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SOME IDENTITIES ON q^{n-m} DESIGNS WITH APPLICATION TO MINIMUM ABERRATION DESIGNS[1]

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Chen and Hedayat and Tang and Wu studied and characterized minimum aberration 2^{n-m} designs in terms of their complemetary designs. Based on a new and more powerful approach, we extend the study to identify minimum aberration q^{n-m} designs through their complementary designs. By using MacWilliams identities and Krawtchouk polynomials in coding theory, we obtain some general and explicit relationships between the wordlength pattern of a q^{n-m} design and that of its complementary design. These identities provide a powerful tool for characterizing minimum aberration q^{n-m} designs. The case of $q = 3$ is studied in more details.

1. Introduction. Fractional factorial designs are arguably the most widely used designs in experimental investigations. An important characteristic of a fractional factorial design is its resolution [Box, Hunter and Hunter (1978)]. Often experimenters prefer to use a design with the highest resolution. Fries and Hunter (1980) introduced the minimum aberration criterion for distinguishing 2^{n-m} designs with the same resolution. Franklin (1984) extended this criterion to q^{n-m} fractional factorial designs, where q is a prime power. Chen and Wu (1991) classified 2^{n-m} designs with minimum aberration for $m = 3, 4$ and Chen (1992) constructed 2^{n-5} designs with minimum aberration by using a combination of theoretical and computational tools. A detailed discussion on the minimum aberration criterion for 2^{n-m} designs can be found in Chen, Sun and Wu (1993). The original motivation of the minimum aberration criterion was to provide a natural refinement of the resolution concept. It may also lead to other good overall properties. For example, Krouse (1994) studied optimal first order 2^{n-m} designs which are locally robust to misspecification of the prior distribution parameter. These designs turn out to have minimum aberration.

Because minimum aberration designs play a fundamental role in the practical choice of factorial designs, their characterization is an important problem in design theory. The approach for characterizing minimum aberration 2^{n-m} designs in terms of their complementary designs was employed by H. Chen and Hedayat (1996) and Tang and Wu (1996). H. Chen and Hedayat (1996) proposed the weak minimum aberration criterion (a modified version of the minimum aberration criterion) and gave a theoretical characterization of 2^{n-m}

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fractional factorial designs of resolution III with weak minimum aberration. Tang and Wu (1996) obtained a general result that relates the wordlength pattern of a 2^{n-m} design to that of its complementary design. The approach is very powerful for identifying minimum aberration 2^{n-m} designs whose complementary designs are small. Using this approach, these authors constructed several families of 2^{n-m} designs with minimum aberration.

In this paper, we extend the approach to identify minimum aberration q^{n-m} designs by classifying wordlength patterns of their complementary designs. By using MacWilliam identities and Krawtchouk polynomials, we obtain combinatorial identities that govern the relationships between the wordlength pattern of a q^{n-m} design and that of its complementary design. These identities have explicit forms so that the wordlength pattern of a q^{n-m} design can be readily calculated from that of its complementary design. Applying these results, we are able to construct families of q^{n-m} designs with minimum aberration. In Section 2, we introduce some notation, definitions and representation of a q^{n-m} design and its complementary design. Main theorems and some rules for characterizing minimum aberration designs are given in Section 3. Using these results we are able to characterize several families of 3^{n-m} designs with minimum aberration in Section 4. Our success in applying coding-theoretic techniques to solve a difficult problem in factorial design theory demonstrates the power of these techniques and should serve as an encouragement to further cross-fertilization between coding theory and statistical design theory.

2. Notation and definitions. The q^n full factorial design has n factors, each with q levels, and q^n runs comprising all possible level combinations of the n factors. A q^{n-m} fractional factorial design is a q^{-m} fraction of the q^n design, so it has n factors but q^{n-m} runs. A word of a q^{n-m} design is an n-dimensional vector with components in the finite field $GF(q)$. Associated with every q^{n-m} design is a set of m independent words $W_1, \dots, W_m$ called generators. The set of distinct words formed by $b_1W_1+\cdots+b_mW_m$, $b_i \in GF(q)$ is the defining relations of the fraction. In the defining relations, the word $W_1 = (a_1, \dots, a_n)$ and all its multiples

$$\lambda W_1 = (\lambda a_1, \dots, \lambda a_n) \quad \text{for any } \lambda \neq 0,\ \lambda \in GF(q)$$

are considered to be the same. The defining relations of a q^{n-m} design consist of $(q^m-1)/(q-1)$ nonzero words which form an $(m-1)$-dimensional subspace of $PG(n-1, q)$, the finite projective geometry of dimension $n-1$ over $GF(q)$ [see Raghavarao (1971) for detailed discussion of finite projective geometry]. The length of a word is the number of its nonzero components. Let $D(q^{n-m})$ be a q^{n-m} fractional factorial design and $A_i(D)$ be the number of words of length i in the defining relations of $D(q^{n-m})$. The vector whose entries are $A_i(D)$'s is denoted by

$$W_P(D) = (A_1(D), A_2(D), \dots, A_n(D)),$$

which is called the *wordlength pattern* of $D(q^{n-m})$. The resolution of $D(q^{n-m})$ is the smallest i with positive $A_i(D)$ in its wordlength pattern. For additional information concerning fractional factorial design, see Bose (1947) and Raktoe, Hedayat and Federer (1981).

DEFINITION 1. A q^{n-m} design has maximum resolution if no other q^{n-m} design has larger resolution. Let D_1 and D_2 be two q^{n-m} designs with wordlength patterns $W_P(D_1)$ and $W_P(D_2)$, and let s be the smallest integer such that $A_s(D_1) \neq A_s(D_2)$ in these two wordlength patterns. Then D_1 is said to have less aberration than D_2 if $A_s(D_1) < A_s(D_2)$. A q^{n-m} design has minimum aberration if no other q^{n-m} design has less aberration.

Bose (1961) studied the connections between fractional factorial designs and algebraic coding theory and pointed out the mathematical equivalence between fractional factorial designs and linear codes. It is well known that the concepts of fractional factorial design, wordlength pattern, resolution and defining relation have their counterparts in the context of linear codes. In the following, we will discuss the representation of a q^{n-m} design and its complementary design and their wordlength patterns in the context of algebraic coding theory.

First we briefly describe some of the basic concepts of algebraic coding theory as they pertain to fractional factorial designs. See Pless (1989) for details of basic concepts and notation of algebraic coding theory. Let $V_n(q)$ be the n-dimensional linear vector space over the finite field $GF(q)$. Let G be an $(n-m) \times n$ matrix over $GF(q)$ of rank $n-m$. The set C of all vectors $\mathbf{c}^t = (c_1, \dots, c_n) \in V_n(q)$ such that $G\mathbf{c} = \mathbf{0}$, where $\mathbf{c}^t$ denotes the transpose of the vector $\mathbf{c}$, is called an $[n, m]$ linear code of length n and dimension m. It is easy to see that C is the null space of the matrix G and an m-dimensional subspace of $V_n(q)$ over $GF(q)$. The weight of a vector of C is defined as the number of nonzero coordinates of the vector and $B_i(C)$ denotes the number of vectors in C with weight i. The vector

$$W_D(C) = (B_1(C), \dots, B_n(C))$$

is called the *weight distribution* of the code C. Let C' be the $(n-m)$-dimensional subspace of $V_n(q)$ generated by the rows of G. Then C' is called the dual code of the linear code C. MacWilliams identities in coding theory give a fundamental relationship between the weight distributions of a code and its dual code. There are a number of versions of MacWilliams identities, one of which is given below.

LEMMA 1 (MacWilliams). *If C is a linear $[n, m]$ code over $GF(q)$ with dual code C', $\{B_i(C)\}$ the weight distribution of C, and $\{B_i(C')\}$, the weight distribution of C', are related by*

$$B_i(C') = q^{-m} \sum_{j=0}^{n} P_i(j; n) B_j(C), \tag{1}$$

$$B_i(C) = q^{m-n} \sum_{j=0}^{n} P_i(j; n) B_j(C') \tag{2}$$

for $i = 0, \dots, n$, *where* $P_i(j; n) = \sum_{s=0}^{i}(-1)^s(q-1)^{i-s}\binom{j}{s}\binom{n-j}{i-s}$ *is a Krawtchouk polynomial.*

Throughout this paper, we extend the definition of $\binom{n}{s}$ to allow n and s to be any integers:

$$\binom{n}{s} = \begin{cases} \dfrac{n(n-1)\cdots(n-s+1)}{s(s-1)\cdots 1}, & \text{for positive } s, \\ 1, & \text{for } s = 0, \\ 0, & \text{for negative } s. \end{cases}$$

A q^{n-m} design D can be considered as an $[n, n-m]$ linear code which is the null space of an $m \times n$ matrix G, whose rows are the m generators of D, that is, $W_1, \dots, W_m$. The dual code D' of the linear code D generated by $W_1, \dots, W_m$ has the same structure as the defining relations of D, except that each word in the defining relations corresponds to $q-1$ different vectors in D'. Therefore, the wordlength pattern $\{A_i(D)\}$ of D and the weight distribution $\{B_i(D')\}$ of the dual code D' have the following relation:

$$B_i(D') = (q-1)A_i(D) \quad \text{for } i = 1, \dots, n. \tag{3}$$

Let $k = n - m$ and P be the $k \times (q^k - 1)/(q-1)$ matrix whose columns consist of all the distinct points of $PG(k-1, q)$. All vectors of the k-dimensional subspace generated by the rows of P correspond to a $q^k \times (q^k-1)/(q-1)$ matrix $H_k(q)$; that is,

$$H_k(q) = \{\underbrace{\mathbf{a}_1, \dots, \mathbf{a}_n}_{D}, \underbrace{\mathbf{a}_{n+1}, \dots, \mathbf{a}_{(q^k-1)/(q-1)}}_{\bar{D}}\} = \{D, \bar{D}\}. \tag{4}$$

A q^{n-m} design D of resolution III or higher corresponds to n distinct columns of $H_k(q)$. Without loss of generality, we represent D by the first n columns; that is, $D = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$. Let $\bar{D}$ consist of the remaining $(q^k-1)/(q-1)-n$ columns of $H_k(q)$. We call $\bar{D} = \{\mathbf{a}_{n+1}, \dots, \mathbf{a}_{(q^k-1)/(q-1)}\}$ the *complementary design* of D. The $q^k \times n$ matrix D can also be considered as a k-dimensional linear code with the weight distribution $\{B_i(D)\}$. Let D' be the dual code of D with the weight distribution $\{B_i(D')\}$. In the following we use MacWilliams identities and (3) to link the various weight distributions and wordlength patterns.

Let k^* be the rank of $\bar{D}$, and $\bar{n} = (q^k - 1)/(q-1) - n$. If k^* equals k, the $q^k \times \bar{n}$ matrix $\bar{D}$ can be considered as a k-dimensional linear code. Otherwise, $\bar{D}$ consists of q^{k-k^*} copies of a k^*-dimensional linear code, say D^*. We define $B_i(\bar{D})$ to be the number of row vectors of weight i in $\bar{D}$. Let $\bar{D}'$ be the dual code of $\bar{D}$ (if $k^* < k$, $\bar{D}'$ is the dual code of D^*) with the weight distribution $\{B_i(\bar{D}')\}$. Then MacWilliams identities still hold between the weight distribution $\{B_i(\bar{D})\}$ and the weight distribution $\{B_i(\bar{D}')\}$. As in (3) the wordlength pattern $W_P(\bar{D})$ of the complementary design is related to $\{B_i(\bar{D}')\}$ as follows:

$$(q-1)A_i(\bar{D}) = B_i(\bar{D}') \quad \text{for } i = 1, \dots, \bar{n}. \tag{5}$$

The k-dimensional subspace $H_k(q)$ is the dual code of the Hamming code with $q^k - 1$ vectors of weight q^{k-1} and one vector of weight zero [Peterson and Weldon (1972), page 75]. Since $\{D, \bar{D}\}$ is a partition of $H_k(q)$, a vector in $H_k(q)$ corresponds to a vector in D and a vector in $\bar{D}$. The weight of a vector in $H_k(q)$ is the sum of the weight of a vector in D and the weight of a vector in $\bar{D}$. This relationship allows us to prove the following lemma, which provides the crucial connection between the weight distributions of D and $\bar{D}$. This connection is essential for identifying the relationship between a fractional factorial design and its complementary design.

LEMMA 2. *If $\{B_i(D)\}$ and $\{B_i(\bar{D})\}$ are the weight distributions of the codes D and $\bar{D}$, then*

$$\begin{aligned} B_i(D) &= B_{q^{k-1}-i}(\bar{D}) \quad \textit{for } i = 1, \dots, q^{k-1} - 1, \\ B_i(D) &= 0 \quad \textit{for } i > q^{k-1}, \\ B_0(D) &= 1 + B_{q^{k-1}}(\bar{D}) \quad \textit{and} \\ B_{q^{k-1}}(D) &= B_0(\bar{D}) - 1. \end{aligned} \tag{6}$$

PROOF. For any nonzero vector $\mathbf{v} \in H_k(q) = \{D, \bar{D}\}$, $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2)$, where $\mathbf{v}_1 \in D$ and $\mathbf{v}_2 \in \bar{D}$. The weight of $\mathbf{v}$ is q^{k-1}, which equals the sum of the weight of $\mathbf{v}_1$ and the weight of $\mathbf{v}_2$. Therefore the number of vectors with weight i in D equals the number of vectors with weight $q^{k-1}-i$ in $\bar{D}$ ($i = 1, \dots, q^{k-1}-1$). Also there is no vector in D with weight larger than q^{k-1}. The last two identities in (6) follow from the fact that $\mathbf{0} \in H_k(q)$. □

3. Main theorems. In this section, we will establish the relationship between the wordlength pattern of a q^{n-m} design and the wordlength pattern of its complementary design. From (3) and (5), it suffices to study the connection between the weight distribution of D' and the weight distribution of $\bar{D}'$. Using MacWilliams identities, these weight distributions can be determined by their dual codes. Combining Lemma 2 and MacWilliams identities, we have the following theorem.

THEOREM 1. *The weight distributions of the linear codes D' and $\bar{D}'$ satisfy the following equations*:

$$B_i(D') = C_i + \sum_{j=0}^{\bar{n}} C_{ij} B_j(\bar{D}') \quad \textit{for } i = 0, \dots, n,$$

where $k = n-m$, $C_i = q^{-k}[P_i(0; n) - P_i(q^{k-1}; n)]$, and $C_{ij} = q^{-\bar{n}} \sum_{s=0}^{\bar{n}} P_i(q^{k-1} - s; n) P_s(j; \bar{n})$, for $i = 0, \dots, n$, and $j = 0, \dots, \bar{n}$.

PROOF. From MacWilliams identities and Lemma 2, we have

$$\begin{aligned}
B_i(D') &= q^{-k}\sum_{s=0}^{n} P_i(s;n)B_s(D) \\
&= q^{-k}\left[P_i(0;n) - P_i(q^{k-1};n) + \sum_{s=0}^{q^{k-1}} P_i(s;n)B_{q^{k-1}-s}(\bar{D})\right] \\
&= q^{-k}\left[P_i(0;n) - P_i(q^{k-1};n) + \sum_{s=0}^{q^{k-1}} P_i(q^{k-1}-s;n)B_s(\bar{D})\right] \\
&= q^{-k}\Bigg[P_i(0;n) - P_i(q^{k-1};n) \\
&\qquad + \sum_{s=0}^{\bar{n}} P_i(q^{k-1}-s;n)q^{k-\bar{n}}\sum_{j=0}^{\bar{n}} P_s(j;\bar{n})B_j(\bar{D}')\Bigg] \\
&= q^{-k}[P_i(0;n) - P_i(q^{k-1};n)] \\
&\qquad + q^{-\bar{n}}\sum_{j=0}^{\bar{n}}\left[\sum_{s=0}^{\bar{n}} P_i(q^{k-1}-s;n)P_s(j;\bar{n})\right]B_j(\bar{D}').
\end{aligned}$$

The theorem is proved. □

To further simplify the coefficients C_{ij} in Theorem 1, we need the following known propertities of Krawtchouk polynomials [see MacWilliams and Sloane (1977) for details].

LEMMA 3. *For Krawtchouk polynomials $P_s(j;n)$, we have the following*:

(i) $P_s(j;n) = \binom{n}{s}(q-1)^{s-j}P_j(s;n)/\binom{n}{j}$;
(ii) $[1+(q-1)y]^{n-s}(1-y)^s = \sum_{j=0}^{\infty} P_j(s;n)y^j$; *and*
(iii) $[1+(q-1)z]^{n-q^{k-1}+s}(1-z)^{q^{k-1}-s} = \sum_{i=0}^{\infty} P_i(q^{k-1}-s;n)z^i$.

By applying Lemma 3, the coefficients C_{ij} in Theorem 1 have the following formulas.

THEOREM 2. *The coefficients C_{ij} in Theorem 1 are the following*:

(i) $C_{ij} = 0$, *when* $j > i$;
(ii)
$$C_{ij} = \sum_{t,\,u,\,v\geq 0,\, t+u+v=i-j} \binom{\frac{q^{k-1}-1}{q-1}-\bar{n}}{t}\binom{q^{k-1}-\bar{n}}{u}\binom{\bar{n}-j}{v} \times (-1)^{j+u}(q-2)^v(q-1)^t,$$

when $j \leq i$;
(iii) $C_{i,i-1} = (-1)^i((i-1)q-2i+3)$; *and*
(iv) $C_{ii} = (-1)^i$.

PROOF. From Lemma 3 we have

$$C_{ij} = q^{-\overline{n}} \sum_{s=0}^{\overline{n}} P_i(q^{k-1} - s; n) \binom{\overline{n}}{s} (q-1)^{s-j} P_j(s; \overline{n}) \Big/ \binom{\overline{n}}{j}$$

$$= \left[q^{\overline{n}} (q-1)^j \binom{\overline{n}}{j} \right]^{-1} \sum_{s=0}^{\overline{n}} \binom{\overline{n}}{s} (q-1)^s P_i(q^{k-1} - s; n) P_j(s; \overline{n}).$$

Consider the following polynomial in y and z

$$\sum_{s=0}^{\overline{n}} \binom{\overline{n}}{s} (q-1)^s [1 + (q-1)y]^{\overline{n}-s} (1-y)^s [1 + (q-1)z]^{n - q^{k-1} + s} (1-z)^{q^{k-1} - s}$$

$$(7) \qquad = \sum_{s=0}^{\overline{n}} \binom{\overline{n}}{s} (q-1)^s \left(\sum_{j=0}^{\infty} P_j(s; \overline{n}) y^j \right) \left(\sum_{i=0}^{\infty} P_i(q^{k-1} - s; n) z^i \right)$$

$$= \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \left(\sum_{s=0}^{\overline{n}} \binom{\overline{n}}{s} (q-1)^s P_j(s; \overline{n}) P_i(q^{k-1} - s; n) \right) y^j z^i.$$

The coefficient of $y^j z^i$ in the polynomial (7) equals $q^{\overline{n}} (q-1)^j \binom{\overline{n}}{j} C_{ij}$. The polynomial (7) can be written as

$$[1 + (q-1)z]^{(q^{k-1}-1)/(q-1)} (1-z)^{q^{k-1}}$$

$$\times \sum_{s=0}^{\overline{n}} \binom{\overline{n}}{s} \left[\frac{1 + (q-1)y}{1 + (q-1)z} \right]^{\overline{n}-s} \left[\frac{(q-1)(1-y)}{1-z} \right]^s$$

$$= [1 + (q-1)z]^{(q^{k-1}-1)/(q-1)} (1-z)^{q^{k-1}}$$

$$\times \left[\frac{1 + (q-1)y}{1 + (q-1)z} + \frac{(q-1)(1-y)}{1-z} \right]^{\overline{n}}$$

$$= q^{\overline{n}} [1 + (q-1)z]^{(q^{k-1}-1)/(q-1) - \overline{n}} (1-z)^{q^{k-1} - \overline{n}}$$

$$\times [1 + (q-2)z - (q-1)yz]^{\overline{n}}$$

$$(8) \qquad = q^{\overline{n}} \sum_{t \geq 0} \binom{\frac{q^{k-1}-1}{q-1} - \overline{n}}{t} [(q-1)z]^t \sum_{u \geq 0} \binom{q^{k-1} - \overline{n}}{u} (-z)^u$$

$$\times \sum_{v,\, j \geq 0} \binom{\overline{n}}{j} \binom{\overline{n} - j}{v} [(q-2)z]^v [-(q-1)yz]^j$$

$$= q^{\overline{n}} \sum_{t,\, u,\, v,\, j \geq 0} \binom{\frac{q^{k-1}-1}{q-1} - \overline{n}}{t} \binom{q^{k-1} - \overline{n}}{u} \binom{\overline{n}}{j} \binom{\overline{n} - j}{v}$$

$$\times (-1)^{u+j} (q-2)^v (q-1)^{t+j} y^j z^{t+u+v+j}.$$

The coefficient of $y^j z^i$ in the above polynomial is 0, when $j > i$. Thus, $C_{ij} = 0$.

If $j \le i$, the coefficient of $y^j z^i$ is equal to

$$q^{\bar{n}} \sum_{t,\,u,\,v\ge 0,\, t+u+v=i-j} \binom{\frac{q^{k-1}-1}{q-1}-\bar{n}}{t}\binom{q^{k-1}-\bar{n}}{u}\binom{\bar{n}}{j}\binom{\bar{n}-j}{v} \times (-1)^{u+j}(q-2)^v(q-1)^{t+j}$$

$$= q^{\bar{n}}(q-1)^j\binom{\bar{n}}{j}C_{ij},$$

which proves part (ii).

From the expression for $C_{i,i-1}$, it is easy to show that the summation has three terms by taking $t=1$, $u=1$ or $v=1$ respectively (so that $t+u+v=1$); that is,

$$(-1)^i C_{i,\,i-1} = -\left(\frac{q^{k-1}-1}{q-1}-\bar{n}\right)(q-1) + (q^{k-1}-\bar{n}) - (\bar{n}-i+1)(q-2)$$

$$= (i-1)q - 2i + 3,$$

which is positive for $q \ge 2$.

For $j = i$, we have $t = u = v = 0$ and $C_{ii} = (-1)^i$. This completes the proof. □

COROLLARY 1. *For $q=2$, the coefficients C_{ij} in Theorem 2 can be expressed as follows*:

(i) $C_{ij} = (-1)^{i-[(i-j)/2]}\binom{2^{k-1}-1-\bar{n}}{[(i-j)/2]}$, *for $j \le i$, where $[x]$ is the largest integer less than or equal to x; and*

(ii) $C_{ii} = C_{i,\,i-1} = (-1)^i$.

PROOF. When $q=2$, the polynomial in (8) becomes

$$2^{\bar{n}}(1+z)^{2^{k-1}-1-\bar{n}}(1-z)^{2^{k-1}-\bar{n}}(1-yz)^{\bar{n}}$$

$$= 2^{\bar{n}}(1-z)(1-z^2)^{2^{k-1}-1-\bar{n}}(1-yz)^{\bar{n}}$$

$$= 2^{\bar{n}}(1-z)\sum_{u\ge 0}\binom{2^{k-1}-1-\bar{n}}{u}(-z^2)^u \sum_{j\ge 0}\binom{\bar{n}}{j}(-yz)^j$$

$$= 2^{\bar{n}}(1-z)\sum_{u,\,j\ge 0}\binom{2^{k-1}-1-\bar{n}}{u}\binom{\bar{n}}{j}(-1)^{u+j}y^j z^{2u+j}$$

$$= 2^{\bar{n}}\sum_{u,\,j\ge 0}\binom{2^{k-1}-1-\bar{n}}{u}\binom{\bar{n}}{j}(-1)^{u+j}y^j z^{2u+j}$$

$$+ 2^{\bar{n}}\sum_{u,\,j\ge 0}\binom{2^{k-1}-1-\bar{n}}{u}\binom{\bar{n}}{j}(-1)^{u+j+1}y^j z^{2u+j+1}.$$

The coefficient of $y^j z^i$ is

$$2^{\bar{n}}\binom{2^{k-1}-1-\bar{n}}{\frac{i-j}{2}}\binom{\bar{n}}{j}(-1)^{(i+j)/2} \quad \text{if } i-j \text{ is even and}$$
$$2^{\bar{n}}\binom{2^{k-1}-1-\bar{n}}{\frac{i-j-1}{2}}\binom{\bar{n}}{j}(-1)^{(i+j+1)/2} \quad \text{if } i-j \text{ is odd.}$$

Therefore, the coefficient of $y^j z^i$ is

$$2^{\bar{n}}\binom{2^{k-1}-1-\bar{n}}{\left[\frac{i-j}{2}\right]}\binom{\bar{n}}{j}(-1)^{i-[(i-j)/2]}.$$

Following the same argument in the proof of Theorem 2, the coefficient of $y^j z^i$ in the polynomial is also equal to $2^{\bar{n}}\binom{\bar{n}}{j}C_{ij}$. Hence,

$$C_{ij} = (-1)^{i-[(i-j)/2]}\binom{2^{k-1}-1-\bar{n}}{\left[\frac{i-j}{2}\right]}.$$

Part (ii) is a special case of Theorem 2(iii) and (iv).

From (3) and (5), Theorems 1 and 2, the wordlength pattern of design D can be calculated in terms of the wordlength pattern of its complementary design $\bar{D}$.

THEOREM 3. *Let $W_P(D)$ and $W_P(\bar{D})$ be the wordlength patterns of a q^{n-m} design D and its complementary design $\bar{D}$ respectively. Then*

$$A_i(D) = (q-1)^{-1}(C_i + C_{i0}) + \sum_{j=3}^{i-2} C_{ij}A_j(\bar{D}) + (-1)^i(((i-1)q-2i+3)A_{i-1}(\bar{D}) + A_i(\bar{D})), \tag{9}$$

for $i = 3,\ldots,n$, where C_i and C_{ij} are given in Theorems 1 and 2.

Note that $A_i(\bar{D})$ in (9) are 0, for $i > \bar{n}$ or $i = 1$ and 2. Because $(i-1)q-2i+3$ in (9) is positive for $q \geq 2$ and $i \geq 3$, the signs of the coefficients for the two highest terms $A_{i-1}(\bar{D})$ and $A_i(\bar{D})$ are both positive for even i and negative for odd i. The equations in (9) generalize the result of Tang and Wu (1996) which only dealt with 2^{n-m} designs. Unlike the latter, the coefficients in (9) are explicit and can be easily computed.

For $q = 2$, (9) takes a simpler and more elegant form. The following identities in (10) were not available in Tang and Wu (1996).

COROLLARY 2. *Let $W_P(D)$ and $W_P(\bar{D})$ be the wordlength patterns of a 2^{n-m} design D and its complementary design $\bar{D}$, respectively. Then*

$$A_i(D) = C_i + C_{i0} + \sum_{j=3}^{i-2} C_{ij} A_j(\bar{D}) + (-1)^i (A_{i-1}(\bar{D}) + A_i(\bar{D})) \quad \text{for } i = 3, \dots, n, \tag{10}$$

where C_i are given in Theorem 1, and

$$C_{ij} = (-1)^{i-[(i-j)/2]} \binom{2^{k-1} - 1 - \bar{n}}{\left[\frac{i-j}{2}\right]}.$$

The signs of the coefficients for $A_i(\bar{D})$ in (10) have an interesting pattern. For even i, the coefficients of the two highest terms (i.e., $A_{i-1}(\bar{D})$ and $A_i(\bar{D})$) are positive, then followed alternately by two negative signs, then two positive signs, and so on. For odd i, the signs have the same alternating pattern, except that it starts with two negative coefficients for $A_{i-1}(\bar{D})$ and $A_i(\bar{D})$.

The identities in Theorem 3 give an important relationship between the wordlength patterns of a q^{n-m} design and its complementary design. This relationship is particularly useful for characterizing minimum aberration designs through their complementary designs when the size of the complementary design $\bar{n}$ is small. From Theorems 2 and 3, we have the following identities:

$$\begin{aligned} A_3(D) &= (q-1)^{-1}(C_3 + C_{30}) - A_3(\bar{D}), \\ A_4(D) &= (q-1)^{-1}(C_4 + C_{40}) + (3q-5)A_3(\bar{D}) + A_4(\bar{D}). \end{aligned} \tag{11}$$

Using the identities in (11), we can establish some general rules for identifying q^{n-m} designs with minimum aberration.

RULE 1. A q^{n-m} design D^* with $\overline{D^*}$ as its complementary design (of size $\bar{n}$) has minimum aberration if the following are given:

(i) $A_3(\overline{D^*})$ is the maximum among all complementary designs of size $\bar{n}$;
(ii) $\overline{D^*}$ is the unique set satisfying (i).

RULE 2. A q^{n-m} design D^* has minimum aberration if the following are given:

(i) $A_3(\overline{D^*})$ is the maximum among all complementary designs of size $\bar{n}$;
(ii) $A_4(\overline{D^*})$ is the minimum among all complementary designs of size $\bar{n}$ whose number of words of length three equals $A_3(\overline{D^*})$;
(iii) $\overline{D^*}$ is the unique set satisfying (ii).

More generally, by noting the relation

$$A_i(D) = (-1)^i A_i(\bar{D}) + \text{lower order terms},$$

we have from (9) the following general rule.

D^* has minimum aberration if its complementary design $\bar{D}^*$ is the unique design of size $\bar{n}$ that maximizes $A_3(\bar{D})$, $A_5(\bar{D}), \ldots, A_{2v-1}(\bar{D})$(or $A_{2v+1}(\bar{D})$) and minimizes $A_4(\bar{D})$, $A_6(\bar{D}), \ldots$, and $A_{2v}(\bar{D})$ lexicographically.

For application to physical experiments whose run size and number of factors cannot be too large, calculation of $A_i(\bar{D})$ with $i \geq 6$ are rarely needed. For application to larger experiments such as computer experiments, the general rule may be useful.

Now we apply Rule 1 to identify a 3^{9-6} design with minimum aberration. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three independent columns of $H_3(3)$, where $H_3(3)$ is a 27×13 matrix as defined in (4), and denote $\mathbf{a}^i\mathbf{b}^j\mathbf{c}^l = i\mathbf{a} + j\mathbf{b} + l\mathbf{c}$, where i, j and l are in $GF(3)$:

$$H_3(3) = \{\mathbf{a}, \mathbf{b}, \mathbf{ab}, \mathbf{ab}^2, \mathbf{c}, \mathbf{ac}, \mathbf{bc}, \mathbf{abc}, \mathbf{ab}^2\mathbf{c}, \mathbf{ac}^2, \mathbf{bc}^2, \mathbf{abc}^2, \mathbf{ab}^2\mathbf{c}^2\}.$$

Any 3^{9-6} design can be determined by a subset of nine columns from $H_3(3)$. Its complementary design $\bar{D}$ has the remaining four columns from $H_3(3)$. There are three different complementary designs, that is, $\bar{D}_1 = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{abc}\}$, $\bar{D}_2 = \{\mathbf{a}, \mathbf{b}, \mathbf{ab}, \mathbf{c}\}$ and $\bar{D}_3 = \{\mathbf{a}, \mathbf{b}, \mathbf{ab}, \mathbf{ab}^2\}$. They have the following wordlength patterns:

$$W_P(\bar{D}_1) = (0, 0, 0, 1),$$
$$W_P(\bar{D}_2) = (0, 0, 1, 0),$$
$$W_P(\bar{D}_3) = (0, 0, 4, 0).$$

Let D_1, D_2 and D_3 be the corresponding designs of nine cloumns. Since $\bar{D}_3$ has the maximum number of words of length three among $\bar{D}_1$, $\bar{D}_2$, and $\bar{D}_3$, the 3^{9-3} design D_3, which consists of the last nine columns of $H_3(3)$, has minimum aberration. From (9), we have

$$A_3(D) = 16 - A_3(\bar{D}),$$
$$A_4(D) = 38 + 4A_3(\bar{D}) + A_4(\bar{D}),$$
$$A_5(D) = 74 - 5A_3(\bar{D}) - 5A_4(\bar{D}),$$
$$A_6(D) = 96 + 10A_4(\bar{D}),$$
$$A_7(D) = 88 + 5A_3(\bar{D}),$$
$$A_8(D) = 43 - 4A_3(\bar{D}) + 5A_4(\bar{D}),$$
$$A_9(D) = 9 + A_3(\bar{D}) - A_4(\bar{D}).$$

The wordlength patterns of D_1, D_2 and D_3 can be calculated from those of their complementary designs,

$$W_P(D_1) = (0, 0, 16, 39, 69, 106, 78, 48, 8),$$
$$W_P(D_2) = (0, 0, 15, 42, 69, 96, 93, 39, 10),$$
$$W_P(D_3) = (0, 0, 12, 54, 54, 96, 108, 27, 13).$$

Using these rules, we are able to identify families of minimum aberration designs. To illustrate the application of these rules, we will study minimum aberration 3^{n-m} designs in the next section.

4. 3^{n-m} designs with minimum aberration. In this section, we identify families of minimum aberration 3^{n-m} designs whose complementary designs have size $\overline{n} = 1, 2, \ldots, 13$.

It is easy to see that designs obtained by deleting any one or two columns of $H_k(q)$ have minimum aberration [see Pu (1989)]. Pu (1989) and Chen, Sun and Wu (1993) classified complementary designs of size $\overline{n}$ for $\overline{n} = 3, \ldots, 13$. Each of the following complementary designs of size $\overline{n}$ is the unique one with the maximum number of words of length three. Their uniqueness can be verified by computer enumerations or combinatorial arguments:

$$\begin{aligned}
\overline{D}_3 &= \{\mathbf{a}, \mathbf{b}, \mathbf{ab}\},\\
W_P(\overline{D}_3) &= (0, 0, 1),\\
\overline{D}_4 &= PG(1, 3),\\
W_P(\overline{D}_4) &= (0, 0, 4, 0),\\
\overline{D}_5 &= \{\mathbf{a}, \mathbf{b}, \mathbf{ab}, \mathbf{ab}^2, \mathbf{c}\},\\
W_P(\overline{D}_5) &= (0, 0, 4, 0, 0),\\
\overline{D}_6 &= \{\mathbf{a}, \mathbf{b}, \mathbf{ab}, \mathbf{ab}^2, \mathbf{c}, \mathbf{ac}\},\\
W_P(\overline{D}_6) &= (0, 0, 5, 3, 3, 2),\\
\overline{D}_7 &= \{\mathbf{a}, \mathbf{b}, \mathbf{ab}, \mathbf{ab}^2, \mathbf{c}, \mathbf{ab}^2\mathbf{c}, \mathbf{ab}^2\mathbf{c}^2\},\\
W_P(\overline{D}_7) &= (0, 0, 8, 9, 9, 14, 0),\\
\overline{D}_8 &= \{\mathbf{a}, \mathbf{b}, \mathbf{ab}, \mathbf{ab}^2, \mathbf{c}, \mathbf{ab}^2\mathbf{c}, \mathbf{bc}^2, \mathbf{ab}^2\mathbf{c}^2\},\\
W_P(\overline{D}_8) &= (0, 0, 11, 21, 30, 38, 15, 6),\\
\overline{D}_9 &= \{\mathbf{a}, \mathbf{b}, \mathbf{ab}^2, \mathbf{c}, \mathbf{ab}^2\mathbf{c}, \mathbf{ac}^2, \mathbf{bc}^2, \mathbf{abc}^2, \mathbf{ab}^2\mathbf{c}^2\},\\
W_P(\overline{D}_9) &= (0, 0, 16, 39, 69, 106, 78, 48, 8),\\
\overline{D}_{10} &= \{\mathbf{a}, \mathbf{b}, \mathbf{ab}, \mathbf{ab}^2, \mathbf{c}, \mathbf{ac}, \mathbf{bc}, \mathbf{abc}, \mathbf{ac}^2, \mathbf{bc}^2\},\\
W_P(\overline{D}_{10}) &= (0, 0, 22, 68, 138, 250, 290, 213, 92, 20),\\
\overline{D}_{11} &= \{\mathbf{a}, \mathbf{b}, \mathbf{ab}, \mathbf{ab}^2, \mathbf{c}, \mathbf{ac}, \mathbf{bc}, \mathbf{abc}, \mathbf{ac}^2, \mathbf{bc}^2, \mathbf{abc}^2\},\\
W_P(\overline{D}_{11}) &= (0, 0, 30, 108, 252, 546, 810, 765, 517, 216, 36),\\
\overline{D}_{12} &= \{\mathbf{a}, \mathbf{b}, \mathbf{ab}, \mathbf{ab}^2, \mathbf{c}, \mathbf{ac}, \mathbf{bc}, \mathbf{abc}, \mathbf{ac}^2, \mathbf{bc}^2, \mathbf{abc}^2, \mathbf{ab}^2\mathbf{c}^2\},\\
W_P(\overline{D}_{12}) &= (0, 0, 40, 162, 432, 1092, 1944, 2295, 2068, 1296, 432, 80),\\
\overline{D}_{13} &= PG(2, 3),\\
W_P(\overline{D}_{13}) &= (0, 0, 52, 234, 702, 2028, 4212, 5967, 6721, 5616, 2808, 1040, 144).
\end{aligned}$$

By deleting complementary designs with these structures from $H_k(3) (k = n - m)$, the resulting matrices give 3^{n-m} designs with minimum aberration.

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