The Annals of Statistics
1996, Vol. 24, No. 2, 753–762

SOME INEQUALITIES FOR SYMMETRIC CONVEX SETS WITH APPLICATIONS

By T. W. Anderson

Stanford University

Under appropriate conditions the probability of a convex symmetric set decreases as the spread or scatter of the distribution increases. This paper studies the conditions when the random vector has a symmetric unimodal distribution.

1. Introduction. This paper is an extension and application of the following theorem about the integral of a symmetric unimodal density over a symmetric convex set [Anderson (1955)]:

THEOREM (Anderson). *Let C be a convex set, symmetric about the origin. Let $f(\mathbf{y}) \geq 0$ be a function such that* (i) $f(\mathbf{y}) = f(-\mathbf{y})$, (ii) $\{\mathbf{y} | f(\mathbf{y}) \geq u\}$ *is convex for every u, $0 < u < \infty$ and* (iii) $\int_C f(\mathbf{y})\, d\mathbf{y} < \infty$. *Then, for $0 \leq k \leq 1$,*

$$\int_C f(\mathbf{y} + \mathbf{x})\, d\mathbf{y} \leq \int_C f(\mathbf{y} + k\mathbf{x})\, d\mathbf{y}. \tag{1}$$

This theorem was deduced from the following lemma (which was proved, but not stated explicitly). Let C and E be convex sets in $\mathbb{R}^p$, symmetric about the origin. Then $V_p\{(E + \mathbf{x}) \cap C\} \leq V_p\{(E + k\mathbf{x}) \cap C\}$ for $0 \leq k \leq 1$. Here $V_p\{\cdot\}$ indicates the volume of the set. These propositions have had many consequences in probability and statistics; see Perlman (1990) for an informative exposition of some of them.

Such an inequality may be interpreted as indicating the effect on the probability of a convex set, symmetric about the origin, of a change in the location parameter of a symmetric unimodal density. The present paper is a study of the effect of a change in spread or scatter of such a density. Suppose the symmetric unimodal density is centered at the origin and $\mathbf{y}$ is linearly transformed by an expansion or dilation. Under some conditions given below the probability of the symmetric convex set C decreases. However, an example is given to show that this inequality is not independent of the conditions.

One implication of the 1955 theorem was that the probability of a symmetric convex set for a random vector normally distributed with mean $\mathbf{0}$ and covariance $\boldsymbol{\Sigma}$ is greater than for a random vector normally distributed with mean $\mathbf{0}$ and covariance matrix $\boldsymbol{\Phi}$ if $\boldsymbol{\Phi} - \boldsymbol{\Sigma}$ is positive semidefinite. (When $\mathbf{Y}$

Received October 1993; revised March 1995.
AMS 1991 *subject classifications.* 60D05, 60E15, 62H15.
Key words and phrases. Convex sets, inequalities, unimodal functions, elliptically contoured distributions, monotonicity of power functions.

has the covariance matrix $\mathbf{\Sigma}$ and $\mathbf{X}$ the covariance matrix $\mathbf{\Phi} - \mathbf{\Sigma}$, the conditional distribution of $\mathbf{Y} + \mathbf{X}$ given $\mathbf{X} = \mathbf{x}$ satisfies the condition of the 1955 inequality.) This inequality was extended to a more general class of elliptically contoured densities by Fefferman, Jodeit and Perlman (1972) and by Das Gupta, Eaton, Olkin, Perlman, Savage and Sobel (1972); the two sets of authors applied Anderson's inequality in different ways. While the general result for elliptically contoured densities is almost a corollary of the basic theorem in this current paper, a broader application can be made.

Among other applications in statistics of the 1955 theorem was the monotonicity of power functions of tests of hypotheses concerning location parameters, such as regression coefficients, in normal distributions. The current theorem can similarly be applied to show the monotonicity of power functions of tests concerning covariance matrices.

To motivate the basic result, consider the probability of a symmetric convex set on the basis of a given distribution and compare that probability with the probability of that set on the basis of a distribution more spread out. Perhaps the simplest case is that of a random variable uniformly distributed over a symmetric convex set compared with a random variable uniformly distributed over the set stretched out in the direction of the first coordinate axis.

For a set C in $\mathbb{R}^p$ let $C(x) = \{\mathbf{y} | \mathbf{y} \in C,\ y_1 = x\}$, where $\mathbf{y} = (y_1, \dots, y_p)'$. For $0 \le \alpha \le 1$ and C_1 and C_2 two convex sets, define

$$\alpha C_1 + (1-\alpha) C_2 = \{\mathbf{y} | \mathbf{y} = \alpha \mathbf{u} + (1-\alpha)\mathbf{v}, \mathbf{u} \in C_1, \mathbf{v} \in C_2\}. \tag{2}$$

Let $V_p\{C\}$ be the Lebesgue measure of $C \subset \mathbb{R}^p$, and let $V_{p-1}\{C(x)\}$ be the Lebesgue measure of $C(x)$ in the hyperplane $y_1 = x$. We disregard sets of measure zero.

2. Inequalities for contents of symmetric convex sets.

THEOREM 1. *Let E and C be convex sets, symmetric about the origin, such that, for a scalar x,*

$$E(x) - x\boldsymbol{\varepsilon}_1 = E(-x) + x\boldsymbol{\varepsilon}_1, \tag{3}$$

where $\boldsymbol{\varepsilon}_1 = (1, 0, \dots, 0)'$. *Let*

$$E_k = \{\mathbf{y} | \mathbf{y} = \mathbf{F}\mathbf{z}, \mathbf{z} \in E\}, \tag{4}$$

where

$$\mathbf{F} = \begin{bmatrix} k & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{p-1} \end{bmatrix}, \qquad 0 \le k \le 1. \tag{5}$$

Then

$$V_{p-1}\{(E \cap C)(x)\} \le V_{p-1}\{(E_k \cap C)(kx)\}. \tag{6}$$

PROOF. For $\alpha = (k+1)/2$, the set

$$\alpha(E \cap C)(x) + (1-\alpha)(E \cap C)(-x) \subset (E_k \cap C)(kx) \tag{7}$$

since E and C are convex and $E_k(kx) = E(x) - (1-k)x\boldsymbol{\varepsilon}_1$. Hence

$$(8)\quad V_{p-1}\{\alpha(E\cap C)(x) + (1-\alpha)(E\cap C)(-x)\} \le V_{p-1}\{(E_k\cap C)(kx)\}.$$

Since $(E\cap C)(-x) = -(E\cap C)(x)$, $V_{p-1}\{(E\cap C)(x)\} = V_{p-1}\{(E\cap C)(-x)\}$. Then

$$(9)\quad V_{p-1}\{(E\cap C)(x)\} \le V_{p-1}\{\alpha(E\cap C)(x) + (1-\alpha)(E\cap C)(-x)\}$$

by the Brunn–Minkowski theorem, which states that $(1-\theta)V_n^{1/n}\{E_0\} + \theta V_n^{1/n}\{E_1\} \le V_n^{1/n}\{(1-\theta)E_0 + \theta E_1\}$ for E_0 and E_1 convex and nonempty sets in $\mathbb{R}^n$ and $0 \le \theta \le 1$. The theorem follows from (8) and (9). □

Note that the Brunn–Minkowski theorem has been applied to $E_1 = (C\cap E)(x) - (k-1)x\boldsymbol{\varepsilon}_1$, $E_0 = (C\cap E)(-x) + (k+1)x\boldsymbol{\varepsilon}_1$ with $n = p-1$ and $\theta = \alpha$.

COROLLARY 1. *Let C be a convex set in $\mathbb{R}^p$, symmetric about the origin. Then $V_{p-1}\{C(x)\} \le V_{p-1}\{C(kx)\}$ for $0 \le k \le 1$.*

PROOF. In Theorem 1 let E and $E_k = \mathbf{F}E$ be such that $C \subset E_k$. Then Theorem 1 implies Corollary 1. □

This corollary also follows from a theorem of Prékopa (1971, 1973), Rinott (1976) and Brascamp and Lieb (1976) to the effect that the marginal integral of a log-concave function is log-concave.

THEOREM 2. *Let E and C be convex sets, symmetric about the origin. Suppose* (3) *holds for almost every x. Define E_k by* (4) *and* (5). *Then*

$$(10)\qquad kV_p\{E\cap C\} \le V_p\{E_k\cap C\}.$$

PROOF. We have

$$(11)\qquad V_p\{E\cap C\} = \int_{-\infty}^{\infty} V_{p-1}\{(E\cap C)(x)\}\,dx$$

and

$$(12)\qquad \begin{aligned} V_p\{E_k\cap C\} &= \int_{-\infty}^{\infty} V_{p-1}\{(E_k\cap C)(z)\}\,dz \\ &= k\int_{-\infty}^{\infty} V_{p-1}\{(E_k\cap C)(kx)\}\,dx \\ &\ge k\int_{-\infty}^{\infty} V_{p-1}\{(E\cap C)(x)\}\,dx \\ &= kV_p\{E\cap C\}. \end{aligned}$$

□

If E is a sphere or, more generally, an ellipsoid with a principal axis along the first coordinate, the property (3) holds for every x. The condition is equivalent to the condition that if $\mathbf{y} = (x, y_2, \dots, y_p)' \in E$, then $(-x, y_2, \dots, y_p)' \in E$; that is, E is symmetric about $y_1 = 0$. Inequality (10) can also be written as

$$\frac{V_p\{E \cap C\}}{V_p\{E\}} \le \frac{V_p(E_k \cap C)}{V_p(E_k)}. \tag{13}$$

If a random vector $\mathbf{Y}$ has the uniform distribution on E and $\mathbf{Z}$ has the uniform distribution on E_k, (13) is equivalent to $\Pr\{\mathbf{Y} \in C\} \le \Pr\{\mathbf{Z} \in C\}$.

The condition of symmetry of E implies that $\mathbf{Y}$ has the same distribution as $-\mathbf{Y}$; hence $\mathbb{E}\mathbf{Y} = \mathbb{E}(-\mathbf{Y}) = \mathbf{0}$. The condition (3) for almost every x implies further that $(Y_1, Y_2, \dots, Y_p)$ has the same distribution as $(-Y_1, Y_2, \dots, Y_p)$; hence $\mathbb{E}Y_1Y_j = \mathbb{E}(-Y_1Y_j) = -\mathbb{E}Y_1Y_j = 0$, $j = 2, \dots, p$; that is, Y_1 is uncorrelated with $Y_2, \dots, Y_p$.

The set E_k defined in (4) will be denoted as $\mathbf{F}E$. The transformation $\mathbf{F}$ is a contraction. We now generalize Theorem 2 to a more general contraction. If

$$\mathbf{F} = \mathbf{H}^{-1}\mathbf{D}\mathbf{H}, \tag{14}$$

where $\mathbf{H}$ is nonsingular and $\mathbf{D}$ is a diagonal matrix $\operatorname{diag}(d_1, \dots, d_p)$ such that $d_1, \dots, d_p$ are real and $d_i \le 1$, $i = 1, \dots, p$, we call $\mathbf{F}$ a diagonalizable contraction. Consider the three sets E, C, and $E_d = \mathbf{F}E = \mathbf{H}^{-1}\mathbf{D}\mathbf{H}E$, each of which is convex and symmetric about the origin. An equivalent model consists of the sets $E^* = \mathbf{H}E$, $C^* = \mathbf{H}C$ and $E_d^* = \mathbf{H}E_d = \mathbf{D}\mathbf{H}E = \mathbf{D}E^*$, each of which is convex and symmetric about the origin.

THEOREM 3. *Let C and E be convex sets, symmetric about the origin. Define E_d by* (4) *for* $\mathbf{F}$ *given by* (14). *Let* $\mathbf{y}^* = \mathbf{H}\mathbf{y}$ *and* $E^* = \mathbf{H}E$. *Suppose* E^* *is symmetric about each* $y_j^* = 0$ *for which* $d_j < 1$, $j = 1, \dots, p$. *Then*

$$|\mathbf{F}| V_p(E \cap C) \le V_p(E_d \cap C), \tag{15}$$

where $|\mathbf{F}| = |\mathbf{D}| = \prod_{i=1}^p d_i$.

PROOF. We transform to E^*, C^* and $E_d^* = \mathbf{D}E^*$ and then transform back to E, C and E_d. Note that $\mathbf{D} = \prod_{j=1}^p \mathbf{D}_j$ with $\mathbf{D}_j = \operatorname{diag}(1, \dots, 1, d_j, 1, \dots, 1)$. Then application of Theorem 2 for each j for which $d_j < 1$ yields the Theorem 3. $\square$

If

$$\mathbf{R}_j = \begin{bmatrix} \mathbf{I}_{j-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{p-j} \end{bmatrix}, \tag{16}$$

the condition that E^* is symmetric about $y_j^* = 0$ is that $\mathbf{R}_j E^* = E^*$, $j = 1, \dots, p$. The set E^* is said to be sign-invariant. An ellipsoid whose principal axes are in the directions of the coordinate axes is sign-invariant; a ball meets the condition in every coordinate system. A rectangular parallelopiped whose edges are parallel to the coordinate axes in the space of $\mathbf{y}^*$ also meets the condition. Note that if a set is sign-invariant it is symmetric about the origin.

If E is a ball, we can use the singular-value decomposition for $\mathbf{F}$:

$$\mathbf{F} = \mathbf{P}\mathbf{\Delta}\mathbf{Q}, \tag{17}$$

where $\mathbf{P}$ and $\mathbf{Q}$ are orthogonal, $\mathbf{\Delta} = \operatorname{diag}(\delta_1, \dots, \delta_p)$ and $\delta_1^2, \dots, \delta_p^2$ are the characteristic roots of $\mathbf{F}\mathbf{F}'$ or, alternatively, of $\mathbf{F}'\mathbf{F}$ ($\delta_1, \dots, \delta_p$ with $\delta_i \geq 0$ are the singular values of $\mathbf{F}$).

COROLLARY 2. *Let C be a convex set, symmetric about the origin; let E be a ball; and let $E_d = \mathbf{F}E$. If the singular values of $\mathbf{F}$ are less than or equal to* 1, (15) *holds.*

PROOF. Let $E^* = \mathbf{P}'E = E$, $E_d^* = \mathbf{P}'E_d = \mathbf{\Delta}\mathbf{Q}E = \mathbf{\Delta}E$ and $C^* = \mathbf{P}'C$, each of which is convex and symmetric about the origin. Then $V_p(E \cap C) = V_p(E^* \cap C^*)$ and $V_p(E_d \cap C) = V_p(E_d^* \cap C^*)$. The corollary follows from Theorem 3. □

REMARK 1. We now give an example to show that (6) and hence (15) are not true for every convex C and E symmetric about the origin. For $p = 2$ let E be an ellipse with major axis (larger than the minor axis) in the first and third quadrant different from each coordinate axis, and let $E_{1/2} = \mathbf{F}E$ with $\mathbf{F} = \operatorname{diag}(\frac{1}{2}, 1, \dots, 1)$. Suppose the maximum y_1-value in E is z_1 and the corresponding y_2-value is z_2 (> 0); then the maximum y_1-value in $E_{1/2}$ is $\frac{1}{2}z_1$ and the corresponding y_2-value is z_2. The line from the origin to (z_1, z_2) intersects the boundary of $E_{1/2}$ at a point (z_1^*, z_2^*) with $z_1^* < \frac{1}{2}z_1$, $z_2^* < \frac{1}{2}z_2$, that is, the length of the line segment to (z_1, z_2) in E is more than 2 times the length of this line segment intersected with E. Now construct C as a narrow rectangle that includes (z_1, z_2) and $(-z_1, -z_2)$ as interior points. If the width of C is small enough, $V_p(E \cap C) > 2V_p(E_{1/2} \cap C)$. Alternatively, C can be an ellipse containing (z_1, z_2) and $(-z_1, -z_2)$ with a large major axis and a small minor axis.

The import of this example is that some condition on E and $\mathbf{F}$ is needed for the validity of the theorems just as in the case of the 1955 theorem the center of E moves away from C along a ray.

REMARK 2. We now give an example to show that (6) and hence (15) are not true for every convex sign-invariant E and every symmetric star-shaped C. (A set C is star-shaped if $\mathbf{y} \in C$ implies $k\mathbf{y} \in C$, $0 \leq k \leq 1$.) Let $p = 2$. Let $E = \{(y_1, y_2) \mid |y_1| \leq 2,\ |y_2| \leq 1\}$ and $C = \{(y_1, y_2) \mid |y_2| \leq |y_1|,\ |y_1| \leq 2\}$. The set E is a rectangle, and C has the shape of a bow tie. In (5) let $k = \frac{1}{2}$. Then $E_{1/2} = \{(y_1, y_2) \mid |y_1| \leq 1,\ |y_2| \leq 1\}$. Since $V_2\{E \cap C\} = 6 > V_2\{E_{1/2} \cap C\} = 2$, inequality (6) is violated.

3. Inequalities for densities. We use the inequalities in Section 2 to obtain inequalities on integrals over sets.

THEOREM 4. *Let C be a convex set, symmetric about the origin. Let $f(\mathbf{y}) \geq 0$ be a function such that the following hold*: (i) $f(\mathbf{y}) = f(-\mathbf{y})$; (ii) *for every u, $0 < u < \infty$, $\{\mathbf{y} | f(\mathbf{y}) \geq u\}$ is convex*; *and* (iii) $\int_C f(\mathbf{y})\, d\mathbf{y} < \infty$. *Then, for $\mathbf{F}$ given by* (14),

$$\int_C f(\mathbf{y})\, d\mathbf{y} \leq \int_C |\mathbf{F}|^{-1} f(\mathbf{F}^{-1}\mathbf{y})\, d\mathbf{y} \tag{18}$$

if $f(\mathbf{H}^{-1}\mathbf{y}^*) = f(\mathbf{H}^{-1}\mathbf{R}_j\mathbf{y}^*)$ for each j for which $d_j < 1$ and $\mathbf{I} - \mathbf{D}$ is positive semidefinite.

PROOF. For each u, $0 < u < \infty$, the set $E^u = \{\mathbf{y} | f(\mathbf{y}) \geq u\}$ satisfies the conditions of Theorem 3; hence

$$g(u) = V_p\{C \cap E^u\} \leq h(u) = |\mathbf{F}|^{-1} V_p\{C \cap E_d^u\}, \tag{19}$$

where $E_d^u = \{\mathbf{y} | f(\mathbf{F}\mathbf{y}) \geq u\}$. Definitions of the Lebesgue and Lebesgue–Stieltjes integrals show that

$$\begin{aligned} \int_C |\mathbf{F}|^{-1} f(\mathbf{F}^{-1}\mathbf{y})\, d\mathbf{y} - \int_C f(\mathbf{y})\, d\mathbf{y} &= -\int_0^\infty u\, dh(u) + \int_0^\infty u\, dg(u) \\ &= \int_0^\infty u\, d[g(u) - h(u)]. \end{aligned} \tag{20}$$

Integration by parts shows that

$$\int_0^b u\, d[g(u) - h(u)] = b[g(u) - h(u)] + \int_0^b [h(u) - g(u)]\, du. \tag{21}$$

Since $f(\mathbf{y})$ has a finite integral over C, $bh(b) \to 0$ as $b \to \infty$ and hence $bg(b) \to 0$. Since the second term on the right-hand side of (21) is nonnegative for every b, (18) results. □

A probability density $f(\mathbf{y})$ for which the set $\{\mathbf{y} | f(\mathbf{y}) \geq u\}$ is convex for every u, $0 < u < \infty$, will be called *unimodal*; if in addition $f(\mathbf{y}) = f(-\mathbf{y})$, the density will be called *symmetric unimodal*. If $f(\mathbf{R}_j\mathbf{y}) = f(\mathbf{y})$, $j = 1, \dots, p$, we say $f(\mathbf{y})$ is sign-invariant. Note that if $f(\mathbf{y})$ is sign-invariant, it is symmetric. In probability terms we can interpret Theorem 4 as stating that the probability of a convex set, symmetric about the origin, is smaller for $f(\mathbf{y})$ than for $|\mathbf{F}|^{-1} f(\mathbf{F}\mathbf{y})$ if $f(\mathbf{H}^{-1}\mathbf{y}^*)$ is sign-invariant and $\mathbf{I} - \mathbf{D}$ is positive definite.

We specialize Theorem 4 to elliptically contoured densities.

THEOREM 5. *Suppose $\mathbf{Z}_1$ and $\mathbf{Z}_2$ have densities $|\boldsymbol{\Lambda}_1|^{-1/2} g(\mathbf{z}'\boldsymbol{\Lambda}_1^{-1}\mathbf{z})$ and $|\boldsymbol{\Lambda}_2|^{-1/2} g(\mathbf{z}'\boldsymbol{\Lambda}_2^{-1}\mathbf{z})$, respectively, where $g(\cdot)$ is monotonically nonincreasing. Let C be a convex set, symmetric about the origin. If $\boldsymbol{\Lambda}_1 \leq \boldsymbol{\Lambda}_2$ in the Loewner sense and $\boldsymbol{\Lambda}_1$ is nonsingular,*

$$\Pr\{\mathbf{Z}_2 \in C\} \leq \Pr\{\mathbf{Z}_1 \in C\}. \tag{22}$$

PROOF. There exists a nonsingular matrix $\mathbf{G}$ such that $\mathbf{\Lambda}_2 = \mathbf{G}\mathbf{G}'$ and $\mathbf{\Lambda}_1 = \mathbf{G}\mathbf{\Delta}\mathbf{G}'$, where $\mathbf{\Delta}$ is diagonal with nonnegative diagonal elements. Since $\mathbf{G}\mathbf{G}' - \mathbf{G}\mathbf{\Delta}\mathbf{G}' = \mathbf{G}(\mathbf{I} - \mathbf{\Delta})\mathbf{G}'$ is positive semidefinite, each diagonal element of $\mathbf{\Delta}$ is not greater than 1. Let $\mathbf{Z}_i^* = \mathbf{G}^{-1}\mathbf{Z}_i$, $i = 1, 2$. The densities of $\mathbf{Z}_2^*$ and $\mathbf{Z}_1^*$ are $g(\mathbf{z}'\mathbf{z})$ and $|\mathbf{\Delta}|^{-1/2}g(\mathbf{z}'\mathbf{\Delta}^{-1}\mathbf{z})$, respectively. Then Theorem 5 follows from Theorem 4. □

Fefferman, Jodeit and Perlman (1972) proved Theorem 5 without the condition that $g(\cdot)$ is monotonically nonincreasing. Their proof is based on their theorem that if E is the unit sphere and $\mathbf{F}$ is given by (17) with $\mathbf{\Delta} \le \mathbf{I}$, then the uniform surface measure of $\mathbf{F}C \cap E$ is not greater than that of $C \cap E$. They used Theorem 2 of Sherman (1955) based on Anderson's theorem.

REMARK 3. We now give an example to show that Theorem 5 does not hold for every symmetric star-shaped C even if the distribution is normal. Let $p = 2$, $\mathbf{\Lambda}_2 = \mathbf{I}$ and $g(\mathbf{z}'\mathbf{z}) = \exp(-\mathbf{z}'\mathbf{z}/2)/2\pi$. Let $C = \{\mathbf{z} \mid |z_2| \le |z_1|\}$. Then $\Pr\{\mathbf{Z} \in C\} = 4\Pr\{0 \le Z_2 \le Z_1\}$. Let $\mathbf{Z}^{*\prime} = (Z_1^*, Z_2^*) = (\frac{1}{2}Z_1, Z_2)$. Then

$$\mathbf{\Lambda}_1 = \mathbb{E}\mathbf{Z}^*\mathbf{Z}^{*\prime} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{bmatrix} < \mathbf{I}, \tag{23}$$

and

$$\begin{aligned} \Pr\{\mathbf{Z}^* \in C\} &= 4\Pr\{0 \le Z_2^* \le Z_1^*\} \\ &= 4\Pr\{0 \le Z_2 \le \tfrac{1}{2}Z_1\} = \Pr\{\mathbf{Z} \in C_{1/2}^*\}, \end{aligned} \tag{24}$$

where $C_{1/2} = \{\mathbf{z} \mid |z_2| \le \frac{1}{2}|z_1|\}$. Since $C_{1/2} \subset C$, $\Pr\{C_{1/2}\} \le \Pr\{C\}$.

Anderson (1955) showed Theorem 5 for $\mathbf{Z}_1$ and $\mathbf{Z}_2$ normally distributed. In the normal case the covariance matrices of $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$, respectively. Then $\mathbf{Z}_2$ has the distribution of $\mathbf{Z}_1 + \mathbf{Y}$, where $\mathbf{Y} \sim N(\mathbf{0}, \mathbf{\Lambda}_2 - \mathbf{\Lambda}_1)$, and the result follows from Anderson's theorem.

If $\mathbf{Z}$ has an elliptically contoured distribution, then $\mathbf{F}\mathbf{Z}$ has an elliptically contoured distribution. [See Anderson (1993), e.g., for a general discussion.] Suppose the vector $\mathbf{Y}$ is distributed as $\mathbf{X}\boldsymbol{\beta} + \mathbf{V}$, where $\mathbf{X}$ is a known nonstochastic matrix, $\boldsymbol{\beta}$ is a parameter vector and $\mathbf{V}$ is an unobservable random vector with an elliptically contoured distribution centered at $\mathbf{0}$. Then linear estimators of $\boldsymbol{\beta}$ have elliptically contoured distributions. Eaton (1988) has used Theorem 5 to show that the generalized least squares (or Markov) estimator maximizes the probability of falling in any symmetric convex sets in the class of linear unbiased estimators. Ali and Ponnapalli (1990) also showed this result for ellipsoidal sets.

REMARK 4. Define $S_i = \{\mathbf{z} \mid \mathbf{z}'\mathbf{\Lambda}_i^{-1}\mathbf{z} \le k\}$, $i = 1, 2$, with $\mathbf{\Lambda}_1 \le \mathbf{\Lambda}_2$ in the Loewner sense. Then $f_i(\mathbf{z}) = I_{S_i}(\mathbf{z})/V_p(S_i)$, $i = 1, 2$, where $I_{\cdot}(\cdot)$ is the index function, are elliptically contoured, and Theorem 5 holds. The fact that $S_1 \subset S_2$ raises the question of whether (22) [or, equivalently, (13)] would be true for $f_i(\mathbf{z}) = I_{S_i}(\mathbf{z})/V_p\{S_i\}$ for any symmetric convex sets such that $S_1 \subset S_2$.

The following example shows that (22) does not hold for arbitrary symmetric convex sets. Let S_1 be the square with vertices $(1,1)$, $(-1,1)$, $(-1,-1)$, $(1,-1)$; let S_2 be the hexagon with vertices $(1,1)$, $(-1,1)$, $(-2,0)$, $(-1,-1)$, $(1,-1)$, $(2,0)$; and let C be the rectangle with vertices $(2,a)$, $(-2,a)$, $(-2,-a)$, $(2,-a)$. Then $V_2\{S_1\}=4$, $V_2\{S_2\}=6$, $V_2\{S_1\cap C\}=a$ and $V_2\{S_2 \cap C\} = 2a(4-a)$. Then (22) does not hold for $0<a<1$.

4. Monotonicity of a power function . Consider using a sample $(\mathbf{x}_1,\dots,\mathbf{x}_n)=\mathbf{X}$ of size n from $N(\mathbf{0},\mathbf{\Sigma})$ to test the null hypothesis H: $\mathbf{\Sigma}=\mathbf{I}$ against alternative $\mathbf{\Sigma}>\mathbf{I}$. The density of $\mathbf{X}$ is

$$f(\mathbf{X}|\mathbf{\Sigma}) = \frac{1}{(2\pi)^{pn/2}|\mathbf{\Sigma}|^{n/2}} \exp\left(-\frac{1}{2}\operatorname{tr}\mathbf{\Sigma}^{-1}\mathbf{X}\mathbf{X}'\right), \tag{25}$$

which is a function of $\operatorname{tr}\mathbf{\Sigma}^{-1}\mathbf{X}\mathbf{X}' = \sum_{\alpha=1}^n \mathbf{x}'_\alpha\mathbf{\Sigma}^{-1}\mathbf{x}_\alpha = (\operatorname{vec}\mathbf{X})'(\mathbf{I}\otimes\mathbf{\Sigma}^{-1})(\operatorname{vec}\mathbf{X})$. Let $\mathbf{\Sigma}=\mathbf{P}\mathbf{\Delta}\mathbf{P}'$, where $\mathbf{P}$ is orthogonal, $\mathbf{\Delta}=\operatorname{diag}(\delta_1,\dots,\delta_p)$ and $\delta_1\ge\cdots\ge\delta_p$ are the characteristic roots of $\mathbf{\Sigma}$. The density of $\mathbf{Y}=\mathbf{P}'\mathbf{X}$ is $f(\mathbf{Y}|\mathbf{\Delta})$, the exponent of which is $-\frac{1}{2}\operatorname{tr}\mathbf{\Delta}^{-1}\mathbf{Y}\mathbf{Y}' = -\frac{1}{2}\sum_{\alpha=1}^n\sum_{i=1}^p y_{i\alpha}^2/\delta_i$. It follows from Theorem 5 that the probability of $\mathbf{Y}$ falling in some symmetric convex set decreases if one or more of $\delta_1,\dots,\delta_p$ increases. A test of H for which the acceptance region is a symmetric convex set in $\mathbf{X}$ has a power function that is monotonically increasing in each ordered characteristic root of $\mathbf{\Sigma}$. In fact, the result holds if the density of $\mathbf{X}$ has the form

$$|\mathbf{\Sigma}|^{-n/2} g(\operatorname{tr}\mathbf{\Sigma}^{-1}\mathbf{X}\mathbf{X}'). \tag{26}$$

THEOREM 6. *Suppose the density of the $p\times n$ random matrix $\mathbf{X}$ is* (26), *and suppose that the acceptance region of a test of the null hypothesis H: $\mathbf{\Sigma}=\mathbf{I}$ is convex in $\mathbf{X}$ and symmetric about the origin. Then the power function is an increasing function of each ordered characteristic root of $\mathbf{\Sigma}$.*

Anderson and Das Gupta (1964) proved the monotonicity of the power function of a test whose acceptance region is monotonic in the characteristic roots of $\mathbf{X}\mathbf{X}'$. Some tests that are both monotonic in the sample roots and convex in $\mathbf{X}$ are based on $\operatorname{tr}\mathbf{X}\mathbf{X}'=\sum_{i=1}^p c_i$ and c_1, where $c_1>c_2>\cdots>c_p$ are the characteristic roots of $\mathbf{X}\mathbf{X}'$. However, a test based on c_p alone is not convex in $\mathbf{X}$.

The test procedures that depend on the characteristic roots of $\mathbf{X}\mathbf{X}'$ are invariant with respect to transformations $\mathbf{X}^*=\mathbf{P}\mathbf{X}$, where $\mathbf{P}$ is orthogonal; this transformation leaves the null hypothesis invariant. An acceptance region that is symmetric and convex in $\mathbf{X}$ cannot necessarily be expressed in terms of the characteristic roots. Hence, the conditions of Theorem 6 are more general than those of Anderson and Das Gupta (1964).

Anderson and Das Gupta also considered testing the hypothesis $\mathbf{\Sigma}_1=\mathbf{\Sigma}_2$ on the basis of observations $\mathbf{X}_1$ and $\mathbf{X}_2$ on $f(\mathbf{X}|\mathbf{\Sigma}_1)$ and $f(\mathbf{X}|\mathbf{\Sigma}_2)$, respectively.

If a test is monotonic in the characteristic roots of $\mathbf{X}_1\mathbf{X}_1'(\mathbf{X}_2\mathbf{X}_2')^{-1}$ the power function is monotonic in the characteristic roots of $\mathbf{\Sigma}_1\mathbf{\Sigma}_2^{-1}$. Theorem 6 can be extended to this problem.

5. Equality in Corollary 1. In this section we study the implications of equality in Corollary 1.

THEOREM 7. *Let C be a convex set in $\mathbb{R}^p$, symmetric about the origin. Then $V_{p-1}\{C(x_2)\} = V_{p-1}\{C(x_1)\}$ for some $x_1 \in [0, x_2)$ implies*

$$C(x) = C(x_2) - \mathbf{z}_x \tag{27}$$

for some $\mathbf{z}_x$ for every $x \in [-x_2, x_2]$.

The proof of Theorem 7 is based on the part of the Brunn–Minkowski theorem that states that equality of the volumes implies $E_1 = E_0 + \mathbf{z}$ for some $\mathbf{z}$. First, (27) is shown for $x = x_1$. Then use of the Brunn–Minkowski theorem for $\theta = (x_1 + x)/(x_2 + x)$, $E_1 = C(-x_2)$ and $E_0 = C(x)$ yields (27) for other x. The details are left to the reader.

The meaning of Theorem 7 is that if two parallel sections of C have equal areas, C is a cylinder set within the region defined.

Acknowledgments. The author wishes to acknowledge the helpful suggestions of Ingram Olkin, Michael Perlman and two referees.

REFERENCES

ALI, M. M. and PONNAPALLI, R. (1990). An optimal property of the Gauss–Markoff estimator. *J. Multivariate Anal.* **32** 171–176.

ANDERSON, T. W. (1955). The integral of a symmetric unimodal function. *Proc. Amer. Math. Soc.* **6** 170–176.

ANDERSON, T. W. (1993). Nonnormal multivariate distributions: inference based on elliptically contoured distributions. In *Multivariate Analysis: Future Directions* (C. R. Rao, ed.) 1–25. North-Holland, Amsterdam.

ANDERSON, T. W. and DAS GUPTA, S. (1964). A monotonicity property of the power functions of some tests of the equality of two covariance matrices. *Ann. Math. Statist.* **35** 1059–1063.

BRASCAMP, H. J. and LIEB, E. H. (1976). On extensions of the Brunn–Minkowski and Prékopa–Leindler theorems, including inequalities for log concave functions and with an application to the diffusion equation. *J. Funct. Anal.* **22** 366–389.

DAS GUPTA, S., EATON, M. L., OLKIN, I., PERLMAN, M., SAVAGE, L. J. and SOBEL, M. (1972). Inequalities on the probability context of convex regions for elliptically contoured distributions. *Proc. Sixth Berkeley Symp. Math. Statist. Probab.* **2** 241–265. Univ. California Press, Berkeley.

EATON, M. L. (1988). Concentration inequalities for Gauss–Markov estimators. *J. Multivariate Anal.* **25** 119–138.

FEFFERMAN, C., JODEIT, M., JR. and PERLMAN, M. D. (1972). A spherical surface measure inequality for convex sets. *Proc. Amer. Math. Soc.* **33** 114–119.

PERLMAN, M. D. (1990). T. W. Anderson's theorem on the integral of a symmetric unimodal function over a symmetric convex set and its applications in probability and statistics. In *The Collected Papers of T. W. Anderson, 1943–1985* (George P. H. Styan, ed.) **2** 1627–1641. Wiley, New York.

PRÉKOPA, A. (1971). Logarithmic concave measures with applications. *Acta Sci. Math. (Szeged)* **32** 301–316.

PRÉKOPA, A. (1973). On logarithmic concave measures and functions. *Acta Sci. Math. (Szeged)* **34** 335–343.

RINOTT, Y. (1976). On convexity of measures. *Ann. Probab.* **4** 1020–1026.

SHERMAN, S. (1955). A theorem on convex sets with applications. *Ann. Math. Statist.* **26** 926–931.

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA 94305