	T_0	Risk for true coefficient	Efficiency
0.1	22.7788	4	0.5732	0.2866	25.2265	0.9885
0.5	23.2897	4	0.6001	0.3000	25.1146	0.9929
1.5	24.4325	3	0.6808	0.3404	24.9451	0.9997
2.0*	24.9367	3	0.7077	0.3539	24.9367*	1.0000
3.0	25.8399	3	0.7884	0.3942	25.0071	0.9972
5.0	27.2715	3	0.9499	0.4749	25.4143	0.9812
10.0	29.2151	2	1.5687	0.7844	27.3580	0.9115

TABLE 4

The minimum Bayes risks and optimal sampling plans as a_2 varies

a_2	$r(n_0, t_0, T_0)$	n_0	t_0	T_0	Risk for true coefficient	Efficiency
0.5	15.0859	0	0	0	35.5938	0.7006
1.0	20.8319	3	0.3848	0.1924	27.1835	0.9173
1.5	23.3494	3	0.5463	0.2731	25.3528	0.9836
2.0*	24.9367	3	0.7077	0.3539	24.9367*	1.0000
3.0	26.8155	3	0.9499	0.4749	25.4143	0.9812
5.0	28.5677	3	1.3804	0.6902	26.9175	0.9264
10.0	29.8049	1	1.7032	1.7032	29.1599	0.8552

TABLE 5

The minimum Bayes risks and optimal sampling plans as C_s varies

C_s	$r(n_0, t_0, T_0)$	n_0	t_0	T_0	Risk for true coefficient	Efficiency
0.1	22.6644	11	0.6270	0.3135	27.0644	0.9214
0.3	24.1116	5	0.6808	0.3404	25.1116	0.9930
0.4	24.5696	4	0.6808	0.3404	24.9696	0.9987
0.5*	24.9367	3	0.7077	0.3539	24.9367*	1.0000
0.6	25.2367	3	0.7077	0.3539	24.9367	1.0000
1.0	26.2303	2	0.7077	0.3539	25.2303	0.9884
2.0	27.7605	1	0.7884	0.3942	26.2605	0.9496

TABLE 6

The minimum Bayes risks and optimal sampling plans as C_r varies

C_r	$r(n_0, t_0, T_0)$	n_0	t_0	T_0	Risk for true coefficient	Efficiency
10	10.0000	0	0	∞	30	0.8312
15	14.8625	1	2.1068	1.0534	28.0265	0.8898
20	18.8574	2	1.1382	0.5691	26.1560	0.9534
30*	24.9367	3	0.7077	0.3539	24.9367*	1.0000
40	29.1674	4	0.5194	0.2597	25.5741	0.9751
50	32.1176	5	0.4117	0.2059	27.1179	0.9196
100	35.5938	0	0	0	35.5938	0.7006

and

$$l = \frac{j}{2}, \qquad j = 1, 2, \ldots, 2n.$$

Finally, the minimum Bayes risk is obtained and the corresponding optimal sampling plan is determined by comparison.

For the loss function given in (2.3) of degree $k > 2$, the finite algorithm and the discretization method can be developed similarly.

4. Discussion. According to our model, the optimal sampling plan (n_0, t_0, T_0) should have a smaller Bayes risk than the existing plans. As an example, we consider the operating characteristic (OC) curve sampling plan. Now, the percentage of defectives is given by

$$\varphi = P(X < T) = 1 - \exp(-\lambda T).$$

To determine an OC curve sampling plan (n, t, T), we should start with three points on the OC curve: the producer's risk point $(\varphi_1, 1 - \alpha)$; the customer's risk point (φ_2, β); and a third point, the central point $(\varphi_3, 0.5)$, say. Then, for each given value $\varphi_i, i = 1, 2, 3$, we can solve the equations $\varphi_i = 1 - \exp(-\lambda_i T)$ for λ_i. Finally, for given α and β, the OC curve sampling plan (n, t, T) is determined by solving the following equations:

(4.1) $P(\hat{\theta}_M \geq T) = 1 - \alpha;$

(4.2) $P(\hat{\theta}_M \geq T) = \beta;$

(4.3) $P(\hat{\theta}_M \geq T) = 0.5$

where λ_1, λ_2 and λ_3 are substituted in (4.1), (4.2) and (4.3), respectively, for λ. In general, the OC curve sampling plan (n, t, T) should have a higher Bayes risk than the optimal sampling plan (n_0, t_0, T_0). This means that a higher expense will normally be involved.

Moreover, from Tables 1–6, the size n_0 of the optimal sampling plan (n_0, t_0, T_0) is quite small; so are the censoring time t_0 and the minimum acceptance time T_0. This implies that using the optimal sampling plan will spare some resources such as money, time, manpower and the like.

From the tabulated results, it appears that when six out of seven parameters or coefficients are fixed, $r(n_0, t_0, T_0), n_0, t_0$ and T_0 are, respectively, monotone or unimodal in the remaining one.

In practice, the parameters and coefficients are usually unknown. However, parameters a and b can be estimated by the standard methods, such as the MLE method. Coefficients $a_0, \ldots, a_k$ can be obtained from the Taylor expansion of (1.1) or (1.2). C_s is the inspection cost per item; if the inspection is destructive, the production cost must be taken into account. C_r is the cost of rejecting the batch; it might include the production cost of the batch, a security deposit and the loss of goodwill. Often, the loss of goodwill is difficult to estimate.

If we are unable to estimate the parameters or the coefficients accurately,

what is the consequence if some of the parameters or coefficients are inaccurately estimated? Sensitivity analysis investigates the behavior of the solution due to a change in parameters or coefficients in the model. In Tables 1–6, we envisage that the estimated parameters or the estimated coefficients are given in column 1, the "minimum" Bayes risks and the "optimal" plans are given in columns 2–5. As the minimum Bayes risks and the optimal plans are evaluated using the estimated parameters or coefficients, they are called the estimated Bayes risks and the estimated plans, respectively. Under the estimated plans, by using the true parameters and coefficients, the true (but not the minimum) Bayes risks are shown in column 6. Moreover, assume that the true parameters are $a = 2.5$ and $b = 0.8$, while the true coefficients are $a_0 = 2, a_1 = 2, a_2 = 2, C_s = 0.5$ and $C_r = 30$. In each table only one or two parameters or one coefficient could be incorrectly estimated, since the others are the true values. The true parameters or coefficients and the true minimum Bayes risks are marked by an asterisk in Tables 1–6. For example, in Table 1, if a and b are inaccurately estimated as 3.5 and 0.8, respectively, the estimated Bayes risk is 29.2789 and the estimated plan is (2, 1.0037, 0.5019). By using the estimated plan, as the true a and b are 2.5 and 0.8, respectively, the true Bayes risk (but not the minimum) is 25.7683. Since the true minimum Bayes risk is 24.9367, the efficiency of the estimated sampling plan (2, 1.0037, 0.5019) is 0.9677 = 24.9307/25.7683 [see Hald (1981) for reference]. The sensitivity analysis for the other values of a or b can be carried out by similar treatment.

The sensitivity analysis for the coefficients is exactly the same. For instance, in Table 6, if C_r is incorrectly estimated as 15, the estimated Bayes risk is 14.8625 and the estimated plan is given by (1, 2.1068, 1.0534). By using the estimated plan, since the true value C_r is 30, the true Bayes risk (but not the minimum) is 28.0265. The efficiency of the estimated sampling plan (1, 2.1068, 1.0534) is 0.8898 = 24.9367/28.0265.

From the results in Tables 1–6, we can conclude that in most cases, the efficiencies are higher than 0.9 even if the errors of the parameter or the coefficient are over 100%. For instance, in Table 2, even if a_0 is incorrectly estimated as 10.0 with relative error 400%, the efficiency is 0.9677. It is still very high.

In conclusion, our model is insensitive to the parameters and the coefficients. This is an additional advantage of the proposed sampling plan. Even if we do not know the parameters and coefficients exactly or we cannot estimate them precisely, our model can be employed. In application, in response to a small change in some parameters or coefficients, an obsolete sampling plan may be retained tentatively until there is a chance to adjust it. For all that, a close examination of Tables 1–6 reveals that the minimum Bayes risk seems to be a bit more sensitive to the cost C_r than to the others. Correspondingly, we have to be more attentive about the estimation of C_r. Also, we must pay more attention to the implementation of an optimal sampling plan as C_r changes.

Although this paper deals mainly with the case of quadratic loss function, the model and the algorithm as well as the discretization method are all applicable to the case of polynomial loss function.

APPENDIX

Consider the integral

$$(A.1) \qquad G_m(x, t) = \int \cdots \int_{\substack{0 < x_1 \leq \cdots \leq x_m \leq t \\ \sum_{j=1}^m x_j < x}} \lambda^m \exp\left\{-\lambda \sum_{j=1}^m x_j\right\} dx_1 \cdots dx_m.$$

To evaluate the value of $G_m(x, t)$, first of all, by symmetry we have

$$(A.2) \qquad G_m(x, t) = \frac{1}{m!} \int \cdots \int_{\substack{\sum_{j=1}^m x_j < x \\ 0 < x_j \leq t \, \forall j}} \lambda^m \exp\left\{-\lambda \sum_{j=1}^m x_j\right\} dx_1 \cdots dx_m.$$

Let

$$g_1(x) = \begin{cases} \lambda \exp(-\lambda x), & 0 < x \leq t, \\ 0, & \text{otherwise.} \end{cases}$$

Then (A.2) becomes

$$(A.3) \qquad G_m(x, \, t) = \frac{1}{m!} \int_0^x g_m(u) \, du,$$

where $g_j(x) = \int_0^x g_1(u) g_{j-1}(x - u) \, du$, $j = 2, 3, \ldots, m$, is the j-fold convolution of g_1 with itself. Now assume that $0 < x \leq mt$ and let $G_j(\omega)$ be the Fourier transform of $g_j(x)$, that is,

$$G_j(\omega) = \int_{-\infty}^{\infty} g_j(x) \exp(-i\omega x) \, dx, \qquad j = 1, 2, \ldots, m.$$

Since $g_1 \in L(-\infty, \infty)$, we have

$$(A.4) \qquad G_m(\omega) = G_1(\omega)^m = \frac{\lambda^m}{(\lambda + i\omega)^m} \left\{1 - \exp\left(-t(\lambda + i\omega)\right)\right\}^m.$$

Assume that $m \geq 2$. Then, obviously, $g_m \in L(-\infty, \infty), G_m \in L(-\infty, \infty)$ and

$$(A.5) \qquad g_m(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_m(\omega) \exp(ix\omega) \, d\omega \quad \text{a.s.}$$

[see, e.g., Rudin (1966)]. Substituting (A.4) into (A.5) yields

$$(A.6) \qquad g_m(x) = \frac{\lambda^m}{2\pi} \sum_{j=0}^m \binom{m}{j} (-1)^j \exp(-j\lambda t) A_j \quad \text{a.s.}$$

where

$$A_j = \int_{-\infty}^{\infty} h_j(\omega) \, d\omega \quad \text{and} \quad h_j(\omega) = \frac{1}{(\lambda + i\omega)^m} \exp\left(i\omega(x - jt)\right).$$

To evaluate A_j, we consider two cases separately.

CASE 1 $(x \geq jt)$.   Let C_R be the upper half of the circle $|z| = R$, where $R > \lambda$. By integrating $h_j(z)$ counterclockwise around the boundary of the semicircular region and letting $R \to \infty$, we have

$$A_j = 2\pi i \, \text{Res}\{h_j(\lambda i)\} = 2\pi \exp\left(-\lambda(x - jt)\right) \frac{(x - jt)^{m-1}}{(m-1)!}.$$

CASE 2 $(x < jt)$.   Let C'_R be the lower half of the circle $|z| = R$. Integrating $h_j(z)$ counterclockwise around the boundary of the semicircular region and then letting $R \to \infty$, it follows that $A_j = 0$ for $x < jt$.

Now, (A.6) becomes

$$(\text{A.7}) \qquad g_m(x) = \lambda^m \exp(-\lambda x) \sum_{j=0}^{k} \binom{m}{j} (-1)^j \frac{(x - jt)^{m-1}}{(m-1)!},$$

where $k = \min(m, [x/t])$, and $[x]$ is the integer part of the real number x. Furthermore, direct checking shows that (A.7) is also true for $m = 1$.

In order to obtain a simpler expression of $G_m(x, t)$ for $0 < x \leq mt$, the following lemma is useful.

LEMMA A.   *Assume that $f_j(x), j = 1, 2, \ldots$ is a sequence of integrable functions. Then*

$$(\text{A.8}) \qquad \int_0^x \sum_{j=0}^{[u]} f_j(u) \, du = \sum_{j=0}^{[x]} \int_j^x f_j(u) \, du.$$

PROOF.

$$\int_0^x \sum_{j=0}^{[u]} f_j(u) \, du = \sum_{k=0}^{[x]-1} \int_k^{k+1} \sum_{j=0}^{[u]} f_j(u) \, du + \int_{[x]}^x \sum_{j=0}^{[u]} f_j(u) \, du$$

$$= \sum_{k=0}^{[x]-1} \sum_{j=0}^{k} \int_k^{k+1} f_j(u) \, du + \int_{[x]}^x \sum_{j=0}^{[x]} f_j(u) \, du$$

$$= \sum_{j=0}^{[x]-1} \sum_{k=j}^{[x]-1} \int_k^{k+1} f_j(u) \, du + \sum_{j=0}^{[x]} \int_{[x]}^x f_j(u) \, du$$

$$= \sum_{j=0}^{[x]} \int_j^x f_j(u) \, du.$$

Then, from (A.3), (A.7) and (A.8), we have

$$G_m(x, t) = \frac{\lambda^m}{m!(m-1)!} \int_0^x \sum_{j=0}^{[u/t]} \binom{m}{j} (-1)^j (u - jt)^{m-1} \exp(-\lambda u)\, du$$

$$= \frac{\lambda^m}{m!(m-1)!} \sum_{j=0}^{[x/t]} \int_{jt}^x \binom{m}{j} (-1)^j (u - jt)^{m-1} \exp(-\lambda u)\, du$$

$$= \frac{\lambda^m}{m!(m-1)!} \sum_{j=0}^{[x/t]} \int_0^{x-jt} \binom{m}{j} (-1)^j u^{m-1} \exp\left[-\lambda(u + jt)\right]\, du.$$

Finally, it is clear that if $x \le 0$, then $G_m(x, t) = 0$; if $x > mt$, then (A.3) gives

$$G_m(x, t) = \frac{1}{m!} \left(\int_0^\infty g_1(u)\, du \right)^m = \frac{1}{m!} \left[1 - \exp(-\lambda t) \right]^m.$$

In conclusion, we have the following important result.

THEOREM A.

$$(A.9) \quad G_m(x, t) = \begin{cases} 0, & x \le 0, \\[2mm] \dfrac{\lambda^m}{m!(m-1)!} \displaystyle\sum_{j=0}^{[x/t]} \int_0^{x-jt} \binom{m}{j} (-1)^j u^{m-1} \\[2mm] \qquad\qquad \times \exp\left[-\lambda(u + jt)\right] du, & 0 < x \le mt, \\[3mm] \dfrac{1}{m!} \left[1 - \exp(-\lambda t) \right]^m, & x > mt. \end{cases}$$

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