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A NOTE ON SEQUENTIAL DETECTION WITH EXPONENTIAL PENALTY FOR THE DELAY

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We study the continuous time analogue of the Bayes problem of Poor. As usual the results on the optimal solution are more explicit than in the discrete time set-up.

1. Introduction. In Poor (1998) the quickest detection problem of Shiryayev (1963) is investigated when the linear penalty on the detection delay is replaced by an exponential delay-penalty. Such a delay-penalty might be more appropriate to describe situations where the losses incurred under the post-change regime exponentiate with time. See the introduction of Poor (1998) for a more detailed discussion related to this question. The optimal stopping rule with respect to the exponential delay-penalty turns out to be a simple modification of the stopping rule considered by Shiryayev. Poor works in a discrete time setting. There the optimal threshold cannot be computed explicitly. We study the corresponding problem in continuous time and are able to obtain a more explicit result.

Let B denote standard Brownian motion. Let θ be a fixed real number. Let τ denote a nonnegative random variable independent of B such that $P(\tau > t) = \exp\{-\lambda t\}$ for all $t > 0$ for some $\lambda > 0$. Put for $0 \le t < \infty$,

$$W_t = B_t + \theta(t - \tau)^+$$

and let $\mathscr{F}_t = \sigma(W_s; 0 \le s \le t)$. The process W is observed sequentially. We assume that θ and λ are unknown. The goal is to detect τ as soon as possible. More precisely we seek a stopping time T of W that minimizes the following Bayes risk:

$$R(T) = P(T < \tau) + cE\left(e^{\alpha(T-\tau)^+} - 1\right),$$

where $c > 0$ and $\alpha > 0$. The technique to solve this problem is similar to the approach in Beibel and Lerche (1997). The main step in our arguments is to establish for a sufficiently large class of stopping times $R(T) = E\{M_T f(\tilde{\psi}_T)\}$, where M is a positive local martingale, f a positive function assuming its minimum over $[0, \infty)$ at a unique point $v^* \ge 0$ and $\tilde{\psi}_t$ a suitable nonnegative stochastic process. It is then optimal to stop as soon as $\tilde{\psi}$ reaches the minimizing argument v^*.

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2. Some lemmas. We first collect some useful auxiliary results. We employ facts from the theory of stochastic processes and Itô calculus. We refer to Karatzas and Shreve (1998) as a convenient reference. Let π_t denote $P(\tau \le t|\mathscr{F}_t)$. Let ψ_t and $\tilde{\psi}_t$ be given by

$$\psi_t = \frac{\pi_t}{1-\pi_t} \quad \text{and} \quad \tilde{\psi}_t = \frac{E(e^{\alpha(t-\tau)^+}|\mathscr{F}_t)}{1-\pi_t} - 1.$$

Bayes' theorem provides for $0 \le s \le t$,

$$P(\tau \in ds|\mathscr{F}_t) = \frac{\lambda e^{-\lambda s} e^{\theta(W_t - W_s) - (\theta^2/2)(t-s)}\, ds}{\int_0^t \lambda e^{-\lambda s} e^{\theta(W_t - W_s) - (\theta^2/2)(t-s)}\, ds + e^{-\lambda t}}.$$

Some algebra therefore yields

$$\psi_t = e^{\lambda t} e^{\theta W_t - (\theta^2/2)t} \int_0^t e^{-\theta W_s + (\theta^2/2)s} \lambda e^{-\lambda s}\, ds$$

and

$$\tilde{\psi}_t = e^{(\alpha+\lambda)t} e^{\theta W_t (\theta^2/2)t} \int_0^t e^{-\theta W_s + (\theta^2/2)s} \lambda e^{-(\alpha+\lambda)s}\, ds.$$

Standard arguments immediately lead to the following representation of $R(T)$ for all stopping times T with $R(T) < \infty$:

$$R(T) = E\{1 - \pi_T + c(1-\pi_T)(\tilde{\psi}_T + 1) - c\}. \tag{1}$$

Let $\alpha' = 2\alpha/\theta^2$ and $\lambda' = 2\lambda/\theta^2$. For $x \ge 0$ let

$$g(x) = \frac{\int_0^\infty e^{-u} u^{\gamma_1 - 1} (\lambda' + xu)^{\gamma_2 - 1}\, du}{(\lambda')^{\gamma_2 - 1} \Gamma(\gamma_1)},$$

where $\Gamma(l) = \int_0^\infty e^{-u} u^{l-1}\, du$,

$$\gamma_1 = \tfrac{1}{2}(\lambda' + \alpha' - 1) + \sqrt{\tfrac{1}{4}(\lambda' + \alpha' - 1)^2 + \lambda'}$$

and

$$\gamma_2 = 1 - \tfrac{1}{2}(\lambda' + \alpha' - 1) + \sqrt{\tfrac{1}{4}(\lambda' + \alpha' - 1)^2 + \lambda'}.$$

Note that $\gamma_1 > 0$ and so g is well-defined. The function g solves [see Abramowitz and Stegun (1965), Section 13.1; in particular 13.1.35 (with $f \equiv 0$, $h(z) = \lambda'/z$, $A = a = \gamma_1$ and $b = \gamma_2 + a$), 13.1.37 and 13.2.5].

$$\frac{\theta^2}{2} x^2 g''(x) + [(\lambda + \alpha)x + \lambda] g'(x) - \lambda g(x) = 0 \tag{2}$$

on $(0, \infty)$. See also Section 2 of Kramkov and Mordecky (1994) for a related discussion. We have $g(0) = 1$ and $g'(0) = \gamma_1(\gamma_2 - 1)/\lambda' = 1$. Obviously, $\sqrt{(\lambda' + \alpha' - 1)^2/4 + \lambda'} > (\lambda' + \alpha' - 1)/2$ and so $\gamma_2 - 1 > 0$. Moreover,

$$\gamma_2 - 2 = -\tfrac{1}{2}(\lambda' + \alpha' + 1) + \sqrt{\tfrac{1}{4}(\lambda' + \alpha' + 1)^2 - \alpha'}$$

and so $\gamma_2 - 2 < 0$. Therefore $\gamma_2 - 1 \in (0, 1)$. Differentiation under the integral hence yields $g'(x) > 0$ and $g''(x) < 0$ for $x > 0$. This implies that g is strictly increasing and strictly concave on $[0, \infty)$. Moreover,

$$g(x) = x^{\gamma_2 - 1} \frac{\Gamma(\gamma_1 + \gamma_2 - 1)}{(\lambda')^{\gamma_2 - 1}\Gamma(\gamma_1)}(1 + o(1))$$

and

$$g'(x) = (\gamma_2 - 1)x^{\gamma_2 - 2} \frac{\Gamma(\gamma_1 + \gamma_2 - 1)}{(\lambda')^{\gamma_2 - 1}\Gamma(\gamma_1)}(1 + o(1))$$

as $x \to \infty$. Therefore g grows slower than the identity and $\lim_{x\to\infty}[x/g(x)] = +\infty$.

Let $M_t = (1 - \pi_t)g(\tilde{\psi}_t)$ and define $S_v = \inf\{t \geq 0 | \tilde{\psi}_t \geq v\}$.

LEMMA 1. *$(M_{s_v \wedge t}, \mathcal{F}_t)$ is for all $0 \leq v < \infty$ a closable martingale on $0 \leq t < \infty$. In particular $E(M_{S_v}) = M_0 = 1$ for all $v \geq 0$.*

PROOF. Let $\overline{W}_t$ denote the innovation process $W_t - \theta \int_0^t \pi_s \, ds$. Then $(\overline{W}_t, \mathcal{F}_t)$ is a standard Brownian motion [see Shiryayev (1963)]. It is easy to see, using Itô's formula [Karatzas and Shreve (1988), Theorem 3.6, page 153], that

$$d\psi_t = [(\lambda + \theta^2 \pi_t)\psi_t + \lambda] \, dt + \theta \psi_t \, d\overline{W}_t$$

and

$$d\widetilde{\psi}_t = [(\lambda + \alpha + \theta^2 \pi_t)\tilde{\psi}_t + \lambda] \, dt + \theta \tilde{\psi}_t \, d\overline{W}_t.$$

Moreover [see Shiryayev (1963)],

$$d\pi_t = \lambda(1 - \pi_t) \, dt + \theta \pi_t (1 - \pi_t) \, d\overline{W}_t.$$

Itô's formula and (2) therefore imply

$$d[(1 - \pi_t)g(\tilde{\psi}_t)] = \theta(1 - \pi_t)[g'(\tilde{\psi}_t)\tilde{\psi}_t - g(\tilde{\psi}_t)\pi_t] \, d\overline{W}_t,$$

and so M_t is a positive local martingale. On $0 \leq t \leq S_v$ we have $0 \leq \tilde{\psi}_t \leq v$ and therefore

$$E\left(\int_0^{S_v} (1 - \pi_t)^2 [g'(\tilde{\psi}_t)\tilde{\psi}_t - g(\tilde{\psi}_t)\pi_t]^2 \, dt\right) \leq \kappa E(S_v)$$

for some constant κ depending on v. Now $\tilde{\psi}_t \geq \psi_t$ implies $S_v \leq \inf\{t \geq 0 | \pi_t \geq v/(1 + v)\}$. Therefore,

$$P(S_v > t) \leq P\left(1 - \pi_t > \frac{1}{1 + v}\right) \leq (1 + v)E(1 - \pi_t) = (1 + v)e^{-\lambda t}.$$

Hence $E(S_v) < \infty$. The assertion now follows either directly from Problem 2.18 on page 144 in Karatzas and Shreve (1988) or from the Burkholder–Davis–Gundy inequalities [Theorem 3.28 on page 166 in Karatzas and Shreve (1988) with $m = 1$].

For $x \geq 0$ let $f(x) = [1 + c(x+1)]/g(x)$. Equation (1) can now be restated as

$$R(T) = E\{M_T f(\tilde{\psi}_T)\} - c \tag{3}$$

for all stopping times T with $R(T) < \infty$.

LEMMA 2. *There exists a unique* $v^* \in (0, \infty)$ *such that* $0 < \inf_{0 \leq v < \infty} f(v) = f(v^*)$.

PROOF. The properties of g yield $\inf_{0 \leq v < \infty} f(v) > 0$ and the existence of at least one $v^* \in [0, \infty)$ with $0 < \inf_{0 \leq v < \infty} f(v) = f(v^*)$. The uniqueness of v^* follows from $[g(x)^2 f'(x)]' = -[1 + c(x+1)]g''(x) > 0$. Since $g'(0) > c/(1+c)$, $g(x)$ crosses the line $1 + xc/[1+c]$. Therefore $1 + cv/(1+c) < g(v)$ for some $v > 0$. This implies $f(v) = [1 + c(1+v)]/g(v) < (1+c) = f(0)$ for some $v > 0$.

REMARK 1. v^* solves the implicit equation

$$\frac{g(x)}{g'(x)} - x - 1 = \frac{1}{c}.$$

Now $g(x)/g'(x) = [x/(\gamma_2 - 1)](1 + o(1))$ as $x \to \infty$. Therefore

$$v^* = \frac{1}{c} \frac{\gamma_2 - 1}{2 - \gamma_2}(1 + o(1))$$

as $c \to 0$ for fixed α, λ and θ.

3. Main result. Let $v^* \in (0, \infty)$ be as in Lemma 2 above. We now prove that $T = S_{v^*}$ minimizes $R(T)$ among all stopping times T of W.

THEOREM 1. *For all stopping times T of W*, $R(T) \geq f(v^*) - c = R(S_{v^*})$.

PROOF. Let T be a stopping time of W with $R(T) < \infty$. Lemma 2 implies for any $n \geq 1$ that $E(M_{T \wedge S_n}) = 1$ and so (3) gives

$$\begin{aligned} R(T \wedge S_n) + c &= E\{M_{T \wedge S_n} f(\tilde{\psi}_{T \wedge S_n})\} \\ &\geq f(v^*) E\{M_{T \wedge S_n}\} = f(v^*). \end{aligned}$$

Now $\lim_{n \to \infty} R(T \wedge S_n) = R(T)$ yields $R(T) \geq f(v^*) - c$. The equality for $T = S_{v^*}$ follows from $R(S_{v^*}) + c = E\{M_{S_{v^*}} f(\tilde{\psi}_{S_{v^*}})\} = f(v^*) E\{M_{S_{v^*}}\} = f(v^*)$.

REMARK 2. If α goes to zero and simultaneously $c\alpha$ goes to some constant $\tilde{c} > 0$, then the exponential delay penalty $c(\exp\{\alpha(t - \tau)^+\} - 1)$ converges to the linear delay penalty $\tilde{c}(t - \tau)^+$ considered in Shiryayev (1963). Let $\widetilde{R}(T) = P(T < \tau) + \tilde{c}E((T - \tau)^+)$ and

$$F(x) = \frac{2}{\theta^2} \left(\frac{1-x}{x} \right)^{\lambda'} e^{\lambda'/x} \int_0^x e^{-\lambda'/w} \left(\frac{w}{1-w} \right)^{\lambda'} \frac{1}{w(1-w)^2} \, dw.$$

In Shiryayev (1965) it is shown that $\widetilde{T} = \inf\{t \geq 0 | \psi_t \geq \tilde{v}\}$ minimizes $\widetilde{R}(T)$, if $\tilde{v}$ is the solution of $F(\tilde{v}/(\tilde{v}+1)) = 1/\tilde{c}$. Obviously, $\tilde{\psi}_t \to \psi_t$ when $\alpha \to 0$. A substitution argument yields

$$F(x/(1+x)) = \frac{2x}{\theta^2(\lambda'+1)}\left(1 + \frac{1}{\lambda'}\int_0^\infty e^{-u}\left(\frac{\lambda'}{\lambda'+xu}\right)^{\lambda'+1} du\right). \tag{4}$$

We will now show that

$$\frac{1}{\alpha}\left(\frac{g(x)}{g'(x)} - x - 1\right) \to F(x/(1+x))$$

for fixed θ and λ. We have

$$\begin{aligned}\frac{g(x)}{g'(x)} - x - 1 = \frac{2-\gamma_2}{\gamma_2 - 1}(x+1) &+ \frac{(\lambda' - \gamma_1)\Gamma(\gamma_1)}{(\gamma_2 - 1)\int_0^\infty e^{-u}u^{\gamma_1}(\lambda'+xu)^{\gamma_2-2}\,du} \\ &+ \frac{\int_0^\infty e^{-u}u^{\gamma_1-1}(\lambda'-u)[(\lambda'+xu)^{\gamma_2-2} - 1]\,du}{(\gamma_2-1)\int_0^\infty e^{-u}u^{\gamma_1}(\lambda'+xu)^{\gamma_2-2}\,du}.\end{aligned}$$

This yields after some algebra

$$\lim_{\alpha\to 0}\frac{1}{\alpha}\left(\frac{g(x)}{g'(x)} - x - 1\right) = \frac{2x}{\theta^2(\lambda'+1)}\left(1 + \frac{1}{\Gamma(\lambda'+1)}\int_0^\infty \frac{e^{-u}u^{\lambda'}}{\lambda'+xu}\,du\right). \tag{5}$$

Equalities of (4) and (5) follows from

$$\begin{aligned}&\int_0^\infty e^{-u}u^{\lambda'-1}\,du \int_0^\infty e^{-u}\left(\frac{\lambda'}{\lambda'+xu}\right)^{\lambda'+1} du \\ &\qquad = (\lambda')^{\lambda'+1}\int_0^\infty e^{-u}\left(\int_0^u \left(\frac{u-z}{\lambda'+xz}\right)^{\lambda'-1}\frac{1}{(\lambda'+xz)^2}\,dz\right) du \\ &\qquad = (\lambda')^{\lambda'+1}\left(\int_0^\infty e^{-u}\frac{1}{\lambda'+xu}\int_0^{u/\lambda'} w^{\lambda'-1}\,dw\right) du \\ &\qquad = \int_0^\infty e^{-u}\frac{u^{\lambda'}}{\lambda'+xu}\,du.\end{aligned}$$

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