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PROBABILISTIC REPRESENTATION FOR SOLUTIONS OF AN IRREGULAR POROUS MEDIA TYPE EQUATION

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We consider a porous media type equation over all of $\mathbb{R}^d$, $d = 1$, with monotone discontinuous coefficient with linear growth, and prove a probabilistic representation of its solution in terms of an associated microscopic diffusion. The interest in such singular porous media equations is due to the fact that they can model systems exhibiting the phenomenon of self-organized criticality. One of the main analytic ingredients of the proof is a new result on uniqueness of distributional solutions of a linear PDE on $\mathbb{R}^1$ with not necessarily continuous coefficients.

1. Introduction. We are interested in the probabilistic representation of the solution to a porous media type equation given by

$$\begin{cases} \partial_t u = \frac{1}{2}\partial_{xx}^2(\beta(u)), & t \in [0, \infty[, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

in the sense of distributions, where u_0 is an initial probability density. We look for a solution of (1.1) with time evolution in $L^1(\mathbb{R})$.

We always make the following general assumption on β.

ASSUMPTION 1.1.

- $\beta : \mathbb{R} \to \mathbb{R}$ is monotone increasing.
- $|\beta(u)| \leq \text{const}|u|$, $u \in \mathbb{R}$.
- There exists $\lambda > 0$ such that $(\beta \mp \lambda id)(x) \to \mp\infty$ when $x \to \mp\infty$, where $id(x) \equiv x$.
- $u_0 \in (L^1 \cap L^\infty)(\mathbb{R})$.

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REMARK 1.2.

1. Since β is monotone, (1.1) implies that $\beta(u) = \Phi^2(u)u$, Φ being a nonnegative bounded Borel function.
2. $\beta(0) = 0$ and β is continuous at zero.

We recall that when $\beta(u) = |u|u^{m-1}$, $m > 1$, (1.1) is nothing but the classical *porous media equation*.

One of our final targets is to consider Φ as continuous, except for a possible jump at one positive point, say $e_c > 0$. A typical example is

$$\Phi(u) = H(u - e_c), \tag{1.2}$$

H being the Heaviside function.

The analysis of (1.1) and its probabilistic representation can be carried out in the framework of monotone partial differential equations (PDEs) allowing multi-valued functions and will be discussed in detail in the main body of the paper. This extension is necessary, among other reasons, to allow the graph associated with β to be a maximal monotone graph. We refer to Assumption 3.1 below. In this introduction, for simplicity, we restrict our presentation to the single-valued case.

DEFINITION 1.3. We will say that equation (1.1) or β is *nondegenerate* if there is a constant $c_0 > 0$ such that $\Phi \geq c_0$.

Of course, if Φ is as in (1.2), then β is not nondegenerate. In order for β to be nondegenerate, one needs to add a positive constant to it.

Several contributions were made in this framework, starting from [9], for existence, [14], for uniqueness in the case of bounded solutions, and [10], for continuous dependence on the coefficients. The authors consider the case where β is continuous, even if their arguments allow some extensions for the discontinuous case.

As mentioned in the abstract, the first motivation of this paper was to discuss a continuous-time model of self-organized criticality (SOC), which is described by equations of type (1.2).

SOC is a property of dynamical systems which have a critical point as an attractor; see [2] for a significant monograph on the subject. SOC is typically observed in slowly driven out-of-equilibrium systems with threshold dynamics relaxing through a hierarchy of avalanches of all sizes. We refer, in particular, to the interesting physics paper [3]. The latter makes reference to a system whose evolution is similar to the evolution of a "snow layer" under the influence of an "avalanche effect" which starts when the top of the layer attains a critical value e_c. Adding a stochastic noise should describe other contingent effects. For instance, an additive perturbation by noise could describe the regular effect of "snow falling."

In Bantay et al. [3] it was proposed to describe this phenomenon by a singular diffusion involving a coefficient precisely of the type (1.2).

In the absence of noise, the density $u(t,x)$, $t>0$, $x\in\mathbb{R}$, of this diffusion is formally described by (1.1) and $\beta(u)=\Phi(u)^2u$, where Φ is given by (1.2).

Such a discontinuous monotone β has to be considered as a multivalued map in order to apply monotonicity methods.

The singular nonlinear diffusion equation (1.1) models the *macroscopic* phenomenon for which we try to give a *microscopic* probabilistic representation via a nonlinear stochastic differential equation (NLSDE) modeling the evolution of a single point on the layer.

Even if the irregular diffusion equation (1.1) can be shown to be well posed, until now we can only prove existence (but not yet uniqueness) of solutions to the corresponding NLSDE. On the other hand, if $\Phi\ge c_0>0$, then uniqueness can be proven. For our applications, this will solve the case $\Phi(u)=H(x-e_c)+\varepsilon$ for some positive ε. The main novelty with respect to the literature is the fact that Φ can be irregular with jumps.

To the best of our knowledge the first author who considered a probabilistic representation (of the type studied in this paper) for the solutions of a nonlinear deterministic PDE was McKean [22], particularly in relation to the so-called propagation of chaos. In his case, however, the coefficients were smooth. Thereafter, the literature steadily grew and there is now a vast amount of contributions to the subject, especially when the nonlinearity is in the first order part, as, for example, in Burgers' equation. We refer the reader to the excellent survey papers [28] and [18].

A probabilistic interpretation of (1.1) when $\beta(u)=|u|u^{m-1}$, $m>1$, was provided in, for instance, [8]. For the same β, although the method could be adapted to the case where β is Lipschitz, in [19], the author studied the evolution equation (1.1) when the initial condition and the evolution take values in the class of probability distribution functions on $\mathbb{R}$. Therefore, instead of an evolution equation in $L^1(\mathbb{R})$, he considers a state space of functions vanishing at $-\infty$ and with value 1 at $+\infty$. He studies both the probabilistic representation and the propagation of chaos.

Let us now describe the principle of the aforementioned probabilistic representation. The stochastic differential equation (in the weak sense), giving rise to the probabilistic representation, is given by the following (random) nonlinear diffusion:

$$\begin{cases} Y_t=Y_0+\displaystyle\int_0^t\Phi(u(s,Y_s))\,dW_s,\\ \text{law density}(Y_t)=u(t,\cdot),\end{cases}\tag{1.3}$$

where W is a classical Brownian motion. The solution of that equation may be visualized as a continuous process Y on some filtered probability space

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ equipped with a Brownian motion W. By looking at a properly chosen version, we can, and shall, assume that $Y:[0,T]\times\Omega\to\mathbb{R}_+$ is $\mathcal{B}([0,T])\otimes\mathcal{F}$-measurable. Of course, we can only have (weak) uniqueness for (1.3) if we fix the initial distribution, that is, we have to fix the distribution (density) u_0 of Y_0.

The connection with (1.1) is then given by the following result.

THEOREM 1.4. (i) *Let us assume the existence of a solution Y for* (1.3). *Then*, $u:[0,T]\times\mathbb{R}\to\mathbb{R}_+$ *provides a solution*, *in the sense of distributions*, *of* (1.1) *with* $u_0 := u(0,\cdot)$.

(ii) *Let u be a solution of* (1.1), *in the sense of distributions*, *and let Y solve the first equation in* (1.3) *with law density $v(t,\cdot)$ and initial law density $u_0 = u(0,\cdot)$. Then*,

$$\partial_t v = \tfrac{1}{2}\partial_{xx}^2(\Phi^2(u)v), \tag{1.4}$$

in the sense of distributions. In particular, if v is the unique solution of (1.4), *with* $v(0,\cdot)=u_0$, *then* $v=u$.

PROOF. Let $\varphi\in C_0^\infty(\mathbb{R})$, Y be a solution to the first line of (1.3) such that $v(t,\cdot)$ is the law density Y_t for positive t. We apply Itô's formula to $\varphi(Y)$, to obtain

$$\varphi(Y_t) = \varphi(Y_0) + \int_0^t \varphi'(Y_s)\Phi(u(s,Y_s))\,dW_s + \frac{1}{2}\int_0^t \varphi''(Y_s)\Phi^2(u(s,Y_s))\,ds.$$

Taking expectation, we obtain

$$\int_{\mathbb{R}} \varphi(y)v(t,y)\,dy = \int_{\mathbb{R}} \varphi(y)u_0(y)\,dy + \frac{1}{2}\int_0^t ds\int_{\mathbb{R}} \varphi''(y)\Phi^2(u(s,y))v(s,y)\,dy.$$

Both assertions (i) and (ii) now follow. □

REMARK 1.5. An immediate consequence of the probabilistic representation of a solution of (1.1) is its positivity at any time. Also, the property that the initial condition is of mass 1 is, in this case, conserved.

The main purpose of this paper is to show existence and uniqueness in law of the probabilistic representation equation (1.3), in the case that β is nondegenerate and not necessarily continuous. In addition, we prove existence for (1.3) in some degenerate cases under certain conditions; see Section 4.2.

Let us now briefly explain the points that we are able to treat and the difficulties which naturally appear in regard to the probabilistic representation.

For simplicity, we do this for β being single-valued and continuous. However, with some technical complications, this generalizes to the multivalued case, as will be shown in the subsequent sections.

1. Monotonicity methods allow us to show existence and uniqueness of solutions to (1.1), in the sense of distributions, under the assumption that β is monotone, that there exists $\lambda > 0$ with $(\beta + \lambda id)(\mathbb{R}) = \mathbb{R}$ and that β is continuous at zero; see Proposition 3.4 below. We emphasize that, for uniqueness, no surjectivity of $\beta + \lambda id$ is required; see Remark 3.6 below.
2. Let $a : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a strictly positive bounded Borel function. Let $\mathcal{M}(\mathbb{R})$ be the set of all signed measures on $\mathbb{R}$ with finite total variation. We prove uniqueness of solutions of

$$\begin{cases} \partial_t v = \partial_{xx}^2 (av), \\ v(0, x) = u_0(x), \end{cases} \tag{1.5}$$

as an evolution problem in $\mathcal{M}(\mathbb{R})$, at least under an additional Assumption (A); see Theorem 3.8 below.
3. If β is *nondegenerate*, then we can construct a unique (weak) solution Y to the nonlinear SDE corresponding to (1.3), for any initial bounded probability density u_0 on $\mathbb{R}$; see Theorem 4.4 below. For this construction, items 1 and 2 above are used in a crucial way.
4. Suppose β is possibly degenerate. We fix a bounded probability density u_0. We set $\Phi_\varepsilon = \Phi + \varepsilon$ and consider the weak solution Y^ε of

$$Y_t^\varepsilon = \int_0^t \Phi_\varepsilon(u^\varepsilon(s, Y_s^\varepsilon))\, dW_s, \tag{1.6}$$

where $u^\varepsilon(t, \cdot)$ is the law of Y_t^ε, $t \geq 0$, and Y_0^ε is distributed according to $u_0(x)\, dx$. The sequence of laws of the processes (Y^ε) are tight. However, the limiting processes Y of convergent subsequences may, in general, not solve the SDE

$$Y_t = \int_0^t \Phi(u(s, Y_s))\, dW_s. \tag{1.7}$$

However, under some additional assumptions (see Properties 4.8 and 4.10 below), processes Y will do it. The analysis of the degenerate case in greater generality [including case (1.2)] will be the subject of the forthcoming paper [6].

In this paper, we proceed as follows. Section 2 is devoted to preliminaries concerning elliptic PDEs satisfying monotonicity conditions.

In Section 3, we first state a general existence and uniqueness result (Proposition 3.4) for (1.1) and provide its proof; see item 1 above. The rest of Section 3 is devoted to the study of the uniqueness of a deterministic, time-inhomogeneous singular linear equation with evolution in the space of probabilities on $\mathbb{R}$. This will be applied to the study of the uniqueness of (1.5) in item 2 above. This is only possible in the nondegenerate case (see Theorem 3.8); if β is not nondegenerate, we give a counterexample in Remark 3.11.

Section 4 is devoted to the probabilistic representation (1.3). In particular, in the nondegenerate (but not smooth) case, Theorem 4.4 gives existence and uniqueness of the nonlinear diffusion (1.3) which represents, probabilistically, (1.1). In the degenerate case, Proposition 4.12 gives an existence result.

Finally, we would like to mention that, in order to keep this paper self-contained and make it accessible to a larger audience, we include the analytic background material and necessary (although standard) definitions. Likewise, we have tried to explain all details of the delicate analytic, and quite technical, parts of the paper which form the backbone of the proofs for our main result.

2. Preliminaries.

We start with the basic analytical framework.

If $f:\mathbb{R}\to\mathbb{R}$ is a bounded function, we will set $\|f\|_\infty := \sup_{x\in\mathbb{R}}|f(x)|$. By $C_b(\mathbb{R})$, we denote the space of bounded and continuous real functions, by $\mathcal{S}(\mathbb{R})$, the space of rapidly decreasing infinitely differentiable functions $\varphi:\mathbb{R}\to\mathbb{R}$ and by $\mathcal{S}'(\mathbb{R})$ its dual (the space of tempered distributions).

Let K_ε be the Green function of $\varepsilon-\Delta$, that is, the kernel of the operator $(\varepsilon-\Delta)^{-1}:L^2(\mathbb{R})\to L^2(\mathbb{R})$. Thus, for all $\varphi\in L^2(\mathbb{R})$, we have

$$B_\varepsilon(\varphi) := (\varepsilon-\Delta)^{-1}\varphi(x) = \int_{\mathbb{R}} K_\varepsilon(x-y)\varphi(y)\,dy. \tag{2.1}$$

The next lemma provides us with an explicit expression for the kernel function K_ε.

LEMMA 2.1.

$$K_\varepsilon(x) = \frac{1}{2\sqrt{\varepsilon}}e^{-\sqrt{\varepsilon}|x|}, \qquad x\in\mathbb{R}. \tag{2.2}$$

PROOF. From Definition 6.27 in [26], we get

$$K_\varepsilon(x) = \frac{1}{(4\pi)^{1/2}}\int_0^\infty t^{-1/2}e^{-|x|^2/(4t)-\varepsilon t}\,dt. \tag{2.3}$$

The result follows by standard calculus. $\square$

Clearly, if $\varphi\in C^2(\mathbb{R})\cap\mathcal{S}'(\mathbb{R})$, then $(\varepsilon-\Delta)\varphi$ coincides with the classical associated PDE operator.

LEMMA 2.2. *Let* $\varepsilon>0, m\in\mathcal{M}(\mathbb{R})$. *There is then a unique solution* $v_\varepsilon\in C_b(\mathbb{R})\cap(\bigcap_{p\geq 1}L^p(\mathbb{R}))$ *of*

$$\varepsilon v_\varepsilon - \Delta v_\varepsilon = m, \tag{2.4}$$

in the sense of distributions, *given by*

$$v_\varepsilon(x) := \int_{\mathbb{R}} K_\varepsilon(x-y)\,dm(y), \qquad x\in\mathbb{R}. \tag{2.5}$$

Moreover, it fulfills

$$\sup_x \sqrt{\varepsilon}|v_\varepsilon(x)| \le \frac{\|m\|_{\mathrm{var}}}{2}, \tag{2.6}$$

where $\|m\|_{\mathrm{var}}$ *denotes the total variation norm. In addition, the derivative* v'_ε *has a bounded cadlag version which is locally of bounded variation.*

In the sequel, in analogy with (2.1), that solution will be denoted by $B_\varepsilon m$.

PROOF OF LEMMA 2.2. Uniqueness follows from an obvious application of a Fourier transform. In fact, it holds even in $\mathcal{S}'(\mathbb{R})$.

v_ε given by (2.5) clearly satisfies (2.4), in the sense of distributions. By Lebesgue's dominated convergence theorem and because K_ε is a bounded continuous function, it follows that $v_\varepsilon \in C_b(\mathbb{R})$.

By Lemma 2.1, we have

$$\sup_x |v_\varepsilon(x)| \le \frac{1}{2\sqrt{\varepsilon}}\|m\|_{\mathrm{var}}. \tag{2.7}$$

By Fubini's theorem and (2.2), it follows that $v_\varepsilon \in L^1(\mathbb{R})$. Hence, $v_\varepsilon \in L^p(\mathbb{R})$, $\forall p \ge 1$, because v_ε is bounded.

Since v''_ε equals $\varepsilon v_\varepsilon - m$, in the sense of distributions, after integration, we can show that

$$v'_\varepsilon(x) = \varepsilon \int_{-\infty}^x v_\varepsilon(y)\,dy - m(]-\infty, x])$$

for dx-a.e. $x \in \mathbb{R}$. In particular, v_ε has a bounded cadlag version which is locally of bounded variation and

$$\|v'_\varepsilon\|_\infty \le \varepsilon\|v\|_{L^1(\mathbb{R})} + \|m\|_{\mathrm{var}}. \qquad \square$$

We now recall some basic notions from the analysis of monotone operators. More information can also be found in, for instance, [25]; see also [5, 13].

Let E be a general Banach space.

One of the most basic notions of this paper is that of a multivalued function (graph). A *multivalued function* (graph) β on E will be a subset of $E \times E$. It can be seen either as a family of couples (e, f), $e, f \in E$, where we will write $f \in \beta(e)$, or as a function $\beta : E \to \mathcal{P}(E)$.

We start with a definition in the case $E = \mathbb{R}$.

DEFINITION 2.3. A multivalued function β defined on $\mathbb{R}$ with values in subsets of $\mathbb{R}$ is said to be *monotone* if $(x_1 - x_2)(y_1 - y_2) \ge 0$ for all $x_1, x_2 \in \mathbb{R}$, $y_i \in \beta(x_i)$, $i = 1, 2$.

We say that β is *maximal monotone* if it is monotone and there exists $\lambda > 0$ such that $\beta + \lambda id$ is surjective, that is,

$$\mathcal{R}(\beta + \lambda id) := \bigcup_{x\in\mathbb{R}} \big(\beta(x) + \lambda x\big) = \mathbb{R}.$$

We recall that one focus of this paper is the case where $\beta(u) = H(u - e_c)u$.

Let us consider a monotone function ψ. Then, all the discontinuities are of jump type. At every discontinuity point x of ψ, it is possible to *complete* ψ, producing a multivalued function, by setting $\psi(x) = [\psi(x-), \psi(x+)]$.

Since ψ is a monotone function, the corresponding multivalued function will, of course, also be monotone.

We now return to the case of our general Banach space E with norm $\|\cdot\|$. An operator $T : E \to E$ is said to be a *contraction* if it is Lipschitz of norm less than or equal to 1 and $T(0) = 0$.

DEFINITION 2.4. A map $A : E \to E$, or, more generally, a multivalued map $A : E \to \mathcal{P}(E)$, is said to be *accretive* if, for all $f_1, f_2, g_1, g_2 \in E$ such that $g_i \in Af_i$, $i = 1, 2$, we have

$$\|f_1 - f_2\| \le \|f_1 - f_2 + \lambda(g_1 - g_2)\|$$

for any $\lambda > 0$.

This is equivalent to saying the following: for any $\lambda > 0$, $(1 + \lambda A)^{-1}$ is a contraction for any $\lambda > 0$ on $Rg(I + \lambda A)$. We remark that a contraction is necessarily single-valued.

REMARK 2.5. Suppose that E is a Hilbert space equipped with the scalar product $(\cdot,\cdot)_H$. Then, A is accretive if and only if it is *monotone*, that is, $(f_1 - f_2, g_1 - g_2)_H \ge 0$ for any $f_1, f_2, g_1, g_2 \in E$ such that $g_i \in Af_i$, $i = 1, 2$; see Corollary 1.3 of [25].

DEFINITION 2.6. A monotone map $A : E \to E$ (possibly multivalued) is said to be *m-accretive* if, for some $\lambda > 0$, $A + \lambda I$ is surjective (as a graph in $E \times E$).

REMARK 2.7. A is therefore m-accretive if and only if, for all λ strictly positive, $(I + \lambda A)^{-1}$ is a contraction on E.

Let us now consider the case $E = L^1(\mathbb{R})$. The following is taken from [10], Section 1.

PROPOSITION 2.8. *Let $\beta : \mathbb{R} \to \mathbb{R}$ be a monotone (possibly multivalued) map such that the corresponding graph is m-accretive. Suppose that $\beta(0) = 0$.*

Let $f \in E = L^1(\mathbb{R})$.

1. *There is a unique $u \in L^1(\mathbb{R})$ for which there exists $w \in L^1_{\mathrm{loc}}(\mathbb{R})$ such that*

$$u - \Delta w = f \qquad \text{in } \mathcal{D}'(\mathbb{R}), \qquad w(x) \in \tfrac{1}{2}\beta(u(x)) \qquad \text{for a.e. } x \in \mathbb{R}; \tag{2.8}$$

   see Proposition 2 *of* [10].
2. *It is then possible to define a* (*multivalued*) *operator $A := A_\beta : E \to E$, where $D(A)$ is the set of $u \in L^1(\mathbb{R})$ for which there exists $w \in L^1_{\mathrm{loc}}(\mathbb{R})$ such that $w(x) \in \frac{1}{2}\beta(u(x))$ for a.e. $x \in \mathbb{R}$ and $\Delta w \in L^1(\mathbb{R})$. For $u \in D(A)$, we set*

$$Au = \left\{-\tfrac{1}{2}w \mid w \text{ as in the definition of } D(A)\right\}.$$

   This is a consequence of the remarks following Theorem 1 *in* [10].

   In particular, if β is single-valued, then $Au = -\frac{1}{2}\Delta\beta(u)$. We will also adopt this notation if β is multi-valued.
3. *The operator A defined in* 2 *above is m-accretive on $E = L^1(\mathbb{R})$; see Proposition* 2 *of* [10].
4. *We set $J_\lambda = (I + \lambda A)^{-1}$, which is a single-valued operator. If $f \in L^\infty(\mathbb{R})$, then $\|J_\lambda f\|_\infty \le \|f\|_\infty$; see Proposition* 2(iii) *of* [10]. *In particular, for every positive integer n, $\|J_\lambda^n f\|_\infty \le \|f\|_\infty$.*

Let us summarize some important results of the theory of nonlinear semigroups; see, for instance, [16, 4, 5, 9], or the more recent monograph [25], which we shall use below. Let $A : E \to E$ be a (possibly multivalued) m-accretive operator. We consider the equation

$$0 \in u'(t) + A(u(t)), \qquad 0 \le t \le T. \tag{2.9}$$

A function $u : [0, T] \to E$ which is absolutely continuous such that for a.e. t, $u(t, \cdot) \in D(A)$ and fulfills (2.9) in the following sense, is called a *strong solution*.

There exists an $\eta : [0, T] \to E$, Bochner integrable, such that $\eta(t) \in A(u(t))$ for a.e. $t \in [0, T]$ and

$$u(t) = u_0 + \int_0^t \eta(s)\, ds, \qquad 0 < t \le T.$$

A weaker notion for (2.9) is the so-called C^0-*solution*; see Chapter IV.8 of [25]. In order to introduce it, one first defines the notion of an ε-solution for (2.9).

An ε-*solution* is a discretization

$$\mathcal{D} = \{0 = t_0 < t_1 < \cdots < t_N = T\}$$

and an E-valued step function

$$u^\varepsilon(t) = \begin{cases} u_0, & t = t_0, \\ u_j \in D(A), & t \in]t_{j-1}, t_j], \end{cases}$$

for which $t_j - t_{j-1} \le \varepsilon$ for $1 \le j \le N$ and

$$0 \in \frac{u_j - u_{j-1}}{t_j - t_{j-1}} + Au_j, \qquad 1 \le j \le N.$$

We remark that, since A is m-accretive, u^ε is determined by $\mathcal{D}$ and u_0; see Proposition 2.8, item 1.

DEFINITION 2.9. A C^0-*solution* of (2.9) is a function $u \in C([0,T]; E)$ such that for every $\varepsilon > 0$, there exists an ε-solution u^ε of (2.9) with

$$\|u(t) - u^\varepsilon(t)\| \le \varepsilon, \qquad 0 \le t \le T.$$

PROPOSITION 2.10. *Let A be an m-accretive* (*multivalued*) *operator on a Banach space E. We again set $J_\lambda := (I + \lambda A)^{-1}$, $\lambda > 0$. Suppose that $u_0 \in \overline{D(A)}$. Then*:

1. *there is a unique C^0-solution $u : [0,T] \to E$ of* (2.9);
2. *$u(t) = \lim_{n\to\infty} J^n_{t/n} u_0$ uniformly in $t \in [0,T]$.*

PROOF. Item 1 is stated in Corollary IV.8.4 of [25] and item 2 is contained in Theorem IV 8.2 of [25]. □

The notion of C^0-(or mild) solution needs to be introduced since the dual E^* of $E = L^1(\mathbb{R})$ is not uniformly convex. If E^* were indeed uniformly convex, then we could have stayed with strong solutions. In fact, according to Theorem IV 7.1 of [25], for a given $u_0 \in D(A)$, there would exist a (strong) solution $u : [0,T] \to E$ of (2.9), which is a simpler notion to deal with. For the comfort of the reader, we recall the following properties:

- a strong solution is a C^0-solution, by Proposition IV 8.2 of [25];
- Theorem 1.2 of [15] states that given $u_0 \in \overline{D(A)}$ and given a sequence (u_0^n) in $D(A)$ converging to u_0, then the sequence of the corresponding strong solutions (u_n) converges to the unique C^0-solution of the same equation.

3. A porous media equation with singular coefficients. In this section, we will first provide an existence and uniqueness result for solutions to the parabolic deterministic equation (1.1), in the sense of distributions, for multivalued m-accretive β. The proof is partly based on the theory of nonlinear semigroups; see [10] for the case where β is continuous.

However, the most important result of this section is an existence and uniqueness result for a "nondegenerate" linear equation for measures; see (1.5). This technical result will be crucial for identifying the law of the process appearing in the probabilistic representation (1.3).

We suppose that β has the same properties as those given in the Introduction. However, β is allowed to be multivalued, hence m-accretive, as a graph, in the sense of Definition 2.3. Furthermore, generalizing Assumption 1.1, we shall assume the following.

ASSUMPTION 3.1. Let $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ be an m-accretive graph with the property that there exists $c > 0$ such that

$$w \in \beta(u) \quad \Rightarrow \quad |w| \le c|u|. \tag{3.1}$$

REMARK 3.2. In particular, $\beta(0) = 0$ and β is continuous at zero. We again use the representation $\beta(u) = \Phi^2(u)u$ with Φ being a nonnegative, bounded, multivalued map $\Phi : \mathbb{R} \to \mathbb{R}$.

REMARK 3.3. As mentioned before, if $\beta : \mathbb{R} \to \mathbb{R}$ is monotone (possibly discontinuous), it is possible to complete β into a monotone graph. For instance, if $\Phi(x) = H(x - e_c)$, then

$$\beta(x) = \begin{cases} 0, & x < e_c, \\ [0, e_c], & x = e_c, \\ x, & x > e_c. \end{cases}$$

Since the function β is monotone, the corresponding graph is monotone. Moreover, $\beta + id$ is surjective so that, by definition, β is m-accretive.

PROPOSITION 3.4. *Let $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. There then exists a unique solution, in the sense of distributions, $u \in (L^1 \cap L^\infty)([0, T] \times \mathbb{R})$ of*

$$\begin{cases} \partial_t u \in \frac{1}{2}\partial_{xx}^2(\beta(u)), \\ u(t, x) = u_0(x); \end{cases} \tag{3.2}$$

that is, there exists a unique couple $(u, \eta_u) \in ((L^1 \cap L^\infty)([0, T] \times \mathbb{R}))^2$ such that

$$\begin{aligned} \int u(t, x)\varphi(x)\,dx &= \int u_0(x)\varphi(x)\,dx \\ &\quad + \frac{1}{2}\int_0^t ds \int \eta_u(s, x)\varphi''(x)\,dx \qquad \forall \varphi \in \mathcal{S}(\mathbb{R}) \quad \textit{and} \end{aligned} \tag{3.3}$$

$$\eta_u(t, x) \in \beta(u(t, x)) \qquad \textit{for } dt \otimes dx\textit{-a.e.}(t, x) \in [0, T] \times \mathbb{R}.$$

Furthermore, $\|u(t, \cdot)\|_\infty \le \|u_0\|_\infty$ for every $t \in [0, T]$ and there exists a unique version of u such that $u \in C([0, T]; L^1(\mathbb{R}))$ $(\subset L^1([0, T] \times \mathbb{R}))$.

REMARK 3.5.

1. We remark that the uniqueness of u determines the uniqueness of $\eta \in \beta(u)$ a.e. In fact, for $s, t \in [0, T]$, we have

$$\left(\frac{1}{2}\int_s^t \eta_u(r, \cdot)\,dr\right)'' = u(t, \cdot) - u(s, \cdot) \qquad \text{a.e.} \tag{3.4}$$

Since $\eta_u \in L^1([0, T] \times \mathbb{R})$, this implies that the function η_u is $dt \otimes dx$-a.e. uniquely determined. Furthermore, since $\beta(0) = 0$ and because β is monotone, for $dt \otimes dx$-a.e. $(t, x) \in [0, T] \times \mathbb{R}$, we have

$$u(t, x) = 0 \quad \Rightarrow \quad \eta_u(t, x) = 0$$

and

$$u(t,x)\eta_u(t,x) \geq 0.$$

2. If β is continuous, then we can take $\eta_u(s,x) = \beta(u(s,x))$.
3. This result applies in the Heaviside case, where $\Phi(x) = H(x - e_c)$, and in the nondegenerate case $\Phi(x) = H(x - e_c) + \varepsilon$.

PROOF OF PROPOSITION 3.4. We first recall that, by our assumptions, we have $(\beta + \lambda id)(\mathbb{R}) = \mathbb{R}$ for every $\lambda > 0$.

1. The first step is to prove the existence of a C^0-solution of the evolution problem (2.9) in $E = L^1(\mathbb{R})$, with A and $D(A)$ as defined in Proposition 2.8, item 2. Suppose that $\overline{D(A)} = L^1(\mathbb{R})$. Then, the existence of a C^0-solution $u \in C([0,T]; L^1(\mathbb{R}))$ is a consequence of Proposition 2.8, item 3 and Proposition 2.10, item 1. In particular, u belongs to $L^1([0,T] \times \mathbb{R})$.
2. We now prove that $D(A)$ is dense in $E = L^1(\mathbb{R})$.

   Let $u \in E$. We must show the existence of a sequence (u_n) in $D(A)$ converging to u in E. We set $u_\lambda = (I + \lambda A)^{-1}u$ so that $u \in u_\lambda - \lambda\Delta\frac{1}{2}\beta(u_\lambda)$. The result follows if we are able to show that

$$\lim_{\lambda \to 0} u_\lambda = u \qquad \text{weakly in } E, \tag{3.5}$$

   because then $D(A)$ is weakly sequentially dense in $L^1(\mathbb{R})$. In fact, we can easily show that $D(A)$ is convex and therefore so is its closure. Hence, by Satz 6.12 of [1], $D(A)$ is also weakly sequentially closed and so the result would follow. We therefore continue proving (3.5). Since $(I + \lambda A)^{-1}$ is a contraction on E, $u_\lambda \in E$ and the sequence (u_λ) is bounded in $L^1(\mathbb{R})$. Since $u_\lambda \in D(A)$, by definition, there exists $w_\lambda \in L^1_{\mathrm{loc}}(\mathbb{R})$ such that $w_\lambda(x) \in \frac{1}{2}\beta(u_\lambda(x))$ for dx-a.e. $x \in \mathbb{R}$, $\Delta w_\lambda \in L^1(\mathbb{R})$ and $u = u_\lambda - \lambda\Delta w_\lambda$. Since β has linear growth, w_λ also belongs to E for every $\lambda > 0$ and the sequence w_λ is bounded in E. Consequently, λw_λ converges to zero in E when $\lambda \to 0$ and it follows that $\lambda\Delta w_\lambda$ converges to zero in the sense of distributions, hence $u_\lambda \to u$, again in the sense of distributions. Because (u_λ) is bounded in $L^1(\mathbb{R})$, it follows that $u_\lambda \to u$ weakly in $E = L^1(\mathbb{R})$ as $\lambda \to 0$.
3. The third step consists of showing that a C^0-solution is a solution, in the sense of distributions, of (3.2).

   Let $\varepsilon > 0$ and consider a family $u^\varepsilon : [0,T] \to E$ of ε-solutions. Note that, for $u_0^\varepsilon := u_0$ and $1 \leq j \leq N$, with A as in Proposition 2.8, item 2, we recursively have

$$u_j^\varepsilon = \left(I - (t_j^\varepsilon - t_{j-1}^\varepsilon)A\right)^{-1} u_{j-1}^\varepsilon, \tag{3.6}$$

   hence

$$\Delta w_j^\varepsilon = -\frac{u_j^\varepsilon - u_{j-1}^\varepsilon}{t_j^\varepsilon - t_{j-1}^\varepsilon}$$

for some $w_j^\varepsilon \in L^1_{\mathrm{loc}}(\mathbb{R})$ such that $w_j^\varepsilon \in \frac{1}{2}\beta(u_j^\varepsilon)$, dx-a.e. Hence, for $t \in]t_{j-1}^\varepsilon, t_j^\varepsilon]$, we have

$$u^\varepsilon(t,\cdot) = u^\varepsilon(t_{j-1}^\varepsilon,\cdot) + \int_{t_{j-1}^\varepsilon}^{t_j^\varepsilon} \Delta w^\varepsilon(s,\cdot)\,ds,$$

where $w^\varepsilon(t) = w_j^\varepsilon$, $t \in]t_{j-1}^\varepsilon, t_j^\varepsilon]$.

Consequently, summing up, for $t \in]t_{j-1}^\varepsilon, t_j^\varepsilon]$, we have

$$u^\varepsilon(t,\cdot) = u_0 + \int_0^t \Delta w^\varepsilon(s,\cdot)\,ds + (t_j^\varepsilon - t)\Delta w^\varepsilon(t_j^\varepsilon,\cdot).$$

We integrate against a test function $\alpha \in \mathcal{S}(\mathbb{R})$ and get

$$\begin{aligned}\int_{\mathbb{R}} u^\varepsilon(t,x)\alpha(x)\,dx &= \int_{\mathbb{R}} u_0(x)\alpha(x)\,dx + \int_0^t \int_{\mathbb{R}} w^\varepsilon(s,x)\alpha''(x)\,dx\,ds \\ &\quad + (t - t_j^\varepsilon)\int_{\mathbb{R}} w^\varepsilon(t_j^\varepsilon,x)\alpha''(x)\,dx.\end{aligned}\tag{3.7}$$

Letting ε go to zero, we use the fact that $u^\varepsilon \to u$ uniformly in t in $L^1(\mathbb{R})$. (u^ε) converges, in particular, to $u \in L^1([0,T]\times\mathbb{R})$ when $\varepsilon \to 0$.

The third term in the right-hand side of (3.7) converges to zero since $t - t_j^\varepsilon$ is smaller than the mesh ε of the subdivision.

Consequently, (3.7) implies

$$\int_{\mathbb{R}} u(t,x)\alpha(x)\,dx = \int_{\mathbb{R}} u_0(x)\alpha(x)\,dx + \lim_{\varepsilon\to 0}\int_0^t \int_{\mathbb{R}} w^\varepsilon(s,x)\alpha''(x)\,dx\,ds.\tag{3.8}$$

According to our assumption on β, there exists a constant $c > 0$ such that $|w^\varepsilon| \le c|u^\varepsilon|$. Therefore, the sequence (w^ε) is equi-integrable on $[0,T]\times\mathbb{R}$. Therefore, there exists a sequence (ε_n) such that w^{ε_n} converges to some $\frac{1}{2}\eta_u \in L^1([0,T]\times\mathbb{R})$ in $\sigma(L^1, L^\infty)$. Taking (3.8) into account, it remains to see that $\eta_u(t,x) \in \beta(u(t,x))$, $dt \otimes dx$-a.e., in order to prove that u solves (3.3).

Let $K > 0$. Using Proposition 2.8, item 4, by (3.6), we conclude that $\|u^\varepsilon(t,\cdot)\|_\infty \le \|u_0\|_\infty$. Consequently, for any $K > 0$, the dominated convergence theorem implies that the sequence u^{ε_n} restricted to $[0,T]\times[-K,K]$ converges to u restricted to $[0,T]\times[-K,K]$ in $L^2([0,T]\times[-K,K])$, and w^{ε_n} restricted to $[0,T]\times[-K,K]$, being bounded by $c|u^{\varepsilon_n}|$, converges (up to a subsequence) weakly in L^2, necessarily to $\frac{1}{2}\eta_u$ restricted to $[0,T]\times[-K,K]$. The map $v \to \frac{1}{2}\beta(v)$ on $L^2([0,T]\times[-K,K])$ is an m-accretive multivalued map; see [25], Example 2c, page 164. Thus, it is weakly-strongly closed because of [5], Proposition 1.1(i) and (ii), page 37. Hence, the result follows.

4. The fourth step consists of showing that the obtained solution is in $L^\infty([0,T]\times\mathbb{R})$.

   Item 2 of Proposition 2.10 tells us that

$$u(t,\cdot) = \lim_{n\to+\infty} J_{t/n}^n u_0$$

in $L^1(\mathbb{R})$. Hence, for every $t \in]0, T]$ and for some subsequence (n_k) depending on t,

$$|u(t, \cdot)| = \lim_{k \to \infty} |J^{n_k}_{t/n_k} u_0| \le \|u_0\|_\infty, \qquad dx\text{-a.e.},$$

where we have again used Proposition 2.8, item 4. It follows, by Fubini's theorem, that $|u(t, x)| \le \|u_0\|_\infty$ for $dt \otimes dx$-a.e. $(t, x) \in [0, T] \times \mathbb{R}$.

5. Finally, uniqueness of the equation in $\mathcal{D}'([0, T] \times \mathbb{R})$ follows from Theorem 1 and Remark 1.20 of [14]. □

REMARK 3.6.

1. Theorem 1 and Remark 1.20 of [14] apply, if β is continuous, to give the uniqueness in item 5 above. However, Remark 1.21 of [14] implies that this holds true even if $\beta(0) = 0$ and β is only continuous at zero and possibly multivalued. This case applies, for instance, when $\Phi(x) = H(x - e_c)$, $e_c > 0$.
2. We would like to mention that there are variants of the results in Proposition 3.4 known from the literature. However, some of them are just for bounded domains, while we work in all of $\mathbb{R}$. For instance, when the domain is bounded and β is continuous, Example 9B in Section IV.9 of [25] remarks that a C^0-solution is a solution in the sense of distributions.

In order to establish the well-posedness of the related probabilistic representation, one needs a uniqueness result for the evolution of probability measures. This will be the subject of Theorem 3.8 below. However, as we will see, it will require some global L^2-integrability for the solutions.

A first step in this direction is Corollary 3.2 of [12], which we quote here for the convenience of the reader.

LEMMA 3.7. *Let $\kappa \in]0, T[$, μ be a finite Borel measure on $[\kappa, T] \times \mathbb{R}$ and $a, b \in L^1([\kappa, T] \times \mathbb{R}; \mu)$. We suppose that*

$$\int_{[\kappa, T] \times \mathbb{R}} \big(\partial_t \varphi(t, x) + a(t, x)\partial^2_{xx}\varphi(t, x) + b(t, x)\partial_x \varphi(t, x)\big)\mu(dt\, dx) = 0$$

for all $\varphi \in C^\infty_0(]0, +\infty[\times\mathbb{R})$. There then exists $\rho \in L^2_{\mathrm{loc}}([\kappa, T] \times \mathbb{R})$ such that

$$\sqrt{a(t, x)}\, d\mu(t, x) = \rho(t, x)\, dt\, dx.$$

We denote the subset of positive measures in $\mathcal{M}(\mathbb{R})$ by $\mathcal{M}_+(\mathbb{R})$.

THEOREM 3.8. *Let a be a Borel, nonnegative, bounded function on $[0, T] \times \mathbb{R}$. Let $z_i : [0, T] \to \mathcal{M}_+(\mathbb{R})$, $i = 1, 2$, be continuous with respect to the weak topology of finite measures on $\mathcal{M}(\mathbb{R})$.*

Let z^0 be an element of $\mathcal{M}_+(\mathbb{R})$. Suppose that both z_1 and z_2 solve the problem $\partial_t z = \partial^2_{xx}(az)$, in the sense of distributions, with initial condition $z(0) = z^0$.

More precisely,

$$\int_{\mathbb{R}} \varphi(x) z(t)(dx) = \int_{\mathbb{R}} \varphi(x) z^0(dx) + \int_0^t ds \int_{\mathbb{R}} \varphi''(x) a(s,x) z(s)(dx) \tag{3.9}$$

for every $t \in [0,T]$ and any $\varphi \in \mathcal{S}(\mathbb{R})$.

Then, $(z_1 - z_2)(t)$ is identically zero for every t if $z := z_1 - z_2$ satisfies the following assumption.

ASSUMPTION (A). There exists $\rho : [0,T] \times \mathbb{R} \to \mathbb{R}$ belonging to $L^2([\kappa, T] \times \mathbb{R})$ for every $\kappa > 0$ such that $\rho(t,\cdot)$ is the density of $z(t)$ for almost all $t \in]0,T]$.

REMARK 3.9. If $a \geq \text{const} > 0$, then ρ, such that $\rho(t,\cdot)$ is a density of $(z_1 - z_2)(t)$ for almost all $t > 0$, always exists, by Lemma 3.7. It remains to check if it is indeed square integrable on every $[\kappa, T] \times \mathbb{R}$.

REMARK 3.10. The weak continuity of $z(t,\cdot)$ implies that

$$\sup_{t \in [0,T]} \|z(t)\|_{\text{var}} < \infty.$$

Indeed, if this were not true, we could find $t_n \in [0,T]$ such that $\|z(t_n)\|_{\text{var}}$ diverges to infinity. We may assume that $\lim_{n\to\infty} t_n = t_0 \in [0,T]$. Then,

$$\lim_{n\to\infty} \int_{\mathbb{R}} f(x) z(t_n)(dx) = \int_{\mathbb{R}} f(x) z(t_0)(dx)$$

for all $f \in C_b(\mathbb{R})$ and, hence, by the uniform boundedness principle, we arrive at the contradiction that

$$\sup_n \|z(t_n)\|_{\text{var}} < \infty.$$

REMARK 3.11. Theorem 3.8 does not hold without Assumption (A), even in the time-homogeneous case.

To explain this, let $\Phi : \mathbb{R} \to \mathbb{R}_+$ be continuous and bounded such that $\Phi(0) = 0$ and Φ is strictly positive on $\mathbb{R} - \{0\}$. We also suppose that $\frac{1}{\Phi^2}$ is integrable in a neighborhood of zero.

We choose $z^0 := \delta_0$, that is, the Dirac measure at zero. It is then possible to exhibit two different solutions to the considered problem with initial condition z^0.

We justify this in the following lines using a probabilistic representation. Let Y_0 be identically zero.

According to the Engelbert–Schmidt criterion (see, e.g., Theorem 5.4 and Remark 5.6 of [20], Chapter 5), it is possible to construct two solutions (in law) to the SDE

$$Y_t = \int_0^t \Phi(Y_s)\, dW_s, \tag{3.10}$$

where W is a Brownian motion on some filtered probability space.

One solution, $Y^{(1)}$, is identically zero. The second one, $Y^{(2)}$, is a nonconstant martingale starting from zero. We recall the construction of $Y^{(2)}$ as it is of independent interest.

Let B be a classical Brownian motion and set

$$T_t = \int_0^t \frac{du}{\Phi^2(B_u)}. \tag{3.11}$$

Problem 6.30 of [20] implies that the increasing process (T_t) diverges to infinity when t goes to infinity. We define, pathwise, (A_t) as the inverse of (T_t) and we set $M_t = B_{A_t}$. M is a martingale since it is a time change of Brownian motion. On one hand, we have $[M]_t = A_t$. However, pathwise, by (3.11), we have

$$A_t = \int_0^{A_t} \Phi^2(B_u)\, dT_u = \int_0^t \Phi^2(B_{A_v})\, dv,$$

via a change of variables $u = A_v$. Consequently, we get

$$A_t = \int_0^t \Phi^2(M_v)\, dv.$$

Theorem 4.2 from [20], Chapter 3 states that there exists a Brownian motion $\tilde{W}$ on a suitable filtered larger probability space and an adapted process (ρ_t) such that $M_t = \int_0^t \rho\, d\tilde{W}$. We have $[M]_t = \int_0^t \rho_s^2\, ds = \int_0^t \Phi^2(M_s)\, ds$ for all $t \geq 0$, hence $\rho_t^2 = \Phi^2(M_t)$ and so $\Phi(M_t)\operatorname{sign}(\rho_t) = \rho_t$.

We define

$$W_t = \int_0^t \operatorname{sign}(\rho_v)\, d\tilde{W}_v.$$

Clearly, $[W]_t = t$. By Lévy's characterization theorem of Brownian motion, W is a standard Brownian motion. Moreover, we obtain $M_t = \int_0^t \Phi(M_s)\, dW_s$ so that $Y^{(2)} := M$ solves the stochastic differential equation (3.10). Now, Y_t^1 and Y_t^2 do not have the same marginal laws $v_i(t,\cdot)$, $i = 1, 2$. In fact, $v_1(t,\cdot)$ is equal to δ_0 for all $t \in [0, T]$.

Using Itô's formula, it is easy to show that the law $v(t,\cdot)$ of a solution Y of (3.10) solves the PDE in Theorem 3.8 with $a := \Phi^2$ and initial condition δ_0. This constitutes a counterexample to Theorem 3.8 without Assumption (A).

PROOF OF THEOREM 3.8. The arguments developed in this proof were inspired by a uniqueness proof for distributional solutions of the porous media equation; see Theorem 1 of [14].

Given a locally integrable function $(t, x) \to u(t, x)$, u' (resp., u'') stands for the first (resp., second) distributional derivative with respect to the second variable x.

In the first part of the proof, we do not use Assumption (A). We will explicitly state when it is needed.

Let z^1, z^2 be two solutions to (3.9) and set $z = z^1 - z^2$. We will study the quantity

$$g_\varepsilon(t) = \int_{\mathbb{R}} B_\varepsilon z(t)(x) z(t)(dx),$$

where $B_\varepsilon z(t) \in (L^1 \cap L^\infty)(\mathbb{R})$ is the continuous function v_ε defined in Lemma 2.2, taking $m = z(t)$. $g_\varepsilon(t)$ is well defined since

$$g_\varepsilon(t) \le \|z(t)\|_{\mathrm{var}} \sup_x |B_\varepsilon z(t)(x)| \qquad \text{for all } t \in [0, T].$$

We assume that we can show

$$\lim_{\varepsilon \to 0} g_\varepsilon(t) = 0 \qquad \text{for all } t \in [0, T]. \tag{3.12}$$

We are then able to prove that $z(t) \equiv 0$ for all $t \in [0, T]$.

Indeed, Lemma 2.2 states that $B_\varepsilon z(t)'$ is bounded, with a locally bounded variation function, and that $B_\varepsilon z(t) \in C_b(\mathbb{R}) \cap L^p(\mathbb{R})$ for all $p \ge 1$.

Now, let $\mathcal{C}, \tilde{\mathcal{C}}$ be positive real constants. Then, since all terms in (2.4) are signed measures of finite total variation, (2.4) implies that

$$\begin{aligned} \int_{]-\tilde{\mathcal{C}}, \mathcal{C}]} B_\varepsilon z(t)(x) z(t)(dx) &= \varepsilon \int_{]-\tilde{\mathcal{C}}, \mathcal{C}]} (B_\varepsilon z(t)(x))^2 \, dx \\ &\quad - \int_{]-\tilde{\mathcal{C}}, \mathcal{C}]} B_\varepsilon z(t)(x) B_\varepsilon z(t)''(dx). \end{aligned} \tag{3.13}$$

If F, G are functions of locally bounded variation, with F continuous and G right-continuous, classical Lebesgue–Stieltjes calculus implies that

$$\int_{]-\tilde{\mathcal{C}}, \mathcal{C}]} F \, dG = FG(\mathcal{C}) - FG(-\tilde{\mathcal{C}}) - \int_{]-\tilde{\mathcal{C}}, \mathcal{C}]} G \, dF. \tag{3.14}$$

Setting $F = B_\varepsilon z(t)$, $G(x) = B_\varepsilon z(t)'$, we get

$$\begin{aligned} &- \int_{]-\tilde{\mathcal{C}}, \mathcal{C}]} B_\varepsilon z(t)(x) B_\varepsilon z(t)''(dx) \\ &\qquad = -B_\varepsilon z(t)(\mathcal{C}) B_\varepsilon z(t)'(\mathcal{C}) + B_\varepsilon z(t)(-\tilde{\mathcal{C}}) B_\varepsilon z(t)'(-\tilde{\mathcal{C}}) \\ &\qquad\quad + \int_{]-\tilde{\mathcal{C}}, \mathcal{C}]} (B_\varepsilon z(t)'(x))^2 \, dx. \end{aligned}$$

Since $B_\varepsilon z(t) \in L^1(\mathbb{R})$, we can choose sequences $(\mathcal{C}_n)$, $(\tilde{\mathcal{C}}_n)$ converging to infinity such that $B_\varepsilon z(t)(\mathcal{C}_n) \to 0$, $B_\varepsilon z(t)(-\tilde{\mathcal{C}}_n) \to 0$ as $n \to \infty$. Then, letting $n \to \infty$ and using the fact that $B_\varepsilon z(t)$ and $B_\varepsilon z(t)'$ are bounded, by the monotone and Lebesgue dominated convergence theorems, we conclude that

$$- \int B_\varepsilon z(t)(x) B_\varepsilon z(t)''(dx) = \int (B_\varepsilon z(t)'(x))^2 \, dx.$$

In particular, $B_\varepsilon z(t)' \in L^2(\mathbb{R})$. Consequently, (3.13) implies that

$$\begin{aligned} g_\varepsilon(t) &= \int B_\varepsilon z(t)(x) z(t)(dx) \\ &= \varepsilon \int (B_\varepsilon z(t)(x))^2\,dx + \int (B_\varepsilon z(t)'(x))^2\,dx. \end{aligned}$$

In particular, the left-hand side is positive. Therefore, if $g_\varepsilon(t) \to 0$ as $\varepsilon \to 0$ for all $t \in [0, T]$, then

$$\begin{aligned} \sqrt{\varepsilon} B_\varepsilon z(t) &\to 0, \\ B_\varepsilon z(t)' &\to 0 \end{aligned}$$

in $L^2(\mathbb{R})$ as $\varepsilon \to 0$ and so, for all $t \in [0, T]$,

$$z(t) = \varepsilon B_\varepsilon z(t) - B_\varepsilon z(t)'' \to 0,$$

in the sense of distributions. Therefore, $z \equiv 0$.

It remains to prove (3.12).

Let $\delta > 0$ and $\phi_\delta \in C_\circ^\infty(\mathbb{R})$, $\phi_\delta \geq 0$, symmetric, with $\int_{\mathbb{R}} \phi_\delta(x)\,dx = 1$ weakly approximating the Dirac measure with mass in $x = 0$. Set

$$z_\delta(t, x) := \big(\phi_\delta \star z(t)\big)(x) := \int_{\mathbb{R}} \phi_\delta(x - y) z(t)(dy), \qquad x \in \mathbb{R}, t \in [0, T].$$

We define $h : [0, T] \to \mathcal{M}(\mathbb{R})$ by $h(t)(dx) = a(t, x) z(t, dx)$. Note that, by (3.9), since $\phi_\delta(x - \cdot) \in \mathcal{S}(\mathbb{R})$ for all $x \in \mathbb{R}$, we have

$$\begin{aligned} z_\delta(t, x) &= \int_0^t \int_{\mathbb{R}} \phi_\delta''(x - y) h(s)(dy)\,ds \\ &= \int_0^t \big(\phi_\delta'' \star h(s)\big)(x)\,ds \qquad \forall t \in [0, T], x \in \mathbb{R}, \end{aligned} \tag{3.15}$$

where we have used the facts that $z_\delta(0) = 0$, because $z(0) = 0$, and that $x \mapsto z_\delta(t, x)$ is continuous for all $t \in [0, T]$. In fact, one can easily prove that z_δ is continuous and bounded on $[0, T] \times \mathbb{R}$.

Let us now consider $w \in \mathcal{S}(\mathbb{R})$. By Fubini's theorem, for all $t \in [0, T]$, it follows that

$$\begin{aligned} \int_{\mathbb{R}} w(x) B_\varepsilon z(t)(x)\,dx &= \int_{\mathbb{R}} w(x) \int_{\mathbb{R}} K_\varepsilon(x - y) z(t)(dy)\,dx \\ &= \int_{\mathbb{R}} (w \star K_\varepsilon)(y) z(t)(dy). \end{aligned}$$

Now, $B_\varepsilon z(0) = 0$ since $z(0) = 0$. Therefore, by (3.9) and the fact that $w \star K_\varepsilon \in \mathcal{S}(\mathbb{R})$, the previous expression is equal to

$$\int_0^t \int_{\mathbb{R}} (w \star K_\varepsilon)''(y) h(s)(dy)\,ds = \int_0^t \int_{\mathbb{R}} w''(x) B_\varepsilon h(s)(x)\,dx\,ds,$$

which, in turn, by Lemma 2.2, is equal to

$$\int_0^t \int_{\mathbb{R}} w(x)\big(\varepsilon B_\varepsilon h(s)(x)\,dx - h(s)(dx)\big)\,ds.$$

Consequently, by approximation,

$$\int_{\mathbb{R}} w(x) B_\varepsilon z(t)(x)\,dx = \int_0^t \int_{\mathbb{R}} w(x)\big(\varepsilon B_\varepsilon h(s)(x)\,dx - h(s)(dx)\big)\,ds \qquad \forall w \in C_b(\mathbb{R}), t \in [0,T]. \tag{3.16}$$

As a consequence of (3.15) and (3.16), and again using Fubini's theorem, for all $t \in [0,T]$, we obtain

$$\begin{aligned}
g_{\varepsilon,\delta}(t) &:= \int_{\mathbb{R}} z_\delta(t,x) B_\varepsilon z(t)(x)\,dx \\
&\underbrace{=}_{(3.16)} \int_0^t \int_{\mathbb{R}} z_\delta(t,x)\big(\varepsilon B_\varepsilon h(s)(x)\,dx - h(s)(dx)\big)\,ds \\
&\underbrace{=}_{(3.15)} \int_0^t \int_{\mathbb{R}} z_\delta(s,x)\big(\varepsilon B_\varepsilon h(s)(x)\,dx - h(s)(dx)\big)\,ds \\
&\qquad + \int_0^t \int_{\mathbb{R}} \int_s^t \big(\phi_\delta'' \star h(r)\big)(x)\,dr\big(\varepsilon B_\varepsilon h(s)(x)\,dx - h(s)(dx)\big)\,ds \\
&= \int_0^t \int_{\mathbb{R}} z_\delta(s,x)\big(\varepsilon B_\varepsilon h(s)(x)\,dx - h(s)(dx)\big)\,ds \\
&\qquad + \int_0^t \int_0^r \int_{\mathbb{R}} \big(\phi_\delta'' \star h(r)\big)(x)\big(\varepsilon B_\varepsilon h(s)(x)\,dx - h(s)(dx)\big)\,ds\,dr \\
&\underbrace{=}_{(3.16)} \int_0^t \int_{\mathbb{R}} z_\delta(s,x)\big(\varepsilon B_\varepsilon h(s)(x)\,dx - h(s)(dx)\big)\,ds \\
&\qquad + \int_0^t \int_{\mathbb{R}} \big(\phi_\delta'' \star h(r)\big)(x) B_\varepsilon z(r)(x)\,dx\,dr.
\end{aligned}$$

The application of Fubini's theorem above is justified since a is bounded, $\sup_{t\in[0,T]}\|z(t)\|_{\mathrm{var}} < \infty$, K_ε is bounded and $\phi_\delta \in \mathcal{S}(\mathbb{R})$. However, the last term is equal to

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \phi_\delta''(x-y) B_\varepsilon z(r)(x)\,dx\,h(r)(dy)\,dr \\
&\quad = \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \phi_\delta(x-y)\big(\varepsilon B_\varepsilon z(r)(x)\,dx - z(r)(dx)\big)h(r)(dy)\,dr \\
&\quad = \int_0^t \int_{\mathbb{R}} \varepsilon B_\varepsilon z(r)(x)\big(\phi_\delta \star h(r)\big)(x)\,dx\,dr - \int_0^t \int_{\mathbb{R}} z_\delta(r,y)h(r)(dy)\,dr,
\end{aligned}$$

where we could use Lemma 2.2 in the first step, since $\phi_\delta(\cdot - y) \in \mathcal{S}(\mathbb{R})$ for all $y \in \mathbb{R}$. Hence, for all $t \in [0, T]$,

$$\begin{aligned} g_{\varepsilon,\delta}(t) = {} & \int_0^t \int_{\mathbb{R}} z_\delta(s, x) \varepsilon B_\varepsilon h(s)(x)\, dx\, ds \\ & + \int_0^t \int_{\mathbb{R}} \varepsilon B_\varepsilon z(s)(x) \big(\phi_\delta \star h(s)\big)(x)\, dx\, ds \\ & - 2 \int_0^t \int_{\mathbb{R}} z_\delta(s, x) h(s)(dx)\, ds. \end{aligned} \tag{3.17}$$

For a signed measure ν, we denote its absolute value by $|\nu|$. By Lemma 2.2, we have

$$\sup_{s \in [0,T]} \int_{\mathbb{R}} \big(|z(s)| \star \phi_\delta\big)(x) \varepsilon B_\varepsilon |h(s)|(x)\, dx \leq C\sqrt{\varepsilon},$$

where

$$C = \frac{1}{2} \|a\|_\infty \sup_{s \in [0,T]} \|z(s)\|_{\mathrm{var}}^2$$

and, likewise, the integrand of the second integral in (3.17) is bounded by the same constant independent of δ. Hence, as $\varepsilon \to 0$, the first and second terms in the right-hand side of (3.17) converge to zero uniformly in δ and uniformly in $t \in [0, T]$. We now use Assumption (A), namely, that $z \in L^2([\kappa, T] \times \mathbb{R})$ for all $\kappa > 0$. Then, since $B_\varepsilon z(t) \in L^2(\mathbb{R})$ for all $t \in [\kappa, T]$, and $\|a\|_\infty < \infty$, (3.17) implies that for all $\kappa > 0$, $t \in [\kappa, T]$,

$$\begin{aligned} g_\varepsilon(t) - g_\epsilon(\kappa) & = \lim_{\delta \to 0} \big(g_{\varepsilon,\delta}(t) - g_{\varepsilon,\delta}(\kappa)\big) \\ & \leq 2\sqrt{\varepsilon} T C - 2 \int_\kappa^t \int_{\mathbb{R}} z^2(s, x) a(s, x)\, dx\, ds \\ & \leq 2\sqrt{\varepsilon} T C. \end{aligned} \tag{3.18}$$

Now, $\lim_{\kappa \to 0} g_\varepsilon(\kappa) = 0$. In fact, $z(\kappa, \cdot) \to z(0, \cdot) = 0$ weakly, according to the assumption of Theorem 3.8. According to [11], Theorem 8.4.10, page 192, the tensor product $z(\kappa, \cdot) \otimes z(\kappa, \cdot)$ converges weakly to zero. On the other hand, $(x, y) \mapsto K_\varepsilon(x - y)$ is bounded and continuous on $\mathbb{R}^2$. By Fubini's theorem,

$$g_\varepsilon(\kappa) = \int_{\mathbb{R}^2} z(\kappa)(dx) z(\kappa)(dy) K_\varepsilon(x - y) \to 0.$$

So, first letting $\kappa \to 0$ in (3.18) and then $\varepsilon \to 0$, (3.12) follows since $g_\varepsilon(t) \geq 0$ for all $t \in [0, T]$. In fact, we even proved that the convergence in (3.12) is uniform in $t \in [0, T]$. $\square$

REMARK 3.12. Since our coefficient in Theorem 3.6 is only measurable and possibly degenerate, to the best of our knowledge, this result is really new. For instance, in the recent contributions [21] and [17], the diffusion coefficient is supposed to satisfy at least Sobolev regularity.

Theorem 3.8 will be useful for the probabilistic representation of the solution of (3.3) when β is nondegenerate.

4. The probabilistic representation of the deterministic equation. Despite the fact that β is multivalued, by its monotonicity and because of (3.1), it is still possible to find a multivalued map $\Phi : \mathbb{R} \to \mathbb{R}_+$ such that

$$\beta(u) = \Phi^2(u)u, \qquad u \in \mathbb{R},$$

which is bounded, that is,

$$\sup_{u \in \mathbb{R}_*} \sup \Phi(u) < \infty.$$

In fact, the value of Φ at zero is not determined by β.

We start with the case where Φ is nondegenerate. The value $\Phi(0)$ being a priori arbitrary, we can set

$$\Phi(0) = \Bigl[\liminf_{u \to 0} \inf \Phi(u), \limsup_{u \to 0} \sup \Phi(u)\Bigr].$$

DEFINITION 4.1. The (possibly) multivalued map β (or equivalently Φ) is called *nondegenerate* if there exists some constant $c_0 > 0$ such that $y \in \Phi(u) \Rightarrow y \geq c_0$ for any $u \in \mathbb{R}$.

4.1. *The nondegenerate case.* In this subsection, we suppose β to be nondegenerate.

First, we need to show that solutions of the linear PDE (3.9), which are laws of solutions to an SDE, are space–time square integrable.

PROPOSITION 4.2. *Suppose $a : [0, T] \times \mathbb{R} \to \mathbb{R}$ to be a bounded, measurable function which is bounded below on any compact set by a strictly positive constant.*

We consider a stochastic process $Y = (Y_t, t \in [0, T])$ on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, being a weak solution of the SDE

$$Y_t = Y_0 + \int_0^t \sqrt{2a(s, Y_s)}\, dW_s,$$

where W is a standard $(\mathcal{F}_t)$-Brownian motion. For $t \in [0, T]$, let $z(t)$ be the law of Y_t and set $z^0 := z(0)$. Then:

1. *z solves equation (3.9) with z^0 as initial condition;*

2. *there exists* $\rho \in L^2([0,T] \times \mathbb{R})$ *such that* $\rho(t,\cdot)$ *is the density of* $z(t)$ *for almost all* $t \in [0,T]$;
3. z *is the unique solution of* (3.9) *with initial condition* z^0 *having the property described in item* 2 *above*.

REMARK 4.3. A necessary and sufficient condition for the existence and uniqueness in law of solutions of the equation in Proposition 4.2 is that Y solves the martingale problem of Stroock–Varadhan (see Chapter 6 of [27]), related to $L_t f = a(t,x) f''$. In our case, existence and uniqueness follow from, for instance, [27], Exercises 7.3.2–7.3.4; see also [20], Chapter 5, Refinements 4.32. We remark that the coefficients are not continuous, but only measurable, so that space dimension 1 is essential.

The reader can also consult [23, 24] for more refined conditions to be able to construct a weak solution; however, those do not apply in our case.

PROOF OF PROPOSITION 4.2.

1. The first point follows from a direct application of Itô's formula to $\varphi(Y_t)$, $\varphi \in \mathcal{S}(\mathbb{R})$ (cf. the proof of Theorem 1.4).
2. We first suppose that $Y_0 = x_0$, where $x_0 \in \mathbb{R}$. In this case, its law z^0 equals δ_{x_0}, that is, Dirac measure in x_0. In Exercise 7.3.3 of [27], the following Krylov-type estimate is provided:

$$\left| E\left(\int_0^T f(t, Y_t)\, dt\right)\right| \le \mathrm{const}\|f\|_{L^2([0,T]\times\mathbb{R})}$$

for every smooth function $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ with compact support. This implies the existence of a density $(t,y) \mapsto p_t(x_0, y)$ for the measure $(t,y) \mapsto E(\int_0^T f(t,Y_t)\, dt)$ and

$$\left| \int_{[0,T]\times\mathbb{R}} f(t,y) p_t(x_0,y)\, dt\, dy \right| \le \mathrm{const}\|f\|_{L^2([0,T]\times\mathbb{R})},$$

and "const" does not depend on x_0, but only on lower and upper bounds of a. This obviously implies that

$$\sup_{x_0\in\mathbb{R}} \int_{[0,T]\times\mathbb{R}} p_t^2(x_0,y)\, dt\, dy < \infty.$$

This implies assertion 2 when Y_0 is deterministic.

If the initial condition Y_0 is any law $z^0(dx)$, then, clearly, the density of Y_t is $z_t(dy) = \rho(t,y)\, dy$, where $\rho(t,y) = \int_{\mathbb{R}} u_0(dx) p_t(x,y)$.

Consequently, by Jensen's inequality and Fubini's theorem,

$$\int_{[0,T]\times\mathbb{R}} \rho^2(t,y)\, dt\, dy \le \int_{\mathbb{R}} u_0(dx) \int_{[0,T]\times\mathbb{R}} p_t^2(t,x,y)\, dt\, dy < \infty.$$

3. The final assertion follows by item 2 of Theorem 3.8. $\square$

We now return to the probabilistic representation of (1.1).

Let us consider the solution $u \in (L^1 \cap L^\infty)([0,T] \times \mathbb{R})$ from Proposition 3.4, that is, u solves equation (3.2), in the sense of (3.3), assuming the initial condition u_0 is an a.e. bounded probability density. Define

$$\chi_u(t,x) := \begin{cases} \sqrt{\dfrac{\eta_u(t,x)}{u(t,x)}}, & \text{if } u(t,x) \neq 0, \\ c_1, & \text{if } u(t,x) = 0, \end{cases} \tag{4.1}$$

where $c_1 \in \Phi(0)$. Note that, because β is nondegenerate and $\chi_u(t,x) \in \Phi(u(t,x))$ $dt \otimes dx$-a.e., we have $\chi_u \geq c_0 > 0$, $dt \otimes dx$-a.e. Since χ_u is only defined $dt \otimes dx$-a.e., let us fix a Borel version. According to Remark 4.3, it is possible to construct a (unique in law) process Y which is the weak solution of

$$Y_t = Y_0 + \int_0^t \chi_u(s, Y_s)\, dW_s, \tag{4.2}$$

where W is a classical Brownian motion on some filtered probability space and Y_0 is a random variable such that u_0 is the density of its law.

Now, consider the law $v(t,\cdot)$ of the process Y_t. We set $a(t,x) = \frac{\chi_u^2(t,x)}{2}$. Since $a \geq c > 0$, Proposition 4.2 implies that $v \in L^2([0,T] \times \mathbb{R})$ and it solves the system

$$\begin{cases} \partial_t v = \partial_{xx}^2(av), \\ v(0,x) = u_0(x). \end{cases} \tag{4.3}$$

On the other hand, u itself, which is a solution of (3.2) [in the sense of (3.3)], is another solution of (4.3). So, being in $(L^1 \cap L^\infty)([0,T] \times \mathbb{R})$, u is also square integrable. Setting $z_1 = v$, $z_2 = u$, Theorem 3.8 implies that $v = u$, $dt \otimes dx$-a.e.

Since $u \in C([0,T], L^1(\mathbb{R}))$ and Y has continuous sample paths, it follows that $u(t,\cdot) = v(t,\cdot)$, dx-a.e., for all $t \in [0,T]$.

The considerations above prove the existence part of the following representation theorem, at least in the nondegenerate case.

THEOREM 4.4. *Suppose that Assumption 3.1 holds. Let $u_0 \in L^1 \cap L^\infty$ such that $u_0 \geq 0$ and $\int_{\mathbb{R}} u_0(x)\, dx = 1$. Suppose the multivalued map Φ is bounded and nondegenerate. There then exists a process Y, unique in law, such that there exists $\chi \in (L^1 \cap L^\infty)([0,T] \times \mathbb{R})$ with*

$$\begin{cases} Y_t = Y_0 + \displaystyle\int_0^t \chi(s, Y_s)\, dW_s, & (\textit{weakly}), \\ \chi(t,x) \in \Phi(u(t,x)), & \textit{for } dt \otimes dx\textit{-a.e. } (t,x) \in [0,T] \times \mathbb{R}, \\ \textit{law density of } Y_t = u(t,\cdot), & \\ u(0,\cdot) = u_0, & \end{cases} \tag{4.4}$$

with $u \in C([0,T]; L^1(\mathbb{R})) \cap L^\infty([0,T] \times \mathbb{R})$.

REMARK 4.5. *If Φ is single-valued, then $\chi_u \equiv \Phi(u)$.*

PROOF. Existence has been established above. Concerning uniqueness, we consider two solutions Y^i, $i = 1, 2$ of (4.4), that is, (1.3). By $u_i(t, \cdot)$, $i = 1, 2$, we denote the law densities of, respectively, Y^i, $i = 1, 2$ with corresponding χ_1 and χ_2.

The multivalued version of Theorem 1.4 implies that u_1 and u_2 solve equation (1.1), in the sense of distributions, so that, by Proposition 3.4 [uniqueness for (3.3)], we have $u_1 = u_2$, and also $\chi_1 = \chi_2$ a.e.

We note that, since Y_t^i has a law density for all $t > 0$, the stochastic integrals in (4.4) are independent of the chosen Borel version of χ. Remark 4.3 now implies that the laws of Y^1 and Y^2 (on path space) coincide. $\square$

COROLLARY 4.6. *Consider the situation of Theorem 4.4 and let $v_0 \in L^1 \cap L^\infty$ be such that $v_0 \geq 0$. The unique solution v to (3.2) with initial condition v_0 is nonnegative for any $t \geq 0$. Moreover, the mass $\int_{\mathbb{R}} v(t, x)\, dx$ does not depend on t.*

PROOF. Set $\mu_0 = \int_{\mathbb{R}} v_0(y)\, dy$, which we can suppose to be greater than 0. The function $u(t, x) = \frac{v(t,x)}{\mu_0}$ then solves equation (3.2):

$$\begin{cases} \partial_t u = \dfrac{1}{2}\partial_{xx}^2\left(\dfrac{\beta(\mu_0 u)}{\mu_0}\right), \\ u(0, \cdot) = \dfrac{v_0}{\mu_0}. \end{cases} \tag{4.5}$$

Hence, the result follows from Theorem 4.4. $\square$

REMARK 4.7. We note that if Φ is merely bounded below by a strictly positive constant on every compact set and if the solutions u are continuous on $[0, T] \times \mathbb{R}$, then Theorem 4.4 and Corollary 4.6 still hold. In fact, the Stroock–Varadhan arguments contained in Remark 4.3 are still valid if χ_u is strictly positive on each compact set.

4.2. *The degenerate case.* The degenerate case is much more difficult and will be analyzed in detail in the forthcoming paper [6]. In this subsection, we only explain the first two steps in the special case where the β from Section 1 is of the form $\beta(u) = \Phi^2(u)u$ and the following property holds.

PROPERTY 4.8. $\Phi : \mathbb{R} \to \mathbb{R}$ *is single-valued and continuous on* $\mathbb{R} - \{0\}$.

REMARK 4.9. A priori, $\Phi(0)$ is an interval; however, by convention, in this subsection, we will set $\Phi(0) := \liminf_{\varepsilon \to 0+} \Phi(u)$. This implies that Φ is always lower semicontinuous.

We furthermore assume that the initial condition u_0 and Φ are such that we have, for the corresponding solution u to (1.1) (in the sense of Proposition 3.4), the following property.

PROPERTY 4.10. $\Phi^2(u(t,\cdot)):\mathbb{R}\to\mathbb{R}$ *is Lebesgue almost everywhere continuous for dt a.e. $t\in[0,T]$.*

REMARK 4.11. As will be shown in [6], Property 4.10 is fulfilled in many interesting cases for a large class of initial conditions. In fact, we expect to be able to show that $u(t,\cdot)$ is even locally of bounded variation if u_0 is.

PROPOSITION 4.12. *Suppose that Property 4.8 holds. Let $u_0\ge 0$ be a bounded, integrable real function such that $\int_{\mathbb{R}} u_0(x)\,dx=1$ and the corresponding solution u of (1.1) satisfies Property 4.10. There is then at least one process Y such that*

$$\begin{cases} Y_t = Y_0 + \displaystyle\int_0^t \Phi(u(s,Y_s))\,dW_s \qquad \textit{in law},\\ \textit{law density}(Y_t) = u(t,\cdot),\\ u(0,\cdot)=u_0. \end{cases} \tag{4.6}$$

COROLLARY 4.13. *Suppose that Property 4.8 holds. Let $u_0\in L^1\cap L^\infty$ be such that $u_0\ge 0$ and that the corresponding solution u of (1.1) satisfies Property 4.10. The unique solution u of (3.2) is nonnegative for any $t\ge 0$. Moreover, the mass $\int_{\mathbb{R}} u(t,x)\,dx$ is constant in $t\in[0,T]$.*

PROOF OF PROPOSITION 4.12. We denote the solution to (3.3) by $u=u(t,x)$.

Let $\varepsilon\in\,]0,1]$ and set $\beta_\varepsilon(u)=(\Phi(u)+\varepsilon)^2u$, $\Phi_\varepsilon(u)=\Phi(u)+\varepsilon$, $u\in\mathbb{R}$. Proposition 3.4 provides the solution $u=u^\varepsilon$ of the deterministic PDE equation (3.3)

$$\begin{cases} \partial_t u = \frac{1}{2}\partial_{xx}^2(\beta_\varepsilon(u)),\\ u(0,x)=u_0(x). \end{cases}$$

We consider the unique solution $Y=Y^\varepsilon$ in law of

$$\begin{cases} Y_t = Y_0 + \displaystyle\int_0^t \Phi_\varepsilon(u(s,Y_s))\,dW_s,\\ \text{law density}(Y_t) = u^\varepsilon(t,\cdot),\\ u^\varepsilon(0,\cdot)=u_0. \end{cases} \tag{4.7}$$

Since $\Phi+\varepsilon$ is nondegenerate, this is possible because of Theorem 4.4.

Since Φ is bounded, using the Burkholder–Davies–Gundy inequality, one obtains

$$\mathbb{E}\{Y_t^\varepsilon - Y_s^\varepsilon\}^4 \le \mathrm{const}(t-s)^2 \qquad \forall\varepsilon>0, \tag{4.8}$$

where "const" does not depend on ε. Using the Garsia–Rodemich–Rumsey lemma (see, e.g., [7], (3.b), page 203), we obtain that

$$\sup_{\varepsilon>0} E\left(\sup_{s,t\in[0,T]} \frac{|Y_t^\varepsilon - Y_s^\varepsilon|^4}{|t-s|}\right) < \infty.$$

Consequently, using Chebyshev's inequality, we have

$$\lim_{\delta\to 0}\sup_{\varepsilon>0} P\left(\left\{\sup_{s,t\in[0,T],|t-s|\le\delta} |Y_t^\varepsilon - Y_s^\varepsilon| > \lambda\right\}\right) = 0 \qquad \forall\lambda > 0.$$

This implies condition (4.7) of Theorem 4.10 in Section 2.4 of [20]. Condition (4.6) of the same theorem requires that

$$\lim_{\lambda\to+\infty}\sup_{\varepsilon>0} P\{|Y_0^\varepsilon| \ge \lambda\} = 0.$$

This is trivially satisfied here since the law of Y_0^ε is the same for all ε. Thus, the same theorem implies that the family of laws of Y^ε, $\varepsilon > 0$, is tight.

Consequently, there exists a subsequence $Y^n := Y^{\varepsilon_n}$ converging in law (as $C[0,T]$-valued random elements) to some process Y. We set $\Phi_n := \Phi_{\varepsilon_n}$ and $u^n := u^{\varepsilon_n}$, where we recall that $u^n(t,\cdot)$ is the law of Y_t^n.

We also set $X_t^n = Y_t^n - Y_0^n$. Since

$$[X^n]_t = \int_0^t \Phi_n^2(u^n(s, Y_s^n))\,ds$$

and $E([X^n]_T)$ is finite, Φ being bounded, the continuous local martingales X^n are indeed martingales.

By Skorokhod's theorem, there exist a new probability space $(\tilde\Omega, \tilde{\mathcal{F}}, \tilde P)$ and processes $\tilde Y^n$ with the same distribution as Y^n, such that $\tilde Y^n$ converge $\tilde P$-a.e. to some process $\tilde Y$, distributed, of course, as Y, as $C([0,T])$-random elements. In particular, the processes $\tilde X^n := \tilde Y^n - \tilde Y_0^n$ remain martingales with respect to the filtrations generated by themselves. We again denote the sequence $\tilde Y^n$ (resp., $\tilde Y$) by Y^n (resp., Y).

REMARK 4.14. We observe that, for each $t \in [0,T]$, $u(t,\cdot)$ is the law density of Y_t. Indeed, for any $t \in [0,T]$, Y_t^n converges in probability to Y_t; on the other hand, $u^n(t,\cdot)$, which is the law of Y_t^n, converges to $u(t,\cdot)$ in $L^1(\mathbb{R})$ uniformly in t (cf. [10], Theorem 3 and the preceding remarks).

REMARK 4.15. Let $\mathcal{Y}^n$ (resp., $\mathcal{Y}$) be the canonical filtration associated with Y^n (resp., Y).

We set

$$W_t^n = \int_0^t \frac{1}{\Phi_n(u^n(s, Y_s^n))}\,dY_s^n.$$

Those processes W^n are standard $(\mathcal{Y}_t^n)$-Wiener processes since $[W^n]_t = t$ and because of Lévy's characterization theorem of Brownian motion. One then has

$$Y_t^n = Y_0^n + \int_0^t \Phi_n(u^n(s, Y_s^n))\, dW_s^n.$$

We aim to prove first that

$$Y_t = Y_0 + \int_0^t \Phi(u(s, Y_s))\, dW_s. \tag{4.9}$$

Once the previous equation is established for the given u, the statement of Proposition 4.12 would be completely proven because of Remark 4.14. In fact, that remark shows, in particular, the third line of (4.6).

We consider the stochastic process X (vanishing at zero) defined by $X_t = Y_t - Y_0$. We also again set $X_t^n = Y_t^n - Y_0^n$.

Taking into account Theorem 4.2 of Chapter 3 of [20], as in Remark 3.11, to establish (4.9), it will be enough to prove that X is a $\mathcal{Y}$-martingale with quadratic variation $[X]_t = \int_0^t \Phi^2(u(s, Y_s))\, ds$.

Let $s, t \in [0, T]$ with $t > s$ and Θ a bounded continuous function from $C([0, s])$ to $\mathbb{R}$.

In order to prove the martingale property for X, we need to show that

$$E\big((X_t - X_s)\Theta(Y_r, r \le s)\big) = 0.$$

However, this follows because $Y^n \to Y$ a.s. (so $X^n \to X$ a.s.) as $C([0, T])$-valued processes; so, for each $t \ge 0$, $X_t^n \to X_t$ in $L^1(\Omega)$ since $(X_t^n, n \in \mathbb{N})$ is bounded in $L^2(\Omega)$ and

$$E\big((X_t^n - X_s^n)\Theta(Y_r^n, r \le s)\big) = 0.$$

It remains to show that $X_t^2 - \int_0^t \Phi^2(u(s, Y_s))\, ds$, $t \in [0, T]$, defines a $\mathcal{Y}$-martingale, that is, we need to verify that

$$E\biggl(\biggl(X_t^2 - X_s^2 - \int_s^t \Phi^2(u(r, Y_r))\, dr\biggr)\Theta(Y_r, r \le s)\biggr) = 0.$$

The left-hand side decomposes into $2(I^1(n) + I^2(n) + I^3(n))$, where

$$\begin{aligned} I^1(n) &= E\biggl(\biggl(X_t^2 - X_s^2 - \int_s^t \Phi^2(u(r, Y_r))\, dr\biggr)\Theta(Y_r, r \le s)\biggr) \\ &\qquad - E\biggl(\biggl((X_t^n)^2 - (X_s^n)^2 - \int_s^t \Phi^2(u(r, Y_r^n))\, dr\biggr)\Theta(Y_r^n, r \le s)\biggr), \\ I^2(n) &= E\biggl(\biggl((X_t^n)^2 - (X_s^n)^2 - \int_s^t \Phi_n^2(u^n(r, Y_r^n))\, dr\biggr)\Theta(Y_r^n, r \le s)\biggr) \end{aligned}$$

and

$$I^3(n) = E\biggl(\int_s^t \bigl(\Phi_n^2(u^n(r, Y_r^n)) - \Phi^2(u(r, Y_r^n))\bigr)\, dr\, \Theta(Y_r^n, r \le s)\biggr).$$

We start by showing the convergence of $I^3(n)$. Now, $\Theta(Y_r^n, r \le s)$ is dominated by a constant. Therefore, since Φ_n, Φ are uniformly bounded and $a^2 - b^2 = (a-b)(a+b)$, by the Cauchy–Schwarz inequality, it suffices to consider the expectation of

$$\int_s^t \big(\Phi_n(u^n(r, Y_r^n)) - \Phi(u(r, Y_r^n))\big)^2 \, dr, \tag{4.10}$$

which is equal to

$$\begin{aligned}\int_s^t E\big(\Phi_n(u^n(r, Y_r^n)) - \Phi(u(r, Y_r^n))\big)^2 \, dr \\ = \int_s^t dr \int_{\mathbb{R}} \big(\Phi_n(u^n(r,y)) - \Phi(u(r,y))\big)^2 u^n(r,y)\, dy.\end{aligned}$$

This equals $J_1(n) + J_2(n) - 2J_3(n)$, where

$$\begin{aligned}J_1(n) &= \int_s^t dr \int_{\mathbb{R}} \Phi_n^2(u^n(r,y)) u^n(r,y)\, dy,\\ J_2(n) &= \int_s^t dr \int_{\mathbb{R}} \Phi^2(u(r,y)) u^n(r,y)\, dy,\\ J_3(n) &= \int_s^t dr \int_{\mathbb{R}} \Phi_n(u^n(r,y)) \Phi(u(r,y)) u^n(r,y)\, dy.\end{aligned}$$

Define

$$J := \int_s^t \int_{\mathbb{R}} \Phi^2(u(r,y)) u(r,y)\, dy = \int_s^t \int_{\mathbb{R}} \beta(u(r,y))\, dy.$$

To show that $I^3(n) \to 0$ as $n \to \infty$, it suffices to show that

$$\lim_{n\to\infty} J_1(n) = \lim_{n\to\infty} J_2(n) = J \tag{4.11}$$

and

$$\liminf_{n\to\infty} J_3(n) \ge J. \tag{4.12}$$

Now, repeating exactly the same arguments as in Point 3 of Proposition 3.4, it follows that $\Phi_n^2(u_n)u_n \to \Phi^2(u)u$ in $\sigma(L^1, L^\infty)$ as $n \to \infty$, which immediately implies (4.11).

Furthermore, by Fatou's lemma and since $\Phi_n \ge \Phi$, we have

$$\liminf_{n\to\infty} J_3(n) \ge \int_0^t \int_{\mathbb{R}} \liminf_{n\to\infty} \Phi(u^n(r,y)) \Phi(u(r,y)) u(r,y)\, dy\, dr,$$

which, by the lower semicontinuity of Φ, implies (4.12).

We now proceed with the analysis of $I^2(n)$ and $I^1(n)$. $I^2(n)$ equals zero because X^n is a martingale with quadratic variation given by $[X^n]_t = \int_0^t \Phi_n^2(u^n(r, Y_r^n))\, dr$.

Finally, we treat $I^1(n)$. We recall that $X^n \to X$ a.s. as a random element in $C([0,T])$ and that the sequence $E((X_t^n)^4)$ is bounded, so $(X_t^n)^2$ are uniformly integrable. Therefore, we have

$$E\big(((X_t^n)^2 - (X_s^n)^2)\Theta(Y_r^n, r \le s)\big) - E\big((X_t^2 - X_s^2)\Theta(Y_r, r \le s)\big) \to 0,$$

when $n \to \infty$. It remains to prove that

$$\int_s^t E\big(\Phi^2(u(r, Y_r)) - \Phi^2(u(r, Y_r^n))\Theta(Y_y^n, y \le s)\big)\, dr \to 0. \tag{4.13}$$

Now, for fixed dr-a.e. $r \in [0,T]$, $\Phi(u(r,\cdot))$ has a Lebesgue zero set of discontinuities. Moreover, the law of Y_r has a density. So, let $N(r)$ be the null event of all $\omega \in \Omega$ such that $Y_r(\omega)$ is a point of discontinuity of $\Phi(u(r,\cdot))$. For $\omega \notin N(r)$, we have

$$\lim_{n\to\infty} \Phi^2(u(r, Y_r^n(\omega))) = \Phi^2(u(r, Y_r(\omega))).$$

Hence, Lebesgue's dominated convergence theorem implies (4.13). □

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REFERENCES

[1] ALT, H. W. (2002). *Lineare Funktionalanalysis*: *Eine Anwendungsorientierte Einführung*, 5th ed. Springer, Berlin.

[2] BAK, P. (1996). *How Nature Works*: *The Science of Self-Organized Criticality*. Copernicus, New York. MR1417042

[3] BANTAY, P. and JANOSI, I. M. (1992). Avalanche dynamics from anomalous diffusion. *Phys. Rev. Lett.* **68** 2058–2061.

[4] BARBU, V. (1976). *Nonlinear Semigroups and Differential Equations in Banach Spaces*. Noordhoff, Leiden. MR0390843

[5] BARBU, V. (1993). *Analysis and Control of Nonlinear Infinite-dimensional Systems*. *Mathematics in Science and Engineering* **190**. Academic Press, San Diego, CA. MR1195128

[6] BARBU, V., RÖCKNER, M. and RUSSO, F. (2010). Probabilistic representation for solutions of a porous media equation: The irregular degenerate case. *Probab. Theory Related Fields*. To appear.

[7] BARLOW, M. T. and YOR, M. (1982). Semimartingale inequalities via the Garsia–Rodemich–Rumsey lemma, and applications to local times. *J. Funct. Anal.* **49** 198–229. MR680660

[8] BENACHOUR, S., CHASSAING, P., ROYNETTE, B. and VALLOIS, P. (1996). Processus associés à l'équation des milieux poreux. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **23** 793–832. MR1469575

[9] BENILAN, P., BREZIS, H. and CRANDALL, M. G. (1975). A semilinear equation in $L^1(R^N)$. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **2** 523–555. MR0390473

[10] BÉNILAN, P. and CRANDALL, M. G. (1981). The continuous dependence on φ of solutions of $u_t - \Delta\varphi(u) = 0$. *Indiana Univ. Math. J.* **30** 161–177. MR604277

[11] BOGACHEV, V. I. (2007). *Measure Theory, Vol. I, II*. Springer, Berlin. MR2267655

[12] BOGACHEV, V. I., KRYLOV, N. V. and RÖCKNER, M. (2001). On regularity of transition probabilities and invariant measures of singular diffusions under minimal conditions. *Comm. Partial Differential Equations* **26** 2037–2080. MR1876411

[13] BREZIS, H. (1977). *Opérateurs Maximaux Monotones et Semigroupes de Contraction dans les Espaces de Hilbert*. North-Holland, Amsterdam.

[14] BRÉZIS, H. and CRANDALL, M. G. (1979). Uniqueness of solutions of the initial-value problem for $u_t - \Delta\varphi(u) = 0$. *J. Math. Pures Appl.* (9) **58** 153–163. MR539218

[15] CRANDALL, M. G. and EVANS, L. C. (1975). On the relation of the operator $\partial/\partial s + \partial/\partial\tau$ to evolution governed by accretive operators. *Israel J. Math.* **21** 261–278. MR0390853

[16] EVANS, L. C. (1977). Nonlinear evolution equations in an arbitrary Banach space. *Israel J. Math.* **26** 1–42. MR0440431

[17] FIGALLI, A. (2008). Existence and uniqueness of martingale solutions for SDEs with rough or degenerate coefficients. *J. Funct. Anal.* **254** 109–153. MR2375067

[18] GRAHAM, C., KURTZ, T. G., MÉLÉARD, S., PROTTER, P. E., PULVIRENTI, M. and TALAY, D. (1996). *Probabilistic Models for Nonlinear Partial Differential Equations*. *Lecture Notes in Math.* **1627**. Springer, Berlin. MR1431297

[19] JOURDAIN, B. (2000). Probabilistic approximation for a porous medium equation. *Stochastic Process. Appl.* **89** 81–99. MR1775228

[20] KARATZAS, I. and SHREVE, S. E. (1991). *Brownian Motion and Stochastic Calculus*, 2nd ed. *Graduate Texts in Mathematics* **113**. Springer, New York. MR1121940

[21] LE BRIS, C. and LIONS, P. L. (2008). Existence and uniqueness of solutions to Fokker-Planck type equations with irregular coefficients. *Comm. Partial Differential Equations* **33** 1272–1317. MR2450159

[22] MCKEAN, H. P. JR. (1967). Propagation of chaos for a class of non-linear parabolic equations. In *Stochastic Differential Equations* (*Lecture Series in Differential Equations*, *Session* 7, *Catholic Univ.*, 1967) 41–57. Air Force Office Sci. Res., Arlington, VA. MR0233437

[23] ROZKOSZ, A. and SLOMINSKI, L. (1993). On weak solutions of one-dimensional SDEs with time-dependent coefficients. *Stochastics Stochastics Rep.* **42** 199–208.

[24] SENF, T. (1993). On one-dimensional stochastic differential equations without drift and with time-dependent diffusion coefficients. *Stochastics Stochastics Rep.* **43** 199–220. MR1277164

[25] SHOWALTER, R. E. (1997). *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*. *Mathematical Surveys and Monographs* **49**. Amer. Math. Soc., Providence, RI. MR1422252

[26] STEIN, E. M. (1970). *Singular Integrals and Differentiability Properties of Functions*. *Princeton Mathematical Series* **30**. Princeton Univ. Press, Princeton, NJ. MR0290095

[27] STROOCK, D. W. and VARADHAN, S. R. S. (1979). *Multidimensional Diffusion Processes*. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **233**. Springer, Berlin. MR532498

[28] SZNITMAN, A.-S. (1991). Topics in propagation of chaos. In *École D'Été de Probabilités de Saint-Flour XIX*—1989. *Lecture Notes in Math*. **1464** 165–251. Springer, Berlin. MR1108185

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