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STOCHASTIC CALCULUS OVER SYMMETRIC MARKOV PROCESSES WITHOUT TIME REVERSAL

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We refine stochastic calculus for symmetric Markov processes without using time reverse operators. Under some conditions on the jump functions of locally square integrable martingale additive functionals, we extend Nakao's divergence-like continuous additive functional of zero energy and the stochastic integral with respect to it under the law for quasi-everywhere starting points, which are refinements of the previous results under the law for almost everywhere starting points. This refinement of stochastic calculus enables us to establish a generalized Fukushima decomposition for a certain class of functions locally in the domain of Dirichlet form and a generalized Itô formula.

1. Introduction. In this paper, under the framework of general symmetric Markov processes without using time reverse operators, we give a refinement of stochastic calculus developed in the previous joint paper [3]. More precisely, we establish stochastic integrals both of Itô-type and of Fisk–Stratonovich-type by Dirichlet processes by extending the Nakao's divergence-like continuous additive functional of zero energy to a continuous additive functional locally of zero energy for a class of locally square integrable martingale additive functionals. Throughout this paper, we use the terminology *Dirichlet process* specifically for an additive functional decomposed into the sum of a locally square integrable martingale additive functional and a continuous additive functional (locally) of zero energy, which is not necessarily a semi-martingale in general; indeed, the notion of Dirichlet process in a more general context was introduced by Föllmer [9]. As in [11], stochastic integrals are defined to be additive functionals admitting exceptional sets. So all formulas in this paper can be regarded as a decomposition of additive functional, which holds for all time (or up to the life time) with probability 1 starting from quasi-everywhere point.

Hereafter, we use the abbreviation CAF (resp., MAF) for continuous additive functional (resp., martingale additive functional). For a Dirichlet process given by

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Fukushima's decomposition, Nakao [22] defined stochastic integrals integrated by his divergence-like CAF of zero energy, which enables us to construct an Itô-type stochastic integral by the Dirichlet process. He also defined a Fisk–Stratonovich-type integral for symmetric diffusion processes with no inside killing in order to establish the stochastic line integral along 1-forms for symmetric diffusion processes over smooth manifolds and gave an application of stochastic line integral to a homogenization problem.

On the other hand, Lyons and Zheng [19] and Lyons and Zheng [18] introduced the notion of Fisk–Stratonovich-type integrals in terms of the sum of forward and backward martingales, which is described by time reverse operators in the framework of symmetric conservative diffusion processes. They proved that their Fisk–Stratonovich-type integrals are consistent with Nakao's one under the law $\mathbb{P}_m$.

In the joint paper [3], we extend Nakao's divergence-like CAF of zero energy in terms of time reverse operators and define a stochastic integral integrated by this extended CAF under some mild conditions, which plays an important role in deducing the perturbation of general symmetric Markov processes, that is, the combination of the Feynman–Kac formula and the Girsanov formula (see [4, 5]); however, still described under the law $\mathbb{P}_m$ except a special case.

We extend Nakao's CAFs of zero energy and stochastic integrals with respect to it for more general integrand and integrator in terms of the the space locally in the Dirichlet space and a subclass of locally square integrable MAF on $[\![0,\zeta[\![$ (Definition 3.1). We will define both the Itô-type and the Fisk–Stratonovich-type stochastic integrals integrated by (not necessarily continuous) Dirichlet processes under the law $\mathbb{P}_x$ for quasi-everywhere starting point $x \in E$, which are described in terms of a subclass of locally square integrable MAF on $[\![0,\zeta[\![$ over general symmetric Markov processes (Definitions 4.2 and 4.3). Our definitions of Fisk–Stratonovich-type integrals are somewhat different from what is defined by Meyer [21] and Protter [23] in the framework of semi-martingales (Remark 4.1).

We further show that our stochastic integrals integrated by the purely discontinuous part of Dirichlet processes have a representation of sum of jumps on Dirichlet processes if the jump function of integrator is anti-symmetric, which enables us to see the pathwise behavior of pure jump processes under the law for quasi-everywhere starting points (Theorem 4.1, Corollary 4.3).

As a corollary, we establish a generalized Fukushima decomposition for a class of functions locally in the domain of forms (Theorem 4.2). We also present a generalized Itô formula in terms of our extended stochastic integrals by Dirichlet processes (Theorem 4.3). Our Itô formula for Fisk–Stratonovich-type integrals has an expression different from what is exposed in Protter [23] (Remark 4.3).

Let us briefly outline the organization of this paper. In Section 2, we describe the setting of the paper and give some basic lemmas. In Section 3, we formulate the extension of Nakao's CAF of zero energy and stochastic integral with respect to it under the law for quasi everywhere starting points. In Section 4, we define our stochastic integrals by Dirichlet processes and expose the result as noted above.

2. Preliminary facts. Let $\mathbf{M} = \{\Omega, \mathcal{F}_\infty, \mathcal{F}_t, X_t, \theta_t, \zeta, \mathbb{P}_x, x \in E\}$ be an m-symmetric right Markov process on a Lusin space E, where m is a σ-finite measure with full support on E. Its associated Dirichlet space $(\mathcal{E}, \mathcal{F})$ on $L^2(E; m)$ is known to be quasi-regular (see [20]). By [8], $(\mathcal{E}, \mathcal{F})$ is quasi-homeomorphic to a regular Dirichlet space on a locally compact separable metric space. Thus using this quasi-homeomorphism, without loss of generality, we may and do assume that $\mathbf{M}$ is an m-symmetric Hunt process on a locally compact metric space E such that its associated Dirichlet space $(\mathcal{E}, \mathcal{F})$ is regular on $L^2(E; m)$ and that m is a positive Radon measure with full topological support on E. But we implicitly use the quasi-left continuity up to ∞, which is not the usual property of right Markov processes. So the strict quasi-regularity of $(\mathcal{E}, \mathcal{F})$ is essentially assumed. However, if we restrict ourselves to state the result that holds up to the life time with probability 1 for quasi-everywhere starting point, then the framework of quasi-regular Dirichlet forms is enough.

Without loss of generality, we can take Ω to be the canonical path space $D([0, \infty[\to E_\partial)$ of right-continuous, left-limited (*rcll*, for short) functions from $[0, \infty[$ to E_∂. For any $\omega \in \Omega$, we set $X_t(\omega) := \omega(t)$. Let $\zeta(\omega) := \inf\{t \geq 0 \mid X_t(\omega) = \partial\}$ be the life time of $\mathbf{M}$. As usual, $\mathcal{F}_\infty$ and $\mathcal{F}_t$ are the minimal completed σ-algebras obtained from $\mathcal{F}^0_\infty := \sigma\{X_s \mid 0 \leq s < \infty\}$ and $\mathcal{F}^0_t := \sigma\{X_s \mid 0 \leq s \leq t\}$, respectively, under $\mathbb{P}_x$. We set $X_t(\omega) := \partial$ for $t \geq \zeta(\omega)$ and use θ_t to denote the shift operator defined by $\theta_t(\omega)(s) := \omega(t+s)$, $t, s \geq 0$. For each $s \geq 0$, the shift operator θ_s is defined by $\theta_s\omega(t) := \omega(t+s)$ for $t \in [0, \infty[$. For a Borel subset B of E, $\sigma_B := \inf\{t > 0 \mid X_t \in B\}$ (the first hitting time to B) and $\tau_B := \inf\{t > 0 \mid X_t \notin B\}$ (the *first exit time* of B) are $(\mathcal{F}_t)$-stopping times. If B is closed, then τ_B is an $(\mathcal{F}^0_{t+})$-stopping time. Also, ζ is an $(\mathcal{F}^0_t)$-stopping time because $\{\zeta \leq t\} = \{X_t = \partial\} \in \mathcal{F}^0_t$, $t \geq 0$.

The transition semigroup of $\mathbf{M}$, $\{P_t, t \geq 0\}$, is defined by

$$P_t f(x) := \mathbb{E}_x[f(X_t)] = \mathbb{E}_x[f(X_t) : t < \zeta], \qquad t \geq 0.$$

Each P_t may be viewed as an operator on $L^2(E; m)$; collectively these operators form a strongly continuous semigroup of self-adjoint contractions. The Dirichlet form associated with $\mathbf{M}$ is the bilinear form

$$\mathcal{E}(u, v) := \lim_{t \downarrow 0} \frac{1}{t}(u - P_t u, v)_m$$

defined on the space

$$\mathcal{F} := \Big\{u \in L^2(E; m) \mid \sup_{t>0} t^{-1}(u - P_t u, u)_m < \infty\Big\}.$$

Here we use the notation $(f, g)_m := \int_E f(x)g(x)m(dx)$ for $f, g \in L^2(E; m)$.

An increasing sequence $\{F_n\}$ of closed sets is called an $\mathcal{E}$-*nest* if $\bigcup_{n=1}^\infty \mathcal{F}_{F_n}$ is $\mathcal{E}_1^{1/2}$-dense in $\mathcal{F}$, where $\mathcal{F}_{F_n} := \{u \in \mathcal{F} \mid u = 0 \ m\text{-a.e. on } E \setminus F_n\}$ and a family $\{F_n\}$ of closed sets is an $\mathcal{E}$-nest if and only if it is a nest, that is, $\mathbb{P}_x(\lim_{n\to\infty} \tau_{F_n} = \zeta) = 1$

q.e. $x \in E$. A function u on E is said to be *$\mathcal{E}$-quasi-continuous* if there exists an $\mathcal{E}$-nest $\{F_n\}$ of closed sets such that u is continuous on each F_n. A subset N of E is called *$\mathcal{E}$-polar or ($\mathcal{E}$-)exceptional* if there exists an $\mathcal{E}$-nest $\{F_n\}$ such that $N \subset \bigcap_{n=1}^{\infty}(E \setminus F_n)$; equivalently there is a Borel set $\tilde{N}$ containing N such that $\mathbb{P}_m(\sigma_{\tilde{N}} < \infty) = 0$. A statement $S(x)$ is said to hold for quasi-everywhere $x \in E$ (q.e. $x \in E$ in short) if there exists an exceptional set N such that $\{x \in E \mid S(x)$ does not hold$\} \subset N$.

An increasing sequence $\{F_n\}$ of closed sets is called a *strict $\mathcal{E}$-nest* if

$$\lim_{n\to\infty} \mathrm{Cap}_{1,G_1\varphi}(E \setminus F_n) = 0,$$

where $\mathrm{Cap}_{1,G_1\varphi}$ is the weighted capacity defined in Chapter V, Definition 2.1 of [20] and a family $\{F_n\}$ of closed sets is a strict $\mathcal{E}$-nest if and only if it is a strict nest, that is, $\mathbb{P}_x(\lim_{n\to\infty} \sigma_{E\setminus F_n} = \infty) = 1$ m-a.e. $x \in E$ in view of Chapter V, (2.5) in [20], equivalently it holds q.e. $x \in E$ by Chapter V, Proposition 2.28(i) and Remark 2.8 in [20]. A function u on E_∂ is said to be *strictly $\mathcal{E}$-quasi-continuous* if there exists a strict $\mathcal{E}$-nest $\{F_n\}$ of closed sets such that u is continuous on each $F_n \cup \{\partial\}$.

An increasing sequence $\{G_n\}$ of (q.e.) finely open Borel sets is called a *nest* (resp., *strict nest*) if $\mathbb{P}_x(\lim_{n\to\infty} \tau_{G_n} = \zeta) = 1$ for q.e. $x \in E$ [resp., $\mathbb{P}_x(\lim_{n\to\infty} \sigma_{E\setminus G_n} = \infty) = 1$ for q.e. $x \in E$]. (The definition of q.e. finely open sets can be found in [11].) In [3], we show that under the quasi-left-continuity up to infinity of $\mathbf{M}$, for an increasing sequence $\{G_n\}$ of (q.e.) finely open Borel sets, $\{G_n\}$ is a nest if and only if it is a strict nest. Denote by Θ the family of (strict) nests $\{G_n\}$ of (q.e.) finely open Borel sets. Note that for an $\mathcal{E}$-nest $\{F_n\}$ of closed sets, $\{G_k\} \in \Theta$ by setting $G_k := F_k^{f\text{-int}}$, $k \in \mathbb{N}$, where $F_k^{f\text{-int}}$ means the fine interior of F_k.

Let $\mathcal{F}_e$ be the family of m-measurable functions u on E such that $|u| < \infty$ m-a.e. and there exists an $\mathcal{E}$-Cauchy sequence $\{u_n\}$ of $\mathcal{F}$ such that $\lim_{n\to\infty} u_n = u$ m-a.e. We call $\{u_n\}$ as above an approximating sequence for $u \in \mathcal{F}_e$. For any $u, v \in \mathcal{F}_e$ and their approximating sequences $\{u_n\}$, $\{v_n\}$ the limit $\mathcal{E}(u, v) = \lim_{n\to\infty} \mathcal{E}(u_n, v_n)$ exists and does not depend on the choices of the approximating sequences for u, v. It is known that $\mathcal{E}^{1/2}$ on $\mathcal{F}_e$ is a semi-norm and $\mathcal{F} = \mathcal{F}_e \cap L^2(E; m)$. We call $(\mathcal{E}, \mathcal{F}_e)$ *the extended Dirichlet space* of $(\mathcal{E}, \mathcal{F})$. Let $L^0(E; m)$ be the family of m-measurable functions on E. We further let

$$\begin{aligned}\dot{\mathcal{F}}_{\mathrm{loc}} := \{u \in L^0(E; m) \mid &\text{ there exist } \{G_n\} \in \Theta \text{ and} \\ &u_n \in \mathcal{F} \text{ such that } u = u_n \ m\text{-a.e. on } G_n \text{ for each } n \in \mathbb{N}\}.\end{aligned}$$

$\dot{\mathcal{F}}_{\mathrm{loc}}$ is called the *space of functions locally in $\mathcal{F}$ in the broad sense*. It is shown in [16] that $\mathcal{F} \subset \mathcal{F}_e \subset \dot{\mathcal{F}}_{\mathrm{loc}}$ and every $u \in \dot{\mathcal{F}}_{\mathrm{loc}}$ admits an $\mathcal{E}$-quasi-continuous m-version $\tilde{u}$. More strongly every $u \in \mathcal{F}$ admits a strictly $\mathcal{E}$-quasi-continuous m-version $\tilde{u}$ on E_∂ with $\tilde{u}(\partial) = 0$. For $u \in \dot{\mathcal{F}}_{\mathrm{loc}}$, we always assume that $\tilde{u}$ is extended to be a real-valued function $\tilde{u}$ on E_∂ if not otherwise specified, where we do not necessarily assume $\tilde{u}(\partial) = 0$. However, we can reduce to this case by setting $\tilde{u} - \tilde{u}(\partial)$ on E_∂, which is in $\dot{\mathcal{F}}_{\text{loc}}$ as a function defined on E.

We need the following lemma:

LEMMA 2.1. *Every $u \in \mathcal{F}_e$ admits a strictly $\mathcal{E}$-quasi-continuous m-version $\tilde{u}$ on E_∂ with $\tilde{u}(\partial) = 0$.*

PROOF. Take $u \in \mathcal{F}_e$. Then there exists an m-a.e. strictly positive bounded function $g \in L^1(E; m)$ such that $u \in (\mathcal{F}^g)_e$, where $(\mathcal{E}^g, \mathcal{F}^g)$ is the Dirichlet form on $L^2(E; m)$ defined by $\mathcal{F}^g := \mathcal{F} \cap L^2(E; gm)$, $\mathcal{E}^g(v, w) := \mathcal{E}(v, w) + (v, w)_{gm}$, $v, w \in \mathcal{F}^g$, and $(\mathcal{F}^g)_e$ is its extended Dirichlet space. Then there exist an increasing sequence $\{F_n\}$ of closed sets and a function $\tilde{u}$ on E_∂ such that

$$\lim_{n\to\infty} \mathrm{Cap}^g_{(0)}(E \setminus F_n) = 0,$$

$\tilde{u} = u$ m-a.e. on E and $\tilde{u}$ is continuous on each $F_n \cup \{\partial\}$ with $\tilde{u}(\partial) = 0$, where $\mathrm{Cap}^g_{(0)}$ is the 0-order capacity with respect to $(\mathcal{E}^g, (\mathcal{F}^g)_e)$. It suffices to prove that $\{F_n\}$ is a strict $\mathcal{E}$-nest with respect to $(\mathcal{E}, \mathcal{F})$. For this, we need that for any open set U,

$$H^g_U 1(x) := \mathbb{E}_x\big[e^{-\int_0^{\sigma_U} g(X_s)\,ds}\big]$$

satisfies $H^g_U 1 \in (\mathcal{F}^g)_e$ and

$$\mathrm{Cap}^g_{(0)}(U) = \mathcal{E}^g(H^g_U 1, H^g_U 1). \tag{2.1}$$

This can be similarly proved along the same way as in Section 4.4 in [11]. We will omit the details.

From (2.1), we have

$$\mathbb{P}_x\left(\lim_{n\to\infty} \int_0^{\sigma_{E\setminus F_n}} g(X_s)\,ds = \infty\right) = 1, \qquad m\text{-a.e. } x \in E,$$

and hence $\{F_n\}$ is a strict nest, because of the boundedness of g. □

As a rule we take u to be represented by its (strictly) $\mathcal{E}$-quasi-continuous m-version (when such exists), and drop the tilde from the notation.

Let $\mathring{\mathcal{M}}$ and $\mathcal{N}_c$ denote, respectively, the space of martingale additive functionals of finite energy and the space of continuous additive functionals of zero energy. More precisely, we set

$$\mathring{\mathcal{M}} := \{M \mid M \text{ is a finite rcll AF}, \mathbb{E}_x[M_t^2] < \infty, \mathbb{E}_x[M_t] = 0 \\ \text{for q.e. } x \in E \text{ and all } t \geq 0\}.$$

For an AF M, if the limit

$$\mathbf{e}(M) := \lim_{t\downarrow 0} \frac{1}{2t}\mathbb{E}_m[M_t^2] \tag{2.2}$$

exists, we call it *energy of* M. When $M \in \mathcal{M}$, $t \mapsto \frac{1}{2t}\mathbb{E}_m[M_t^2]$ is increasing and the limit may diverge in general. Then we define

$$\overset{\circ}{\mathcal{M}} := \{M \in \mathcal{M} \mid \mathbf{e}(M) < \infty\},$$
$$\mathcal{N}_c := \{N \mid N \text{ is a finite CAF}, \mathbb{E}_x[|N_t|] < \infty \text{ q.e. } x \in E \text{ for each } t > 0, \text{ and } \mathbf{e}(N) = 0\}.$$

For $M, N \in \overset{\circ}{\mathcal{M}}$, we set

$$\mathbf{e}(M, N) := \lim_{t \downarrow 0} \frac{1}{2t}\mathbb{E}_m[M_t N_t]$$

and call it *mutual energy* of M, N. It is well known that $(\overset{\circ}{\mathcal{M}}, \mathbf{e})$ is a real Hilbert space with inner product $\mathbf{e}$.

For $u \in \mathcal{F}_e$, the following Fukushima decomposition holds:

$$u(X_t) - u(X_0) = M_t^u + N_t^u \tag{2.3}$$

for all $t \in [0, \infty[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$, where $M^u \in \overset{\circ}{\mathcal{M}}$ and $N^u \in \mathcal{N}_c$.

A positive continuous additive functional (PCAF) of $\mathbf{M}$ (call it A) determines a measure $\nu = \nu_A$ on the Borel subsets of E via the formula

$$\nu(f) = \uparrow \lim_{t \to 0} \frac{1}{t}\mathbb{E}_m\left[\int_0^t f(X_s)\, dA_s\right], \tag{2.4}$$

in which $f : E \to [0, \infty]$ is Borel measurable. The measure ν is necessarily *smooth* (denote by $\nu \in S$), in the sense that ν charges no exceptional set of E, and there is an $\mathcal{E}$-nest $\{F_n\}$ of closed subsets of E such that $\nu(F_n) < \infty$ for each $n \in \mathbb{N}$. Conversely, given a smooth measure ν, there is a unique PCAF A^ν such that (2.4) holds with $A = A^\nu$. In the sequel we refer to this bijection between smooth measures and PCAFs as the *Revuz correspondence*, and to ν as the Revuz measure of A^ν.

LEMMA 2.2. *$\mu \in S$ if and only if μ charges no exceptional set and there exists $\{G_n\} \in \Theta$ such that $\mu(G_n) < \infty$ for each $n \in \mathbb{N}$.*

PROOF. The "only if" part is trivial by setting $G_n := F_n^{f\text{-int}}$. We only prove the "if" part. Take an $f \in L^2(E; m)$ with $0 < f \leq 1$ on E and set $R_1^{G_n} f(x) := \mathbb{E}_x[\int_0^{\tau_{G_n}} e^{-s} f(X_s)\, ds]$. Then $R_1^{G_n} f(x) > 0$ on G_n and $R_1^{G_n} f$ is $\mathcal{E}$-quasi-continuous for each $n \in \mathbb{N}$. Take a common $\mathcal{E}$-nest $\{A_k\}$ such that all $R_1^{G_n} f$, $n \geq 1$ are continuous on each A_k. We set $F_n := \{x \in A_n \mid R_1^{G_n} f(x) \geq 1/n\}$. Then $\{F_n\}$ is an $\mathcal{E}$-nest by use of Lemma 3.3 in [16], where we observe $B_n := \{R_1^{G_n} f > 1/n\}$ is increasing and $E \setminus \bigcup_{n=1}^\infty B_n$ is exceptional. For each $n \in \mathbb{N}$, we have $(E \setminus G_n)^r \subset E \setminus F_n$, where $(E \setminus G_n)^r = \{x \in E \mid R_1^{G_n} f(x) = 0\}$ is the set of regular points for $E \setminus G_n$. Since $(E \setminus G_n) \setminus (E \setminus G_n)^r$ is exceptional, we obtain $\mu(F_n) \le \mu(G_n) < \infty$ for each $n \in \mathbb{N}$. $\square$

A (positive) Radon measure μ on E is said to be a measure *of finite energy integral* if there exists $C > 0$ depending on μ such that

$$\int_E |u(x)|\mu(dx) \le C\sqrt{\mathcal{E}_1(u,u)} \qquad \text{for all } u \in \mathcal{F} \cap C_0(E).$$

Let S_0 be the family of measures of finite energy integrals. For $\mu \in S_0$ and $\alpha > 0$, there exists a unique element $U_\alpha\mu \in \mathcal{F}$ such that

$$\mathcal{E}_\alpha(U_\alpha\mu, v) = \int_E v(x)\mu(dx) \qquad \text{for } v \in \mathcal{F} \cap C_0(E).$$

It is known that every $\mu \in S_0$ is a smooth measure. If we set $S_{00} := \{\mu \in S_0 \mid \mu(E) < \infty, U_1\mu \in L^\infty(E; m)\}$, then N is exceptional if and only if $\nu(N) = 0$ for all $\nu \in S_{00}$.

For any $\mu \in S$, $\nu \in S_{00}$, a (q.e.) finely open Borel set G and $t > 0$, we have the following formula:

$$\mathbb{E}_\nu[A^\mu_{t\wedge\sigma_{E\setminus G}}] \le (1+t)\|U_1\nu\|_\infty \mu(G), \tag{2.5}$$

which can be similarly proved as in the proof of Lemma 5.1.9 in [11] with the help of Lemma 5.1.10(ii) in [11].

Take $M, N \in \overset{\circ}{\mathcal{M}}$ and denote by $\langle M, N\rangle$ its quadratic covariational process, which is a CAF of bounded variation, and let $\mu_{\langle M,N\rangle}$ be its Revuz measure. In view of Theorem 2.2 in [22], for $M, N \in \overset{\circ}{\mathcal{M}}$, $\mathbf{e}(M,N) = 0$ implies that $\langle M, N\rangle \equiv 0$ on $[0, \infty[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$. For $M \in \overset{\circ}{\mathcal{M}}$ and $f \in L^2(E; \mu_{\langle M\rangle})$, there exists a unique $f * M \in \overset{\circ}{\mathcal{M}}$ such that

$$\mathbf{e}(f * M, N) = \frac{1}{2}\int_E f(x)\mu_{\langle M,N\rangle}(dx) \qquad \text{for } N \in \overset{\circ}{\mathcal{M}}.$$

Moreover, we have the following.

LEMMA 2.3. *Let $M \in \overset{\circ}{\mathcal{M}}$ and $f \in L^2(E; \mu_{\langle M\rangle})$. If f is a strictly $\mathcal{E}$-quasi-continuous function, then $f * M$ admits a Riemann sum approximation: for each $t > 0$*

$$(f * M)_t = \lim_{n\to\infty} \sum_{\ell=0}^{n-1} f(X_{\ell t/n})\big(M_{(\ell+1)t/n} - M_{\ell t/n}\big)$$

holds $\mathbb{P}_x$-a.s. for q.e. $x \in E$, where the convergence of the right-hand side is in $\mathbb{P}_x$-probability for q.e. $x \in E$.

PROOF. This is well known for experts and shown for the case $f \in C_0(E)$ in Lemma 5.6.2 of [11]. We shall show it for the reader's convenience. By assumption, we have for $\nu \in S_{00}$

$$\mathbb{E}_\nu\left[\int_0^t f^2(X_s)\,d\langle M\rangle_s\right] \le (1+t)\|U_1\nu\|_\infty \int_E f^2(x)\mu_{\langle M\rangle}(dx) < \infty.$$

In particular,

$$\mathbb{E}_x\left[\int_0^t f^2(X_s)\,d\langle M\rangle_s\right] < \infty \qquad \text{for q.e. } x \in E.$$

Then by Theorem A.3.19 in [11], for $x \in E \setminus N$ with an adequate properly exceptional set N, we can define the stochastic integral $f \bullet M := \int_0^t f(X_{s-})\,dM_s$ under $\mathbb{P}_x$, which is characterized by

$$\mathbb{E}_x[(f \bullet M)_t^2] = \mathbb{E}_x\left[\int_0^t f^2(X_{s-})\,d\langle M\rangle_s\right] = \mathbb{E}_x\left[\int_0^t f^2(X_s)\,d\langle M\rangle_s\right].$$

From this, we can get $f \bullet M \in \mathring{\mathcal{M}}$ and $\mu_{\langle f \bullet M, N\rangle} = f\mu_{\langle M,N\rangle}$ for $N \in \mathring{\mathcal{M}}$, hence we have $f * M = f \bullet M$. On the other hand, since $t \mapsto f(X_{t-})$ is left-continuous $\mathbb{P}_x$-a.s. for q.e. $x \in E$, $f \bullet M$ admits the Riemann-sum approximation in view of Chapter I, Proposition 4.44 in [13]. Therefore, we obtain the result. □

REMARK 2.1. From Lemma 2.3, we may write $(f * M)_t = \int_0^t f(X_{s-})\,dM_s$ if f is strict $\mathcal{E}$-quasi-continuous on E_∂.

Let $(N(x,dy), H_t)$ be a Lévy system for $\mathbf{M}$; that is, $N(x,dy)$ is a kernel on $(E_\partial, \mathcal{B}(E_\partial))$ and H_t is a PCAF with bounded 1-potential such that for any nonnegative Borel function ϕ on $E_\partial \times E_\partial$ vanishing on the diagonal and any $x \in E_\partial$,

$$\mathbb{E}_x\left[\sum_{s \le t} \phi(X_{s-}, X_s)\right] = \mathbb{E}_x\left[\int_0^t \int_{E_\partial} \phi(X_s, y) N(X_s, dy)\,dH_s\right].$$

To simplify notation, we will write

$$N\phi(x) := \int_{E_\partial} \phi(x,y) N(x,dy).$$

Let μ_H be the Revuz measure of the PCAF H. Then the jump measure J and the killing measure κ of $\mathbf{M}$ are given by

$$J(dx\,dy) = \tfrac{1}{2} N(x,dy)\mu_H(dx) \quad \text{and} \quad \kappa(dx) = N(x,\{\partial\})\mu_H(dx).$$

These measures feature in the Beurling–Deny decomposition of $\mathcal{E}$: for $f, g \in \mathcal{F}_e$,

$$\mathcal{E}(f,g) = \mathcal{E}^c(f,g) + \int_{E\times E} \bigl(f(x)-f(y)\bigr)\bigl(g(x)-g(y)\bigr) J(dx\,dy) + \int_E f(x)g(x)\kappa(dx),$$

where $\mathcal{E}^c$ is the strongly local part of $\mathcal{E}$.

For $u \in \mathcal{F}_e$, the martingale part M_t^u in (2.3) can be decomposed as

$$M_t^u = M_t^{u,c} + M_t^{u,j} + M_t^{u,\kappa} \qquad \text{for every } t \in [0, \infty[,$$

$\mathbb{P}_x$-a.s. for q.e. $x \in E$, where $M_t^{u,c}$ is the continuous part of martingale M^u, and

$$\begin{aligned} M_t^{u,j} &= \lim_{\varepsilon \downarrow 0} \biggl\{ \sum_{0<s\le t} \bigl(u(X_s) - u(X_{s-})\bigr) \mathbf{1}_{\{|u(X_s)-u(X_{s-})|>\varepsilon\}} \mathbf{1}_{\{s<\zeta\}} \\ &\qquad - \int_0^t \biggl(\int_{\{y\in E \mid |u(y)-u(X_s)|>\varepsilon\}} \bigl(u(y) - u(X_s)\bigr) N(X_s, dy) \biggr) dH_s \biggr\}, \\ M_t^{u,\kappa} &= \int_0^t u(X_s) N(X_s, \{\partial\})\, dH_s - u(X_{\zeta-}) \mathbf{1}_{\{t\ge\zeta\}} \end{aligned}$$

are the jump and killing parts of M^u in $\mathring{\mathcal{M}}$, respectively. The limit in the expression for $M^{u,j}$ is in the sense of convergence in $\mathring{\mathcal{M}}$ and of convergence in probability under $\mathbb{P}_x$ for q.e. $x \in E$ for each fixed $t > 0$. (See Theorem A.3.9 and page 341 in [11].)

If we let

$$\begin{aligned} \mathring{\mathcal{M}}^c &:= \{M \in \mathring{\mathcal{M}} \mid M \text{ is a continuous MAF}\}, \\ \mathring{\mathcal{M}}^d &:= (\mathring{\mathcal{M}}^c)^\perp = \{M \in \mathring{\mathcal{M}} \mid \mathbf{e}(M, N) = 0 \text{ for } N \in \mathring{\mathcal{M}}^c\}, \end{aligned}$$

then every M has an orthogonal decomposition

$$M = M^c + M^d$$

in the Hilbert space $(\mathring{\mathcal{M}}, \mathbf{e})$. $M^c \in \mathring{\mathcal{M}}^c$ (resp., $M^d \in \mathring{\mathcal{M}}^d$) is nothing but the *continuous part* (resp., *purely discontinuous part*) of M discussed in [11]. Moreover, set

$$\mathring{\mathcal{M}}^j := \{M \in \mathring{\mathcal{M}}^d \mid \mathbf{e}(M, M^{u,\kappa}) = 0 \text{ for } u \in \mathcal{F}_e\}, \qquad \mathring{\mathcal{M}}^\kappa := \mathring{\mathcal{M}}^d \cap (\mathring{\mathcal{M}}^j)^\perp.$$

Then $\mathring{\mathcal{M}}^j$ is a closed subspace of $\mathring{\mathcal{M}}$, hence M^d has a unique orthogonal decomposition in $(\mathring{\mathcal{M}}, \mathbf{e})$ as

$$M^d = M^j + M^\kappa,$$

where $M^j \in \mathring{\mathcal{M}}^j$ and $M^\kappa \in \mathring{\mathcal{M}}^\kappa$. For simplicity of notation, we will use the convention $\Delta F_s := F_s - F_{s-}$ for any rcll $(\mathcal{F}_t)$-adapted process F. The square bracket $[M, N]$ for $M, N \in \mathring{\mathcal{M}}$ is defined by

$$[M, N]_t := \langle M^c, N^c\rangle_t + \sum_{0<s\le t} \Delta M_s \Delta N_s.$$

Then $\langle M, N\rangle$ is the dual predictable projection of $[M, N]$ (see (A.3.7) in [11]). We further set for each $i = \varnothing, c, d, j, \kappa$

$$\mathring{\mathcal{M}}^i_{f\text{-loc}} := \bigl\{M \mid \text{there exist } \{G_n\} \in \Theta \text{ and } \bigl\{M^{(n)}\bigr\} \subset \mathring{\mathcal{M}}^i \text{ such that}$$
$$M_t = M_t^{(n)} \text{ for all } t < \sigma_{E\setminus G_n} \text{ and } n \in \mathbb{N}, \mathbb{P}_x\text{-a.s. for q.e. } x \in E\bigr\},$$
$$\mathcal{N}_{c,f\text{-loc}} := \bigl\{N \mid \text{there exist } \{G_n\} \in \Theta \text{ and } \bigl\{N^{(n)}\bigr\} \subset \mathcal{N}_c \text{ such that}$$
$$N_t = N_t^{(n)} \text{ for all } t < \sigma_{E\setminus G_n} \text{ and } n \in \mathbb{N}, \mathbb{P}_x\text{-a.s. for q.e. } x \in E\bigr\}.$$

Similarly, we can define $\mathring{\mathcal{M}}^i_{\text{loc}}$ and $\mathcal{N}_{c,\text{loc}}$ as subclasses of local AFs (or AFs on $[\![0, \zeta[\![$) in terms of first exit times τ_{G_n} (see [3, 11] for the notion of local AF). Here $i = \varnothing$ means $\mathring{\mathcal{M}}^{\varnothing} := \mathring{\mathcal{M}}$ and write $\mathring{\mathcal{M}}_{f\text{-loc}}$ (resp., $\mathring{\mathcal{M}}_{\text{loc}}$) instead of $\mathring{\mathcal{M}}^{\varnothing}_{f\text{-loc}}$ (resp., $\mathring{\mathcal{M}}^{\varnothing}_{\text{loc}}$). Every PCAF is an element of $\mathcal{N}_{c,\text{loc}}$. Our $\mathring{\mathcal{M}}_{f\text{-loc}}$ (resp., $\mathcal{N}_{c,f\text{-loc}}$) is slightly narrower than $\mathring{\mathcal{M}}_{\text{loc}}$ (resp., $\mathcal{N}_{c,\text{loc}}$) treated in [11] (in [3] we use the same symbol $\mathring{\mathcal{M}}_{f\text{-loc}}$ (resp., $\mathcal{N}_{c,f\text{-loc}}$) to denote $\mathring{\mathcal{M}}_{\text{loc}}$ (resp., $\mathcal{N}_{c,\text{loc}}$)). However, Fukushima's decomposition (2.3) for $u \in \dot{\mathcal{F}}_{\text{loc}}$ with $J = \kappa = 0$ can be characterized by $\mathring{\mathcal{M}}_{f\text{-loc}}$ and $\mathcal{N}_{c,f\text{-loc}}$. Before seeing this, we need the following lemma:

LEMMA 2.4. *Let G be a* (*q.e.*) *finely open Borel set.*

(1) *If $u \in \mathcal{F}$ satisfies $u = 0$ q.e. on G, then $\mu_{\langle M^{u,c}\rangle}(G) = 0$ and*

$$M_t^{u,c} = 0 \qquad \text{for any } t \le \sigma_{E\setminus G}\ \mathbb{P}_x\text{-a.s. for q.e. } x \in E.$$

(2) *For $M \in \mathring{\mathcal{M}}$, $\mu_{\langle M\rangle}(G) = 0$ implies $M_t = 0$ for any $t < \sigma_{E\setminus G}$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$.*

PROOF. The proof of (1) is quite similar to the proof of Lemma 5.3.1 in [11]. Note that $t < \sigma_{E\setminus G} \le \widehat{\sigma}_{E\setminus G}$ implies $X_s, X_{s-} \in G \cup \{\partial\}$ for all $s \in]0, t]$, which means $u(X_s) - u(X_{s-}) = 0$ for all $s \in]0, t]$, because of $u(\partial) = 0$. Here $\widehat{\sigma}_{E\setminus G} := \inf\{t > 0 \mid X_{t-} \in E \setminus G\}$ (see (A.2.6) and Theorem A.2.3 in [11]). Next we prove (2). Suppose $\mu_{\langle M\rangle}(G) = 0$ for $M \in \mathring{\mathcal{M}}$. Note that $\int_0^t \mathbf{1}_{\{\partial\}}(X_s)\, d\langle M\rangle_s = 0$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$. Combining this and Theorem 5.1.3(i) in [11], we have $\mathbb{E}_m[\int_0^t \mathbf{1}_{G\cup\{\partial\}}(X_s)\, d\langle M\rangle_s] = 0$ for each $t > 0$, hence $\langle M\rangle_t = 0$ for all $t < \sigma_{E\setminus G}$ $\mathbb{P}_m$-a.e. Then by Lemma 5.1.10(iii) in [11], we obtain the result. $\square$

REMARK 2.2. Our method of the proof of Lemma 2.4(1) does not work to show the same assertion in the case that u is only constant q.e. on G.

From this lemma, we can construct $M^{u,c} \in \mathring{\mathcal{M}}^c_{f\text{-loc}}$ for $u \in \dot{\mathcal{F}}_{\text{loc}}$. Under $J = \kappa = 0$, for $u \in \dot{\mathcal{F}}_{\text{loc}}$, (2.3) holds for all $t \in [0, \zeta[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$, where $M^u \in \mathring{\mathcal{M}}^c_{f\text{-loc}}$ and $N^u \in \mathcal{N}_{c,f\text{-loc}}$. If, further, u can be extended to be a real-valued function on E_∂ [without assuming $u(\partial)=0$], then the decomposition (2.3) holds for all $t \in [0,\infty[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$.

In order to define the stochastic integrals by Dirichlet processes, we have to investigate the structure of $\mathring{\mathcal{M}}^d$. For this we introduce the spaces $\mathring{\mathcal{J}}, \mathcal{J}, \widehat{\mathcal{J}}$ of jump functions

$$\widehat{\mathcal{J}} := \{\phi : E_\partial \times E_\partial \to \mathbb{R} \mid \phi \text{ is a Borel measurable function such that } \phi(x,x)=0 \text{ for } x \in E_\partial \text{ and } N(\mathbf{1}_{E\times E}\phi^2)\mu_H \in S\},$$

$$\mathcal{J} := \{\phi : E_\partial \times E_\partial \to \mathbb{R} \mid \phi \text{ is a Borel measurable function such that } \phi(x,x)=0 \text{ for } x \in E_\partial \text{ and } N(\phi^2)\mu_H \in S\}$$

and $\mathring{\mathcal{J}} := \{\phi \in \mathcal{J} \mid \int_E N(\phi^2)\,d\mu_H < \infty\}$. Clearly $\mathring{\mathcal{J}} \subset \mathcal{J} \subset \widehat{\mathcal{J}}$, and for $\phi \in \widehat{\mathcal{J}}$, we see $\mathbf{1}_{E\times E}\phi \in \mathcal{J}$. Further we set $\mathcal{J}_{\mathrm{as}} := \{\phi \in \mathcal{J} \mid \widetilde{\phi} = 0\ J\text{-a.e. on } E\times E\}$ and $\mathcal{J}_* := \{\phi \in \mathcal{J} \mid N(\mathbf{1}_{E\times E}|\overline{\phi}|^2)\mu_H \in S\}$, $\mathring{\mathcal{J}}_{\mathrm{as}} = \mathring{\mathcal{J}} \cap \mathcal{J}_{\mathrm{as}}$ and $\mathring{\mathcal{J}}_* = \mathring{\mathcal{J}} \cap \mathcal{J}_*$. Here $\overline{\phi}(x,y) := \phi(y,x)$ for $x,y \in E_\partial$, $\widetilde{\phi} := (\phi+\overline{\phi})/2$ on $E_\partial \times E_\partial$. Clearly, $\mathcal{J}_{\mathrm{as}} \subset \mathcal{J}_*$ and $\mathring{\mathcal{J}}_{\mathrm{as}} \subset \mathring{\mathcal{J}}_*$. Similarly, we can define $\widehat{\mathcal{J}}_{\mathrm{as}}$ and $\widehat{\mathcal{J}}_*$ by replacing $\mathcal{J}$ with $\widehat{\mathcal{J}}$ in its definitions. Moreover, for $\phi \in \mathcal{J}_*$ (resp., $\phi \in \widehat{\mathcal{J}}_*$), we see $\mathbf{1}_{E\times E}\overline{\phi} \in \mathcal{J}_*$ (resp., $\overline{\phi} \in \widehat{\mathcal{J}}_*$). For $\phi \in \widehat{\mathcal{J}}$ and $\ell \in \mathbb{N}$, we write $\phi_\ell := \phi\mathbf{1}_{\{|\phi|>1/\ell\}}$. For $\phi,\psi \in \mathcal{J}$ (resp., $\phi,\psi \in \widehat{\mathcal{J}}$), we write $\phi \sim \psi$ if $\phi=\psi$ J^*-a.e. on $E\times E_\partial$ (resp., $\phi=\psi$ J-a.e. on $E\times E$), where J^* is the measure on $E\times E_\partial$ defined by $J^*(dx\,dy) := \frac{1}{2}N(x,dy)\mu_H(dx)$. Then $\sim$ is an equivalence relation and denote by $\mathring{\mathcal{J}}/\sim$, $\mathcal{J}/\sim$, $\widehat{\mathcal{J}}/\sim$ the families of equivalence classes.

LEMMA 2.5. *There exists a one-to-one correspondence between $\mathring{\mathcal{J}}/\sim$ and $\mathring{\mathcal{M}}^d$ which is characterized by the relation that for $\phi \in \mathring{\mathcal{J}}$ (resp., $M \in \mathring{\mathcal{M}}^d$), there exists $M \in \mathring{\mathcal{M}}^d$ (resp., $\phi \in \mathring{\mathcal{J}}$) such that $\mathbf{e}(M) = \frac{1}{2}\int_E N(\phi^2)(x)\mu_H(dx)$ and $M_t - M_{t-} = \phi(X_{t-},X_t)$ for all $t \in [0,\infty[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$. Moreover, $\langle M\rangle_t = \int_0^t N(\phi^2)(X_s)\,dH_s$ for all $t \in [0,\infty[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$.*

PROOF. Take $\phi \in \mathring{\mathcal{J}}$ and set

$$M_t^\ell := \sum_{0<s\le t}\phi_\ell(X_{s-},X_s) - \int_0^t\int_{E_\partial}\phi_\ell(X_s,y)N(X_s,dy)\,dH_s.$$

Then we can obtain $M^\ell \in \mathring{\mathcal{M}}^d$ and

$$\langle M^\ell\rangle_t = \int_0^t\int_{E_\partial}\phi_\ell^2(X_s,y)N(X_s,dy)\,dH_s, \tag{2.6}$$

$$\mathbf{e}(M^\ell) = \frac{1}{2}\int_E\int_{E_\partial}\phi_\ell^2(x,y)N(x,dy)\mu_H(dx). \tag{2.7}$$

Indeed, we easily see that M^ℓ is an MAF. If we let $T_n^\ell := \inf\{t > 0 \mid |M_t^\ell| \geq n\}$, then $\{T_n^\ell\}$ is an increasing sequence of $(\mathcal{F}_t)$-stopping times such that $\lim_{n\to\infty} T_n^\ell = \infty$. Then we have $|M^\ell_{t\wedge T_n^\ell}| \leq |M^\ell_{t\wedge T_n^\ell -}| + |\phi(X_{t\wedge T_n^\ell -}, X_{t\wedge T_n^\ell})| \leq n + |\phi(X_{t\wedge T_n^\ell -}, X_{t\wedge T_n^\ell})|$, which implies that for $\nu \in S_{00}$

$$\begin{aligned}\mathbb{E}_\nu[(M^\ell_{t\wedge T_n^\ell})^2] &\leq 2n^2\nu(E) + 2\mathbb{E}_\nu\Big[\sum_{s\leq t}\phi^2(X_{s-}, X_s)\Big]\\ &= 2n^2\nu(E) + 2\mathbb{E}_\nu\Big[\int_0^t\int_{E_\partial}\phi^2(X_s, y)N(X_s, dy)\,dH_s\Big]\\ &\leq 2n^2\nu(E) + 2(1+t)\|U_1\nu\|_\infty\int_E N(\phi^2)(x)\mu_H(dx) < \infty.\end{aligned}$$

That is, $t\mapsto M^\ell_{t\wedge T_n^\ell}$ is a square integrable purely discontinuous $\mathbb{P}_\nu$-martingale for each n. By Corollary A.3.1 in [11],

$$(M^\ell_{t\wedge T_n^\ell})^2 - \sum_{s\leq t}(\Delta M^\ell_{s\wedge T_n^\ell})^2 = (M^\ell_{t\wedge T_n^\ell})^2 - \sum_{s\leq t\wedge T_n^\ell}\phi_\ell^2(X_{s-}, X_s)$$

is a $\mathbb{P}_\nu$-martingale (also a $\mathbb{P}_x$-martingale for q.e. $x \in E$), which yields that

$$\begin{aligned}\mathbb{E}_\nu[(M_t^\ell)^2] &\leq \varliminf_{n\to\infty}\mathbb{E}_\nu[(M^\ell_{t\wedge T_n^\ell})^2] = \varliminf_{n\to\infty}\mathbb{E}_\nu\Big[\sum_{s\leq t\wedge T_n^\ell}\phi_\ell^2(X_{s-}, X_s)\Big]\\ &= \mathbb{E}_\nu\Big[\sum_{s\leq t}\phi_\ell^2(X_{s-}, X_s)\Big] \leq \mathbb{E}_\nu\Big[\int_0^t\int_{E_\partial}\phi^2(X_s, y)N(X_s, dy)\,dH_s\Big]\\ &\leq (1+t)\|U_1\nu\|_\infty\int_E\int_{E_\partial}\phi^2(x, y)N(x, dy)\mu_H(dx) < \infty.\end{aligned}$$

Thus, M_t^ℓ is a square integrable MAF. Since $\{M^\ell_{t\wedge T_n^\ell}\}_{n=1}^\infty$ is $L^2(\mathbb{P}_\nu)$-bounded, by use of the Banach–Saks theorem, we have the equality

$$\mathbb{E}_\nu[(M_t^\ell)^2] = \mathbb{E}_\nu\Big[\int_0^t\int_{E_\partial}\phi_\ell^2(X_s, y)N(X_s, dy)\,dH_s\Big]$$

for all $\nu \in S_{00}$. We then have the same equation for q.e. $x \in E$ by replacing ν with x. Hence $M^\ell \in \mathring{\mathcal{M}}$, (2.6) and (2.7). Note that there exists a sequence $\{T_n\}$ of totally inaccessible times such that $\{(t,\omega) \mid M_t^\ell - M_{t-}^\ell \neq 0\} = \bigcup_{n=1}^\infty [\![T_n]\!]$. This yields that $M^\ell = M^{\ell,d} \in \mathring{\mathcal{M}}^d$ in view of Theorem A.3.9 in [11]. Moreover, we see that $\{M^\ell\}_{\ell=1}^\infty$ is an $\mathbf{e}$-Cauchy sequence in $\mathring{\mathcal{M}}^d$. Denote by $M \in \mathring{\mathcal{M}}^d$ its limit. Then there exists a subsequence $\{\ell_k\}$ such that M^{ℓ_k} converges to M uniformly on each compact subinterval of $[0,\infty[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$. We see for each ℓ, $M_t^\ell - M_{t-}^\ell = \phi_\ell(X_{t-}, X_t)$ for all $t \in [0,\infty[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$. Therefore we have the desired result. Conversely take an $M \in \mathring{\mathcal{M}}^d$. Then, by Lemma 3.2 in [6], there exists a Borel function ϕ defined on $E_\partial \times E_\partial$ with $\phi(x,x)=0$ for $x \in E_\partial$ such that $M_t - M_{t-} = \phi(X_{t-}, X_t)$, $t \in [0, \zeta_p[$ $\mathbb{P}_m$-a.e., where ζ_p is the predictable part of ζ defined by $\zeta_p := \zeta$ if $\zeta < \infty$ and $X_{\zeta-} = \partial$, and $\zeta_p := \infty$ otherwise. For $L = f * M^u$ with $f \in C_0(E)$, $u \in \mathcal{F} \cap C_0(E)$, we have $\Delta L_{\zeta_p} = f(X_{\zeta_p -})\Delta M^u_{\zeta_p} = 0$ for $\zeta_p < \infty$. In view of Lemma 5.6.3 and Theorem 5.2.1 in [11], we see $\Delta M_{\zeta_p} = 0$ for $\zeta_p < \infty$, which implies $M_t - M_{t-} = \phi(X_{t-}, X_t)$, $t \in [0, \infty[$ $\mathbb{P}_m$-a.e. From this, we have

$$\begin{aligned}\int_E \int_{E_\partial} \phi^2(x,y) N(x,dy)\mu_H(dx) &= \lim_{t\to 0} \frac{1}{t}\mathbb{E}_m\left[\int_0^t \int_{E_\partial} \phi^2(X_s, y) N(X_s, dy)\, dH_s\right] \\ &= \lim_{t\to 0} \frac{1}{t}\mathbb{E}_m\left[\sum_{s\le t} \phi^2(X_{s-}, X_s)\right] \\ &= \lim_{t\to 0} \frac{1}{t}\mathbb{E}_m\left[\sum_{s\le t} (M_s - M_{s-})^2\right] \\ &= \lim_{t\to 0} \frac{1}{t}\mathbb{E}_m[M_t^2] = 2\mathbf{e}(M) < \infty,\end{aligned}$$

where we use Corollary A.3.1 in [11]. Going back to the first argument, we can construct $\widetilde{M} \in \mathring{\mathcal{M}}^d$ such that $\widetilde{M}_t - \widetilde{M}_{t-} = \phi(X_{t-}, X_t)$, $t \in [0, \infty[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$. Applying Corollary A.3.1 in [11] to $M - \widetilde{M} \in \mathring{\mathcal{M}}^d$, we obtain

$$\mathbf{e}(M - \widetilde{M}) = \lim_{t\to 0} \frac{1}{t}\mathbb{E}_m[(M_t - \widetilde{M}_t)^2] = \lim_{t\to 0} \frac{1}{t}\mathbb{E}_m\left[\sum_{s\le t} (\Delta(M - \widetilde{M})_s)^2\right] = 0,$$

which implies the converse assertion. □

COROLLARY 2.1. *Take $\phi \in \mathring{\mathcal{J}}$ and set $\overline{\phi}(x,y) := \phi(y,x)$ for $x, y \in E_\partial$. Then $\mathbf{1}_{E\times E}\overline{\phi} \in \mathring{\mathcal{J}}$, in particular, there exists $K \in \mathring{\mathcal{M}}^j$ such that $K_t - K_{t-} = -\mathbf{1}_{E\times E}(\phi + \overline{\phi})(X_{t-}, X_t)$ $t \in]0, \infty[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$.*

PROOF. The assertion is clear from

$$\begin{aligned}\int_E \int_E \overline{\phi}^2(x,y) N(x,dy)\mu_H(dx) &= \int_E \int_E \phi^2(x,y) N(x,dy)\mu_H(dx) \\ &\le \int_E N(\phi^2)(x)\mu_H(dx) < \infty. \qquad \square\end{aligned}$$

From this corollary, we have $\mathring{\mathcal{J}} \subset \mathcal{J}_*$.

LEMMA 2.6. *Take a Borel function $\phi : E_\partial \times E_\partial \to \mathbb{R}$ with $\phi(x,x) = 0$ for $x \in E_\partial$. The following are equivalent under $\phi \in \mathcal{J}$:*

(1) *$\phi(X_{t-}, X_t) = 0$ for all $t \le \sigma_{E\setminus G}$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$.*

(2) $\phi(X_{t-}, X_t) = 0$ *for all* $t \le \sigma_{E\setminus G}$ $\mathbb{P}_m$*-a.e.*
(3) $\int_0^t N(\phi^2)(X_s)\,dH_s = 0$ *for all* $t < \sigma_{E\setminus G}$ $\mathbb{P}_x$*-a.s. for q.e.* $x \in E$.
(4) $\int_0^t N(\phi^2)(X_s)\,dH_s = 0$ *for all* $t < \sigma_{E\setminus G}$ $\mathbb{P}_m$*-a.e.*
(5) $\phi = 0$ J^**-a.e. on* $G \times E_\partial$.

Replacing $\sigma_{E\setminus G}$ [*resp.*, ϕ *in* (3), (4)] *with* τ_G (*resp.*, $\mathbf{1}_{E\times E}\phi$), *we have a similar equivalence under* $\mathbf{1}_{E\times E}\phi \in \mathcal{J}$, *where the last condition is that* $\phi = 0$ J^**-a.e. on* $G \times E$.

PROOF. The implication (1) $\Longrightarrow$ (2) is trivial and (3) $\Longleftrightarrow$ (4) follows from Lemma 5.1.10(iii) in [11]. We first show (2) $\Longrightarrow$ (3). Suppose $\phi(X_{t-}, X_t) = 0$ for all $t \in]0, \sigma_{E\setminus G}]$ $\mathbb{P}_m$-a.e. Then we see $\phi(X_{\sigma_{E\setminus G}-}, X_{\sigma_{E\setminus G}}) = 0$ $\mathbb{P}_m$-a.e. on $\{\sigma_{X\setminus G} < \infty\}$. So $\phi(X_{t\wedge\sigma_{E\setminus G}-}, X_{t\wedge\sigma_{E\setminus G}}) = 0$ for all $t \in]0, \infty[$ $\mathbb{P}_m$-a.e. From the property of Lévy system (see Appendix (A) in [7] or the formula with $Y_t = 1_{]0,T]}(t)$ at line -9 on page 346 in [25]), we have for each $t > 0$

$$\mathbb{E}_m\left[\int_0^{t\wedge\sigma_{E\setminus G}} N(\phi^2)(X_s)\,dH_s\right] = \mathbb{E}_m\left[\sum_{s\le t\wedge\sigma_{E\setminus G}} \phi^2(X_{s-}, X_s)\right] = 0,$$

which implies (4), hence (3). (3) also yields $\int_G N(\phi^2)\,d\mu_H = 0$ by Lemma 5.1.10(iii) in [11], and in particular, we obtain (5). Conversely suppose (5), that is, $\int_G N(\phi^2)\,d\mu_H = 0$. Then, we can obtain (1) by way of the inequality (2.5) and the property of Lévy system used above. $\square$

COROLLARY 2.2. *Take an MAF* $M \in \mathring{\mathcal{M}}^d$ *and the associated* $\phi \in \mathring{\mathcal{J}}$. *Set* $\phi_\partial(x, y) := \phi(x, y)\mathbf{1}_{\{\partial\}}(y)$. *Then the following are equivalent*:

(1) $M \in \mathring{\mathcal{M}}^j$.
(2) $\phi(x, \partial) = 0$ κ*-a.e.* $x \in E$.
(3) $\int_0^\cdot N(\phi_\partial^2)(X_s)\,dH_s \equiv 0$ $\mathbb{P}_x$*-a.s. for q.e.* $x \in E$.
(4) $\int_0^\cdot N(\phi_\partial^2)(X_s)\,dH_s \equiv 0$ $\mathbb{P}_m$*-a.e.*

Set $\phi_E(x, y) := \phi(x, y)\mathbf{1}_E(y)$. *Then the following are equivalent*:

(1*) $M \in \mathring{\mathcal{M}}^\kappa$.
(2*) $\phi(x, y) = 0$ J*-a.e.* $(x, y) \in E \times E$.
(3*) $\int_0^\cdot N(\phi_E^2)(X_s)\,dH_s \equiv 0$ $\mathbb{P}_x$*-a.s. for q.e.* $x \in E$.
(4*) $\int_0^\cdot N(\phi_E^2)(X_s)\,dH_s \equiv 0$ $\mathbb{P}_m$*-a.e.*

PROOF. (1) $\Longleftrightarrow$ (2) is clear from $\mathbf{e}(M, M^{u,\kappa}) = -\frac{1}{2}\int_E \phi(x, \partial)u(x)\kappa(dx)$ for $u \in \mathcal{F}_e$. Here we use the fact that $\mathcal{F}$ is dense in $L^2(E; \kappa)$. (1*) $\Longleftrightarrow$ (2*) is clear from (1) $\Longleftrightarrow$ (2) and $\mathbf{e}(M, N) = \frac{1}{2}\int_E\int_{E_\partial} \phi(x, y)\psi(x, y)N(x, dy)\mu_H(dx)$. The rest implications hold true for general $\phi \in \mathcal{J}$ and are clear in view of the uniqueness of the Revuz correspondence and Lemma 5.1.10(iii) in [11]. $\square$

Let $\mathcal{M}_{\mathrm{loc}}$ be the space of locally square integrable MAFs and $\mathcal{M}_{\mathrm{loc}}^{[\![0,\zeta[\![}$ the space of locally square integrable MAFs on $[\![0,\zeta[\![$. That is, $M\in\mathcal{M}_{\mathrm{loc}}$ (resp., $M\in\mathcal{M}_{\mathrm{loc}}^{[\![0,\zeta[\![}$) if and only if there exists an increasing sequence $\{T_n\}$ (resp., $\{S_n\}$) of $(\mathcal{F}_t)$-stopping times and $\{M^{(n)}\}\subset\mathcal{M}$ such that $\lim_{n\to\infty}T_n=\infty$ (resp., $\lim_{n\to\infty}S_n=\zeta$) and for each $n\in\mathbb{N}$, $M_{t\wedge T_n}=M^{(n)}_{t\wedge T_n}$ (resp., $M_{t\wedge S_n}\mathbf{1}_{\{t\wedge S_n<\zeta\}}=M^{(n)}_{t\wedge S_n}\mathbf{1}_{\{t\wedge S_n<\zeta\}}$) for all $t\in[0,\infty[$ $\mathbb{P}_x$-a.s. for q.e. $x\in E$. Let $\mathcal{M}^c_{\mathrm{loc}}$ (resp., $\mathcal{M}^d_{\mathrm{loc}}$) be the space of locally square integrable continuous (resp., purely discontinuous) MAFs. That is, for $M\in\mathcal{M}^c_{\mathrm{loc}}$ (resp., $M\in\mathcal{M}^d_{\mathrm{loc}}$), we can take $\{M^{(n)}\}$ from $\mathcal{M}^c$ (resp., $\mathcal{M}^d$) in the above definition. Similarly, we can define the space $\mathcal{M}_{\mathrm{loc}}^{c,[\![0,\zeta[\![}$ (resp., $\mathcal{M}_{\mathrm{loc}}^{d,[\![0,\zeta[\![}$) of locally square integrable continuous (resp., purely discontinuous) MAFs on $[\![0,\zeta[\![$. For every $M\in\mathcal{M}_{\mathrm{loc}}^{[\![0,\zeta[\![}$, its quadratic variational process $\langle M\rangle$ can be defined to be a PCAF (Proposition 2.8 in [3]), and M is decomposed to $M=M^c+M^d$ (Theorem 8.23 in [12]), where $M^c\in\mathcal{M}_{\mathrm{loc}}^{c,[\![0,\zeta[\![}$, $M^d\in\mathcal{M}_{\mathrm{loc}}^{d,[\![0,\zeta[\![}$ have the property $\langle M^c,M^d\rangle\equiv 0$. The next theorem is a natural extension of Lemma 2.5.

THEOREM 2.1. *There exists a one-to-one correspondence between* $\mathcal{J}/\sim$ (*resp.*, $\widehat{\mathcal{J}}/\sim$) *and* $\mathcal{M}^d_{\mathrm{loc}}$ (*resp.*, $\mathcal{M}_{\mathrm{loc}}^{d,[\![0,\zeta[\![}$) *which is characterized by the relation that for* $\phi\in\mathcal{J}$ (*resp.*, $\widehat{\mathcal{J}}$) *there exists* $M\in\mathcal{M}^d_{\mathrm{loc}}$ (*resp.*, $\mathcal{M}_{\mathrm{loc}}^{d,[\![0,\zeta[\![}$) *such that* $M_t-M_{t-}=\phi(X_{t-},X_t)$ *for all* $t\in[0,\infty[$ (*resp.*, $t\in[0,\zeta[$) $\mathbb{P}_x$*-a.s. for q.e.* $x\in E$. *Conversely for each* $M\in\mathcal{M}^d_{\mathrm{loc}}$ (*resp.*, $\mathcal{M}_{\mathrm{loc}}^{d,[\![0,\zeta[\![}$), *there exists a* $\phi\in\mathcal{J}$ (*resp.*, $\widehat{\mathcal{J}}$) *such that the same equation holds. Moreover, we have* $\langle M\rangle_t=\int_0^t\int_{E_\partial}\phi^2(X_s,y)N(X_s,dy)\,dH_s$ *for all* $t\in[0,\infty[$ $\mathbb{P}_x$*-a.s. for q.e.* $x\in E$.

PROOF. We only prove the correspondence between $\mathcal{J}/\sim$ and $\mathcal{M}^d_{\mathrm{loc}}$. The proof of the correspondence between $\widehat{\mathcal{J}}/\sim$ and $\mathcal{M}_{\mathrm{loc}}^{d,[\![0,\zeta[\![}$ is similar by replacing $\sigma_{E\setminus F_k}$ with τ_{F_k}. Suppose $\phi\in\mathcal{J}$. Take an $\mathcal{E}$-nest $\{F_k\}$ of closed sets such that $\mathbf{1}_{F_k}N(|\phi|^2)\mu_H\in S_{00}$. Then $\mathbf{1}_{F_k}\phi\in\mathring{\mathcal{J}}$ for each $k\in\mathbb{N}$, where $(\mathbf{1}_{F_k}\phi)(x,y):=\mathbf{1}_{F_k}(x)\phi(x,y)$, $x,y\in E_\partial$, and there exists an $M^{(k)}\in\mathring{\mathcal{M}}^d$ such that $M^{(k)}_t-M^{(k)}_{t-}=\mathbf{1}_{F_k}(X_{t-})\phi(X_{t-},X_t)$ for all $t\in[0,\infty[$ $\mathbb{P}_x$-a.s. for q.e. $x\in E$. Such $M^{(k)}$ is an **e**-convergent limit of $\{M^{(k),\ell}\}_{\ell=1}^\infty$, where

$$M^{(k),\ell}_t:=\sum_{0<s\le t}\mathbf{1}_{F_k}(X_{s-})\phi_\ell(X_{s-},X_s)-\int_0^t\mathbf{1}_{F_k}(X_s)N(\phi_\ell)(X_s)\,dH_s.$$

This yields that for $j>k$, $M^{(j)}_t=M^{(k)}_t$ for $t<\sigma_{E\setminus F_k}$, more strongly $M^{(j)}_{t\wedge\sigma_{E\setminus F_k}}=M^{(k)}_{t\wedge\sigma_{E\setminus F_k}}$ because of $X_{\sigma_{E\setminus F_k}-}\in F_k\cup\{\partial\}$. Hence M defined by $M_t:=M^{(k)}_t$ for $t<\sigma_{E\setminus F_k}$ satisfies $M_{t\wedge\sigma_{E\setminus F_k}}=M^{(k)}_{t\wedge\sigma_{E\setminus F_k}}$, which implies $M\in\mathcal{M}^d_{\mathrm{loc}}$, because $\{F_k\}$ is also a strict $\mathcal{E}$-nest.

Conversely suppose $M \in \mathcal{M}^d_{\mathrm{loc}}$. Then there exists a sequence $\{M^{(n)}\}$ of square integrable purely discontinuous MAFs and an increasing sequence $\{T_n\}$ of stopping times such that $\lim_{n\to\infty} T_n = \infty$ and $M_{t\wedge T_n} = M^{(n)}_{t\wedge T_n}$ for all $t \in [0,\infty[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$. By an argument in the proof of Proposition 2.8 in [3], we can construct a quadratic variational process $\langle M\rangle$, which is a PCAF, and a nest $\{F_k\}$ of closed sets such that $\mathbf{1}_{F_k\cup\{\partial\}} * M \in \mathring{\mathcal{M}}$ and $\mathbf{e}(\mathbf{1}_{F_k\cup\{\partial\}} * M) = \frac{1}{2}\mu_{\langle M\rangle}(F_k)$. Note that $\mathbf{1}_{\{\partial\}} * M = 0$ because $\int_0^t \mathbf{1}_{\{\partial\}}(X_s)\, d\langle M\rangle_s = 0$. We remark that $\langle M, N\rangle \equiv 0$ for all $N \in \mathcal{M}^c_{\mathrm{loc}}$, which implies $\mathbf{1}_{F_k\cup\{\partial\}} * M \in \mathring{\mathcal{M}}^d$, hence $M \in \mathring{\mathcal{M}}^d_{f\text{-loc}}$. As in the proof of Proposition 2.8 in [3], we see $M_t = (\mathbf{1}_{F_k\cup\{\partial\}} * M)_t$ for $t \le \sigma_{E\setminus F_k}$. Indeed, we have this from the assertion for $t < \sigma_{E\setminus F_k}$ and $\Delta M_{\sigma_{E\setminus F_k}} = \mathbf{1}_{F_k\cup\{\partial\}}(X_{\sigma_{E\setminus F_n}-})\Delta M_{\sigma_{E\setminus F_k}} = \Delta(\mathbf{1}_{F_k\cup\{\partial\}} * M)_{\sigma_{E\setminus F_k}}$. By Lemma 2.5, there exists a Borel function $\phi_k \in \mathring{\mathcal{J}}$ such that $\mathbf{1}_{F_k\cup\{\partial\}}(X_{t-})(M_t - M_{t-}) = \phi_k(X_{t-}, X_t)$ for $t \in [0,\infty[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$. From this, for $j > k$, we see that $\phi_k(X_{t-}, X_t) = \phi_j(X_{t-}, X_t)$ for $t \le \sigma_{E\setminus F_k}$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$. Let G_k be the fine interior of F_k. By Lemma 2.6, for $j > k$ we have $\phi_k = \phi_j$ J^*-a.e. on $G_k \times E_\partial$. So we can define ϕ on $E_\partial \times E_\partial$ such that $\phi = \phi_k$ J^*-a.e. on $G_k \times E_\partial$. From Lemma 2.2, we see $N(\phi^2)\mu_H \in S$. Applying Lemma 2.6 again, $\phi(X_{t-}, X_t) = \phi_k(X_{t-}, X_t) = M_t - M_{t-}$ for all $t \le \sigma_{E\setminus G_k}$. Moreover, we see $\langle \mathbf{1}_{F_k\cup\{\partial\}} * M\rangle_t = \int_0^t \int_{E_\partial} \phi_k^2(X_s, y) N(X_s, dy)\, dH_s$. Therefore we obtain the desired assertion. $\square$

COROLLARY 2.3. *For $\phi \in \widehat{\mathcal{J}}_*$, there exists a $K \in \mathcal{M}^d_{\mathrm{loc}}$ such that $K_t - K_{t-} = -\mathbf{1}_{E\times E}(\phi + \overline{\phi})(X_{t-}, X_t)$ for all $t \in\,]0,\infty[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$.*

PROOF. This is clear from that $\phi \in \widehat{\mathcal{J}}_*$ implies $\mathbf{1}_{E\times E}\phi, \mathbf{1}_{E\times E}\overline{\phi} \in \mathcal{J}_*$. $\square$

REMARK 2.3. A similar argument of the proof of Theorem 2.1 yields

$$\begin{aligned}
\mathcal{M}^d_{\mathrm{loc}} = \big\{M \mid & \text{ there exists } \{G_n\} \in \Theta \text{ and } M^{(n)} \in \mathring{\mathcal{M}}^d \text{ such that} \\
& M_t = M^{(n)}_t \text{ for all } t \le \sigma_{E\setminus G_n} \text{ and } n \in \mathbb{N}, \mathbb{P}_x\text{-a.s. for q.e. } x \in E\big\}, \\
\mathcal{M}^{d,[\![0,\zeta[\![}_{\mathrm{loc}} = \big\{M \mid & \text{ there exists } \{G_n\} \in \Theta \text{ and } M^{(n)} \in \mathring{\mathcal{M}}^d \text{ such that} \\
& M_t = M^{(n)}_t \text{ for all } t \le \tau_{G_n} \text{ and } n \in \mathbb{N}, \mathbb{P}_x\text{-a.s. for q.e. } x \in E\big\}.
\end{aligned}$$

These show $\mathcal{M}^d_{\mathrm{loc}} \subset \mathring{\mathcal{M}}^d_{f\text{-loc}}$ and $\mathcal{M}^{d,[\![0,\zeta[\![}_{\mathrm{loc}} \subset \mathring{\mathcal{M}}^d_{\mathrm{loc}}$. We also have the coincidences $\mathcal{M}^c_{\mathrm{loc}} = \mathring{\mathcal{M}}^c_{f\text{-loc}}$ and $\mathcal{M}^{c,[\![0,\zeta[\![}_{\mathrm{loc}} = \mathring{\mathcal{M}}^c_{\mathrm{loc}}$. Indeed, the inclusion $\mathcal{M}^c_{\mathrm{loc}} \subset \mathring{\mathcal{M}}^c_{f\text{-loc}}$ can be obtained in the same way of the proof of Theorem 2.1. The converse inclusion is easily confirmed from the continuity of $M \in \mathring{\mathcal{M}}^c_{f\text{-loc}}$ and $\mathbb{P}_x(\lim_{n\to\infty} \sigma_{E\setminus G_n} = \infty) = 1$ for q.e. $x \in E$.

The next corollary is needed to assure the uniqueness of the generalized Fukushima decomposition later.

COROLLARY 2.4. *We have* $\mathcal{M}_{\mathrm{loc}} \cap \mathcal{N}_{c,f\text{-loc}} = \{0\}$ *and* $\mathcal{M}_{\mathrm{loc}}^{[\![0,\zeta[\![} \cap \mathcal{N}_{c,\mathrm{loc}} = \{0\}$.

PROOF. We only prove $\mathcal{M}_{\mathrm{loc}} \cap \mathcal{N}_{c,f\text{-loc}} = \{0\}$. The proof of $\mathcal{M}_{\mathrm{loc}}^{[\![0,\zeta[\![} \cap \mathcal{N}_{c,\mathrm{loc}} = \{0\}$ is similar to this by replacing $\sigma_{E\setminus F_k}$ with τ_{F_k}. Take $M \in \mathcal{M}_{\mathrm{loc}}$, $N \in \mathcal{N}_{c,f\text{-loc}}$ and suppose $M_t + N_t = 0$ for all $t \in [0,\infty[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$. In particular, M is continuous. Let $\phi \in \mathcal{J}$ be the jump function associated to M^d. As in the proof of Theorem 2.1, we can construct a common $\{G_k\} \in \Theta$, $M^{(k)} \in \mathring{\mathcal{M}}$ with its jump function $\phi_k \in \mathring{\mathcal{J}}$ and $N^{(k)} \in \mathcal{N}_c$ such that $M_t = M_t^{(k)}$, $N_t = N_t^{(k)}$ for all $t < \sigma_{E\setminus G_k}$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$, and $\phi = \phi_k$ J^*-a.e. on $G_k \times E_\partial$. The continuity of M yields $\phi(X_{t-}, X_t) = 0$ for all $t \in]0,\infty[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$. Then we can conclude that $\phi_k(X_{t-}, X_t) = 0$ for all $t \in]0, \sigma_{E\setminus G_k}]$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$ by way of Lemma 2.6. This implies that $M^{(k)}_{t\wedge\sigma_{E\setminus G_k}} = M^{(k)}_{t\wedge\sigma_{E\setminus G_k}-}$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$. On the other hand, we see that $M^{(k)}_{t\wedge\sigma_{E\setminus G_k}-} + N^{(k)}_{t\wedge\sigma_{E\setminus G_k}} = 0$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$ because $M_t^{(k)} + N_t^{(k)} = 0$ for all $t \in [0, \sigma_{E\setminus G_k}[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$. Therefore we obtain

$$M^{(k)}_{t\wedge\sigma_{E\setminus G_k}} + N^{(k)}_{t\wedge\sigma_{E\setminus G_k}} = 0 \qquad \text{for all } t \in [0,\infty[\ \mathbb{P}_x\text{-a.s. for q.e. } x \in E.$$

Then we can conclude that $M_t = N_t = 0$ for all $t \in [0,\infty[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$ in view of the argument of the proof of the uniqueness of Fukushima decomposition as in Theorem 5.5.1 of [11]. □

We define subclasses of $\mathcal{M}_{\mathrm{loc}}^d$ as follows:

$$\mathcal{M}_{\mathrm{loc}}^j := \{M \in \mathcal{M}_{\mathrm{loc}}^d \mid \phi(\cdot, \partial) = 0\ \kappa\text{-a.e. on } E\},$$
$$\mathcal{M}_{\mathrm{loc}}^\kappa := \{M \in \mathcal{M}_{\mathrm{loc}}^d \mid \phi = 0\ J\text{-a.e. on } E \times E\}.$$

Then we have a similar statement as in Corollary 2.2. From this, we see that $M \in \mathcal{M}_{\mathrm{loc}}^j$, $N \in \mathcal{M}_{\mathrm{loc}}^\kappa$ implies $\langle M, N\rangle \equiv 0$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$. Every $M \in \mathcal{M}_{\mathrm{loc}}$ is decomposed to $M = M^c + M^j + M^\kappa$, where $M^c \in \mathcal{M}_{\mathrm{loc}}^c$, $M^j \in M_{\mathrm{loc}}^j$, $M^\kappa \in \mathcal{M}_{\mathrm{loc}}^\kappa$ have the properties $\langle M^c, M^j\rangle \equiv \langle M^j, M^\kappa\rangle \equiv \langle M^\kappa, M^c\rangle \equiv 0$. For $M \in \mathcal{M}_{\mathrm{loc}}^{[\![0,\zeta[\![}$ with its jump function $\phi \in \widehat{\mathcal{J}}$, we can consider $M^j \in \mathcal{M}_{\mathrm{loc}}^j$ (resp., $K \in \mathcal{M}_{\mathrm{loc}}^j$) associated to $\mathbf{1}_{E\times E}\phi \in \mathcal{J}$ [resp., $-\mathbf{1}_{E\times E}(\phi + \overline{\phi}) \in \mathcal{J}$], where K is constructed in Corollary 2.3.

We introduce the subclasses $\dot{\mathcal{F}}_{\mathrm{loc}}^\dagger$, $\dot{\mathcal{F}}_{\mathrm{loc}}^\ddagger$ of $\dot{\mathcal{F}}_{\mathrm{loc}}$ as follows:

$$\dot{\mathcal{F}}_{\mathrm{loc}}^\dagger := \{u \in \dot{\mathcal{F}}_{\mathrm{loc}} \mid N(\mathbf{1}_{E\times E}(u(\cdot) - u)^2)\mu_H \in S\},$$
$$\dot{\mathcal{F}}_{\mathrm{loc}}^\ddagger := \{u \in \dot{\mathcal{F}}_{\mathrm{loc}}^\dagger \mid u(\partial) \in \mathbb{R} \text{ and } (u(\cdot) - u(\partial))^2\kappa \in S\}.$$

Clearly, $\dot{\mathcal{F}}_{\mathrm{loc}}^\dagger$ and $\dot{\mathcal{F}}_{\mathrm{loc}}^\ddagger$ are linear subspaces of $\dot{\mathcal{F}}_{\mathrm{loc}}$, and $\mathbf{1}_{E_\partial}, \mathbf{1}_E \in \dot{\mathcal{F}}_{\mathrm{loc}}^\ddagger$. By Remark 3.9 of [3] and $\kappa \in S$, we see $\mathcal{F}_e \cup (\dot{\mathcal{F}}_{\mathrm{loc}})_b \subset \dot{\mathcal{F}}_{\mathrm{loc}}^\ddagger$ by regarding $u(\partial) \in \mathbb{R}$ for

$u \in \dot{\mathcal{F}}_{\mathrm{loc}}$. For $u, v \in \dot{\mathcal{F}}^{\ddagger}_{\mathrm{loc}}$ (resp., $u, v \in \dot{\mathcal{F}}^{\dagger}_{\mathrm{loc}}$), we see $uv \in \dot{\mathcal{F}}^{\ddagger}_{\mathrm{loc}}$ (resp., $uv \in \dot{\mathcal{F}}^{\dagger}_{\mathrm{loc}}$) provided u or v is bounded. From Theorem 2.1, for $u \in \dot{\mathcal{F}}^{\ddagger}_{\mathrm{loc}}$ (resp., $u \in \dot{\mathcal{F}}^{\dagger}_{\mathrm{loc}}$), there exists a $M^{u,d} \in \mathcal{M}^d_{\mathrm{loc}}(\subset \mathring{\mathcal{M}}^d_{f\text{-loc}})$ (resp., $M^{u,d} \in \mathcal{M}^{d,[\![0,\zeta[\![}_{\mathrm{loc}}$) such that $\Delta M^{u,d}_t = \Delta u(X_t)$ for all $t \in [0,\infty[$ (resp., $[0,\zeta[$) $\mathbb{P}_x$-a.s. for q.e. $x \in E$.

Moreover, we define

$$\mathcal{F}^{\dagger}_{\mathrm{loc}} := \left\{ u \in \mathcal{F}_{\mathrm{loc}} \,\middle|\, \forall K \in \mathcal{K}, \int_{K\times E} \bigl(u(y)-u(x)\bigr)^2 J(dx\,dy) < \infty \right\},$$

$$\mathcal{F}^{\ddagger}_{\mathrm{loc}} := \left\{ u \in \mathcal{F}^{\dagger}_{\mathrm{loc}} \,\middle|\, u(\partial) \in \mathbb{R} \text{ and } \forall K \in \mathcal{K}, \int_{K} \bigl(u(x)-u(\partial)\bigr)^2 \kappa(dx) < \infty \right\}.$$

Here $\mathcal{K}$ denotes the family of all compact sets and $\mathcal{F}_{\mathrm{loc}}$ is the space of functions locally in $\mathcal{F}$ in the ordinary sense (see [11]). Clearly, $\mathcal{F}^{\dagger}_{\mathrm{loc}} \subset \dot{\mathcal{F}}^{\dagger}_{\mathrm{loc}}$ and $\mathcal{F}^{\ddagger}_{\mathrm{loc}} \subset \dot{\mathcal{F}}^{\ddagger}_{\mathrm{loc}}$. For $u \in \mathcal{F}_{\mathrm{loc}}$, $u \in \mathcal{F}^{\dagger}_{\mathrm{loc}}$ if and only if that for any compact set K with its relatively compact open neighborhood G

$$\int_{K\times G^c} \bigl(u(y)-u(x)\bigr)^2 J(dx\,dy) < \infty.$$

We see $\mathcal{F}_e \cup (\mathcal{F}_{\mathrm{loc}})_b \subset \mathcal{F}^{\ddagger}_{\mathrm{loc}}$, because of $J(K\times G^c) < \infty$ and $\kappa(K) < \infty$ (see Corollary 5.1 in [16]), where K and G are noted as above.

EXAMPLE 2.1. Let $(\mathcal{E},\mathcal{F})$ be a regular Dirichlet form on $L^2(\mathbb{R}^N)$ whose jumping measure J has an expression $J(dx\,dy) = f(|x-y|)\,dx\,dy$ such that f is a locally bounded Borel function on $[0,\infty[$ satisfying

$$\int_c^{\infty} f(r) r^{N+1}\,dr < \infty \qquad \text{for some } c > 0. \tag{2.8}$$

For instance, relativistic symmetric α-stable processes satisfy (2.8) (see [7]). Then each coordinate function $\ell_k(x) := x_k$ satisfies $\ell_k \in \mathcal{F}^{\dagger}_{\mathrm{loc}}$ $(k = 1,2,\dots,N)$ under (2.8). Indeed, for any compact set K and its relatively compact open neighborhood G with $d(K,G^c) := \inf_{x\in K, y\in G^c}|x-y| > 0$,

$$\begin{aligned}\int_{K\times G^c} |\ell_k(x)-\ell_k(y)|^2 J(dx\,dy) &\le \int_K \int_{G^c} |x-y|^2 f(|x-y|)\,dy\,dx \\ &\le |K|\sigma(\mathbb{S}^{N-1}) \int_{d(K,G^c)}^{\infty} r^{N+1} f(r)\,dr < \infty,\end{aligned}$$

where $|K|$ is the volume of K and $\sigma(\mathbb{S}^{N-1})$ is the area of unit sphere.

3. Nakao integrals. Now we are in a position to define an extension of Nakao's divergence-like CAF of zero energy and stochastic integrals with respect to it in our setting.

Let $\mathcal{N}^*_c \subset \mathcal{N}_c$ denote the class of continuous additive functionals of the form $N^u + \int_0^{\cdot} g(X_s)\,ds$ for some $u \in \mathcal{F}$ and $g \in L^2(E;m)$. Nakao [22] constructed a linear operator Γ from $\overset{\circ}{\mathcal{M}}$ into $\mathcal{N}_c^*$ in the following way: for every $Z \in \overset{\circ}{\mathcal{M}}$, there is a unique $w \in \mathcal{F}$ such that

$$\mathcal{E}_1(w, f) = \tfrac{1}{2}\mu_{\langle M^f + M^{f,\kappa}, Z\rangle}(E) \qquad \text{for every } f \in \mathcal{F}. \tag{3.1}$$

This unique w is denoted by $\gamma(Z)$. The operator Γ is defined by

$$\Gamma(Z)_t := N_t^{\gamma(Z)} - \int_0^t \gamma(Z)(X_s)\, ds \qquad \text{for } Z \in \overset{\circ}{\mathcal{M}}. \tag{3.2}$$

It is shown in Nakao [22] that $\Gamma(Z)$ can be characterized by the following equation:

$$\lim_{t \downarrow 0} \frac{1}{t}\mathbb{E}_{g\cdot m}[\Gamma(Z)_t] = -\frac{1}{2}\mu_{\langle M^g + M^{g,\kappa}, Z\rangle}(E) \qquad \text{for every } g \in \mathcal{F}_b. \tag{3.3}$$

Here $\mathcal{F}_b := \mathcal{F} \cap L^\infty(E; m)$. So, in particular, we have $\Gamma(M^u) = N^u$ for $u \in \mathcal{F}$. Moreover, we have the following:

LEMMA 3.1. *It holds that $\Gamma(M^u) = N^u$ for $u \in \mathcal{F}_e$.*

PROOF. Fix $u \in \mathcal{F}_e$ and let $\{u_n\}$ be an approximating $\mathcal{E}$-Cauchy sequence such that $u_n \to u$ m-a.e. In view of the proof of Theorem 5.2.2 in [11], by taking a subsequence $\{n_k\}$, $\{u_{n_k}(X_t)\}$, $M_t^{u_{n_k}}$ and $N_t^{u_{n_k}}$ uniformly converges to $u(X_t)$, M_t^u and N_t^u, respectively, on any compact subinterval of $[0, \infty[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$. From Theorem 3.2 in [22], by taking another subsequence, $\Gamma(M^{u_{n_k}})$ converges to $\Gamma(M^u)$ uniformly on any finite interval $\mathbb{P}_x$-a.s. for q.e. $x \in E$. Since $\Gamma(M^{u_{n_k}}) = N^{u_{n_k}}$, we have $\Gamma(M^u) = N^u$. $\square$

In the same way of Nakao [22] (cf. (3.13) in [3]), we can define a stochastic integral used by the operator Γ: for $M \in \overset{\circ}{\mathcal{M}}$ with its jump function $\varphi \in \overset{\circ}{\mathcal{J}}$ and $f \in \mathcal{F}_e \cap L^2(E; \mu_{\langle M\rangle})$, we set

$$\begin{aligned} &\int_0^t f(X_s)\, d\Gamma(M)_s \\ &\quad := \Gamma(f * M)_t - \frac{1}{2}\langle M^{f,c} + M^{f,j}, M^c + M^j + K\rangle_t, \qquad t \in [0, \infty[, \end{aligned} \tag{3.4}$$

where $(f * M)_t = \int_0^t f(X_{s-})\, dM_s$ and $K \in \overset{\circ}{\mathcal{M}}{}^d$ with $K_t - K_{t-} = -\mathbf{1}_{E\times E}(\varphi + \overline{\varphi})(X_{t-}, X_t)$ $t \in]0, \infty[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$. Equation (3.4) is well defined under $\mathbb{P}_x$ for q.e. $x \in E$. In this paper, we call the operator Γ *Nakao operator* and the integral (3.4) *Nakao integral*.

REMARK 3.1. Equation (3.4) is consistent with the extension of Nakao integral developed in [3] up to ζ under $\mathbb{P}_m$ (see Theorem 6.3 in [17]).

For any $M \in \mathring{\mathcal{M}}^c_{\mathrm{loc}} = \mathcal{M}^{c,[\![0,\zeta[\![}_{\mathrm{loc}}$ (in particular for $M \in \mathring{\mathcal{M}}^c_{f\text{-loc}} = \mathcal{M}^c_{\mathrm{loc}}$), $\Gamma(M)$ can be defined as an element in $\mathcal{N}_{c,f\text{-loc}}$. To see this, we need the following lemma extending Lemma 3.4 in [22]:

LEMMA 3.2 (Local property of Γ on $\mathring{\mathcal{M}}^c$). *Let $M \in \mathring{\mathcal{M}}^c$ and G be a (q.e.) finely open Borel set. Suppose that $M_t = 0$ $\mathbb{P}_m$-a.e. on $\{t < \tau_G\}$ for each $t > 0$. Then $\Gamma(M)_t = 0$ for all $t \in [0, \sigma_{E\setminus G}]$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$.*

PROOF. By assumption, $\mathbb{E}_m[\langle M\rangle_{t\wedge\sigma_{E\setminus G}}] = \mathbb{E}_m[\langle M\rangle_{t\wedge\tau_G}] = \mathbb{E}_m[M^2_{t\wedge\tau_G}] = 0$ for each fixed $t > 0$. Then $\mu_{\langle M\rangle}(G) = 0$ by Lemma 5.1.10(iii) in [11]. Let $h \in \mathcal{F}_F$ for a closed set F with $F \subset G$. Then $\mu_{\langle M^{h,c}\rangle}(E \setminus G) = 0$ by Lemma 2.4(1). From this, we have

$$\mu_{\langle M^{h,c},M\rangle}(E)^2 \le 2\mu_{\langle M^{h,c}\rangle}(E)\mu_{\langle M\rangle}(G) + 2\mu_{\langle M\rangle}(E)\mu_{\langle M^{h,c}\rangle}(E \setminus G) = 0.$$

Hence $\mathcal{E}_1(\gamma(M), h) = 0$ for any $h \in \mathcal{F}_F$ with $F \subset G$. Since $(\mathcal{E}_G, \mathcal{F}_G)$ is a quasi-regular Dirichlet form on $L^2(G; m)$, there exists an $\mathcal{E}_G$-nest $\{F_n\}$ of compact sets of G (see Lemma 3.4 in [16]). From this, for any $h \in \mathcal{F}_G$, there exists $h_k \in \bigcup_{n=1}^\infty \mathcal{F}_{F_n}$ such that $\{h_k\}$ $\mathcal{E}_1^{1/2}$-converges to h as $k \to \infty$. Therefore, $\mathcal{E}_1(\gamma(M), h) = 0$ for any $h \in \mathcal{F}_G$, which implies $\Gamma(M)_t = N_t^{\gamma(M)} - \int_0^t \gamma(M)(X_s)\,ds = 0$ for $t < \sigma_{E\setminus G}$ by way of Lemma 5.4.2(ii) in [11]. $\square$

Let $(\mathcal{M}^d_{\mathrm{loc}})_*$ [resp., $(\mathcal{M}^d_{\mathrm{loc}})_{\mathrm{as}}$] be the subclass of $\mathcal{M}^d_{\mathrm{loc}}$ associated to $\mathcal{J}_*/\sim$ (resp., $\mathcal{J}_{\mathrm{as}}/\sim$) and $(\mathcal{M}^{d,[\![0,\zeta[\![}_{\mathrm{loc}})_*$ [resp., $(\mathcal{M}^{d,[\![0,\zeta[\![}_{\mathrm{loc}})_{\mathrm{as}}$] the subclass of $\mathcal{M}^{d,[\![0,\zeta[\![}_{\mathrm{loc}}$ associated to $\widehat{\mathcal{J}}_*/\sim$ (resp., $\widehat{\mathcal{J}}_{\mathrm{as}}/\sim$).

We say that $M \in (\mathcal{M}_{\mathrm{loc}})_*$ [resp., $M \in (\mathcal{M}_{\mathrm{loc}})_{\mathrm{as}}$] if and only if its purely discontinuous part M^d is in $(\mathcal{M}^d_{\mathrm{loc}})_*$ [resp., $(\mathcal{M}^d_{\mathrm{loc}})_{\mathrm{as}}$], and the classes $(\mathcal{M}^{[\![0,\zeta[\![}_{\mathrm{loc}})_*$ and $(\mathcal{M}^{[\![0,\zeta[\![}_{\mathrm{loc}})_{\mathrm{as}}$ are similarly defined. For $M \in (\mathcal{M}^{[\![0,\zeta[\![}_{\mathrm{loc}})_*$ with its jump function $\varphi \in \widehat{\mathcal{J}}_*$, let $M^c \in \mathcal{M}^{c,[\![0,\zeta[\![}_{\mathrm{loc}}$ be its continuous part and take $M^j \in \mathcal{M}^j_{\mathrm{loc}}$ associated with $\mathbf{1}_{E\times E}\varphi \in \mathcal{J}_*$ and $K \in \mathcal{M}^j_{\mathrm{loc}}$ constructed in Corollary 2.3 associated with $-\mathbf{1}_{E\times E}(\varphi + \overline{\varphi}) \in \mathcal{J}_*$.

We shall extend Γ over $(\mathcal{M}^{[\![0,\zeta[\![}_{\mathrm{loc}})_*$ and establish (3.4) for more general integrands and integrators under $\mathbb{P}_x$ for q.e. $x \in E$. To do this we need the following lemma:

LEMMA 3.3. *Take $M \in (\mathcal{M}^{[\![0,\zeta[\![}_{\mathrm{loc}})_*$ with its jump function $\varphi \in \widehat{\mathcal{J}}_*$. Let G be a (q.e.) finely open Borel set satisfying $\mathbf{1}_{G\times E}\varphi, \mathbf{1}_{G\times E}\overline{\varphi} \in \mathring{\mathcal{J}}$. Take $f \in \mathcal{F}_e \cap L^2(E; \mu_{\langle M\rangle})$ and suppose that $f = 0$ m-a.e. on G. Then we have $\Gamma(f * M)_t = \frac{1}{2}\langle M^{f,c} + M^{f,j}, M^c + M^j + K\rangle_t$ for $t \in [0, \sigma_{E\setminus G}[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$.*

PROOF. We show that for any $g \in (\mathcal{F}_G)_b$

$$\lim_{t\to 0} \frac{1}{t}\mathbb{E}_{gm}\left[\Gamma(f * M)_t - \frac{1}{2}\langle M^{f,c} + M^{f,j}, M^c + M^j + K\rangle_t\right] = 0. \tag{3.5}$$

Then we can obtain the assertion from (3.5) in view of the proof of Theorem 2.2 in [22] and Lemma 5.4.4 in [11]. We know

$$\lim_{t\to 0}\frac{1}{t}\mathbb{E}_{gm}[\Gamma(f*M)_t]=-\frac{1}{2}\int_E f(x)\mu_{\langle M^g+M^{g,\kappa},M\rangle}(dx).$$

So for (3.5) it suffices to show

$$\int_E f(x)\mu_{\langle M^g+M^{g,\kappa},M\rangle}(dx)=-\int_E g(x)\mu_{\langle M^{f,c}+M^{f,j},M^c+M^j+K\rangle}(dx). \tag{3.6}$$

Noting $fg=0$ q.e. on E and the derivation properties of continuous part and jumping part of energy measures (see the proof of Lemma 3.1 in [22]), we see $\int_E f\,d\mu_{\langle M^{g,c},M^c\rangle}+\int_E g\,d\mu_{\langle M^{f,c},M^c\rangle}=0$, $\int_E f\,d\mu_{\langle M^{g,j},M^j\rangle}+\int_E g\,d\mu_{\langle M^{f,j},M^j+K\rangle}=0$ and $\int_E f\,d\mu_{\langle M^{g,\kappa},M^\kappa\rangle}=0$, which imply (3.6). □

DEFINITION 3.1 (Extensions of Nakao operators and Nakao integrals). Fix $M\in(\mathcal{M}_{\mathrm{loc}}^{[\![0,\zeta[\![})_*$ with its jump function $\varphi\in\widehat{\mathcal{J}}_*$ and $f\in\dot{\mathcal{F}}_{\mathrm{loc}}$. Let $\{G_k\}\in\Theta$ be a common nest such that $\mu_{\langle M\rangle}(G_k)<\infty$, $f=f_k$ m-a.e. on G_k for some $f_k\in\mathcal{F}_b$, $\mathbf{1}_{G_k\times E}\varphi,\mathbf{1}_{G_k\times E}\overline{\varphi}\in\mathring{\mathcal{J}}$ for each $k\in\mathbb{N}$. Set $E_k:=\{x\in E\mid\mathbb{E}_x[\int_0^{\tau_{G_k}}e^{-t}g(X_t)\,dt]>1/k\}$ for $g\in L^2(E;m)$ with $0<g\le 1$ m-a.e. Then $e_k:=k\mathbb{E}_x[\int_0^{\tau_{G_k}}e^{-t}g(X_t)\,dt]\wedge 1\in\mathcal{F}_{G_k}$ satisfies $\mathbf{1}_{E_k}\le e_k\le\mathbf{1}_{G_k}$ q.e. on E. In view of Lemma 3.3 in [16], we have $\{E_k\}\in\Theta$.

We now define

$$\Gamma(M)_t:=\Gamma(e_k*M)_t-\tfrac{1}{2}\langle M^{e_k,c}+M^{e_k,j},M^c+M^j+K\rangle_t\qquad\text{for } t\in[0,\sigma_{E\setminus E_k}[$$

for each $k\in\mathbb{N}$ $\mathbb{P}_x$-a.s. for q.e. $x\in E$. For $M\in(\mathcal{M}_{\mathrm{loc}}^{[\![0,\zeta[\![})_*$ and $f\in\mathcal{F}_e\cap L^2(E;\mu_{\langle M\rangle})$, we set

$$\int_0^t f(X_s)\,d\Gamma(M)_s:=\Gamma(f*M)_t-\frac{1}{2}\langle M^{f,c}+M^{f,j},M^c+M^j+K\rangle_t$$

for $t\in[0,\infty[$ $\mathbb{P}_x$-a.s. for q.e. $x\in E$. For general $f\in\dot{\mathcal{F}}_{\mathrm{loc}}$ and $M\in(\mathcal{M}_{\mathrm{loc}}^{[\![0,\zeta[\![})_*$ as above, we set

$$\int_0^t f(X_s)\,d\Gamma(M)_s:=\int_0^t(fe_k)(X_s)\,d\Gamma(M)_s\qquad\text{for } t\in[0,\sigma_{E\setminus E_k}[$$

for each $k\in\mathbb{N}$ $\mathbb{P}_x$-a.s. for q.e. $x\in E$. Note that $fe_k\in\mathcal{F}_b\cap L^2(E;\mu_{\langle M\rangle})$ for each $k\in\mathbb{N}$. These are well defined for all $t\in[0,\infty[$ $\mathbb{P}_x$-a.s. for q.e. $x\in E$ in view of Lemma 3.3 and are elements in $\mathcal{N}_{c,f\text{-loc}}$.

For $f\in\dot{\mathcal{F}}_{\mathrm{loc}}$ and $M\in(\mathcal{M}_{\mathrm{loc}}^{[\![0,\zeta[\![})_*$, we see

$$\int_0^t f(X_s)\,d\Gamma(M^c)_s=\Gamma(f*M^c)_t-\frac{1}{2}\langle M^{f,c},M^c\rangle_t \tag{3.7}$$

for all $t\in[0,\infty[$ $\mathbb{P}_x$-a.s. for q.e. $x\in E$, where $\Gamma(f*M^c)_t$ can be defined by way of Lemma 3.2.

REMARK 3.2. (1) In [3], we define extensions of Nakao operators and Nakao integrals in terms of time reverse operators, which are defined up to ζ under $\mathbb{P}_m$, and the Nakao integral is also refined for integrator $\Gamma(M)$, $M \in \mathring{\mathcal{M}}$ and integrand $f(X)$ for $f \in \dot{\mathcal{F}}_{\text{loc}}$ under $\mathbb{P}_x$ for q.e. $x \in E$. So the Nakao integral in Definition 3.1 is a pure extension of this refinement. Though the condition on the integrator of our Nakao integrals is rather restrictive than theirs described to be up to ζ under $\mathbb{P}_m$, it is defined for all time under the law for quasi-everywhere starting points.

(2) The extensions of Nakao operators and Nakao integrals in [3] are consistent with our corresponding notions up to ζ under $\mathbb{P}_m$ (see Theorem 6.3 in [17]).

The following lemma is needed to establish the generalized Itô formula.

LEMMA 3.4 (Local property of extended Nakao integral). *Take*

$$M \in \big(\mathcal{M}_{\text{loc}}^{[\![0,\zeta[\![}\big)_*$$

with its jump function $\varphi \in \widehat{\mathcal{J}}_*$ and $f \in \dot{\mathcal{F}}_{\text{loc}}$. Let G be a (q.e.) finely open Borel set. Suppose that $f = 0$ m-a.e. on G. Then

$$\int_0^t f(X_s)\,d\Gamma(M)_s = 0$$

holds for all $t \in [0, \sigma_{E\setminus G}[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$.

PROOF. Let $\{E_k\} \in \Theta$ and $e_k \in \mathcal{F}$ be constructed as in Definition 3.1. Since $fe_k \in \mathcal{F}_b \cap L^2(E; \mu_{\langle M\rangle})$ and $\mathbf{1}_{(G\cap E_k)\times E}\varphi, \mathbf{1}_{(G\cap E_k)\times E}\overline{\varphi} \in \mathring{\mathcal{J}}$, we can apply Lemma 3.3 so that

$$\int_0^t (fe_k)(X_s)\,d\Gamma(M)_s = 0 \qquad \text{holds for } t \in [0, \sigma_{E\setminus G} \wedge \sigma_{E\setminus E_k}[,$$

$\mathbb{P}_x$-a.s. for q.e. $x \in E$. Therefore we obtain the desired assertion. □

The following propositions are an addendum (cf. Theorems 4.1 and 4.2 in [3]). We omit its proofs.

PROPOSITION 3.1. *Take $M \in (\mathcal{M}_{\text{loc}}^{[\![0,\zeta[\![})_*$ and $f \in \dot{\mathcal{F}}_{\text{loc}}$. Suppose that $\Gamma(M)$ is a CAF A of finite variation on $[\![0,\zeta[\![$. Then*

$$\int_0^t f(X_s)\,d\Gamma(M)_s = \int_0^t f(X_s)\,dA_s$$

holds for all $t \in [0, \zeta[$ (for all $t \in [0, \infty[$ provided $M \in (\mathcal{M}_{\text{loc}})_*$) $\mathbb{P}_x$-a.s. for q.e. $x \in E$.

PROPOSITION 3.2. *Take* $M \in (\mathcal{M}_{\mathrm{loc}}^{[\![0,\zeta[\![})_*$ *and* $f, g \in \dot{\mathcal{F}}_{\mathrm{loc}}$. *Then*

$$\int_0^t g(X_s)\, d\left(\int_0^\cdot f(X_u)\, d\Gamma(M)_u\right)_s = \int_0^t (fg)(X_s)\, d\Gamma(M)_s$$

holds for all $t \in [0, \zeta[$ (*for all* $t \in [0, \infty[$ *provided* $M \in (\mathcal{M}_{\mathrm{loc}})_*$) $\mathbb{P}_x$-*a.s. for q.e.* $x \in E$.

4. Stochastic integrals by Dirichlet processes.

DEFINITION 4.1 (Dirichlet processes). For $M \in (\mathcal{M}_{\mathrm{loc}}^{[\![0,\zeta[\![})_*$, we set $A := M + \Gamma(M)$, which can be defined by way of Definition 3.1. Note that A is defined on $[0, \infty[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$ if $M \in (\mathcal{M}_{\mathrm{loc}})_*$; otherwise, it is only defined on $[0, \zeta[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$. For $M = M^u$ with $u \in \mathcal{F}_e$, we see $A = A^u$, where $A_t^u := u(X_t) - u(X_0)$. For $M \in \overset{\circ}{\mathcal{M}}$ and each $i = c, d, j, \kappa$, we further set $A_t^i := M_t^i + \Gamma(M^i)_t$ and write $A^{u,i} := A^i$ if $M = M^u$, $u \in \mathcal{F}_e$.

We see $A = A^c + A^d = A^c + A^j + A^\kappa$ for $A = M + \Gamma(M)$, $M \in \overset{\circ}{\mathcal{M}}$. By (2.3) and Lemma 3.1, we have $A^u = A^{u,c} + A^{u,d} = A^{u,c} + A^{u,j} + A^{u,\kappa}$ for $u \in \mathcal{F}_e$.

DEFINITION 4.2 (Stochastic integrals by Dirichlet processes). Take and fix $M \in \overset{\circ}{\mathcal{M}}$. For $f \in L^2(E; \mu_{\langle M^c \rangle})$, we set

$$\int_0^t f(X_s) \circ dA_s^c := (f * M^c)_t + \Gamma(f * M^c)_t$$

for all $t \in [0, \infty[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$. For $f \in \mathcal{F}_e \cap L^2(E; \mu_{\langle M \rangle})$, we set

$$\int_0^t f(X_s) \circ dM_s := (f * M)_t + \frac{1}{2}[M^f, M]_t,$$

$$\int_0^t f(X_{s-})\, dA_s := (f * M)_t + \int_0^t f(X_s)\, d\Gamma(M)_s,$$

$$\int_0^t f(X_s) \circ dA_s := \int_0^t f(X_s) \circ dM_s + \int_0^t f(X_s)\, d\Gamma(M)_s$$

for all $t \in [0, \infty[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$. Recall $(f * M)_t = \int_0^t f(X_{s-})\, dM_s$ for $f \in \mathcal{F}_e \cap L^2(E; \mu_{\langle M \rangle})$ (see the proof of Lemma 2.3). We call $(f * M)_t$ [resp., $\int_0^t f(X_{s-})\, dA_s$] the *Itô integral* of $f(X)$ with integrator M (resp., A) and $\int_0^t f(X_s) \circ dM_s$ [resp., $\int_0^t f(X_s) \circ dA_s$] the *Fisk–Stratonovich integral* of $f(X)$ with integrator M (resp., A).

REMARK 4.1. (1) For the definition of $\int_0^t f(X_s) \circ dM_s^c$ for $f \in \mathcal{F}_e \cap L^2(E; \mu_{\langle M^c \rangle})$, we need $f \in \mathcal{F}_e$, which is unnecessary to define $\int_0^t f(X_s) \circ dA_s^c$.

(2) Our definitions of Fisk–Stratonovich-type integrals are somewhat different from what is found in Protter [23] or in Chapter VI of Meyer [21] except for the case of diffusions. The definition of $\int_0^t f(X_s)\circ dM_s$ in [23] or [21] is given by $(f*M)_t+\frac{1}{2}\langle M^{f,c},M^c\rangle_t$, which has an advantage to give a version of Itô's formula in terms of their Fisk–Stratonovich integrals (see II. Theorem 34, V. Theorems 20 and 21 in [23]), but it only has a Riemann-sum approximation under that $f(X_s)$ and $u(X_s)$ have no jumps in common (see V. Theorem 26 in [23]). Our definition of $\int_0^t f(X_s)\circ dM_s$ admits such an approximation in the framework of semi-martingales at least (cf. Definition 3.9.21 in [2] or Problems 9.12 and 9.13 in [12]). On the other hand, Kurtz, Pardoux and Protter [15] give a different definition for Fisk–Stratonovich-type integrals provided the underlying process is a solution of an SDE driven by semimartingales. Our definitions are also different from theirs.

Now take a jump function $\varphi\in\mathcal{J}$ associated to a given $M\in\mathcal{M}_{\mathrm{loc}}$. We set for each $\ell\in\mathbb{N}$

$$M_t^{d,\ell}:=\sum_{0<s\le t}\varphi_\ell(X_{s-},X_s)-\int_0^t N(\varphi_\ell)(X_s)\,dH_s.$$

In the same way of the proof of Lemma 2.5, if $M\in\overset{\circ}{\mathcal{M}}$, then

$$\mathbf{e}(M^d-M^{d,\ell})=\frac{1}{2}\int_E\int_{E_\partial}\varphi^2(x,y)\mathbf{1}_{\{|\varphi(x,y)|\le 1/\ell\}}N(x,dy)\mu_H(dx).$$

The stochastic integrals $f*M^d$ and $f*M^{d,\ell}$ for $M\in\mathcal{M}_{\mathrm{loc}}$ with $f\in\dot{\mathcal{F}}_{\mathrm{loc}}\cap L^2(E;\mu_{\langle M^d\rangle})$ and $f(\partial)=0$ belong to $\overset{\circ}{\mathcal{M}}$, and satisfy that

$$(f*M^{d,\ell})_t=\sum_{0<s\le t}f(X_{s-})\varphi_\ell(X_{s-},X_s)-\int_0^t f(X_s)N(\varphi_\ell)(X_s)\,dH_s$$

holds for all $t\in[0,\infty[$ $\mathbb{P}_x$-a.s. for q.e. $x\in E$ and

$$\begin{aligned}&\mathbf{e}(f*M^d-f*M^{d,\ell})\\&\quad=\frac{1}{2}\int_E f^2(x)\int_{E_\partial}\varphi^2(x,y)\mathbf{1}_{\{|\varphi(x,y)|\le 1/\ell\}}N(x,dy)\mu_H(dx).\end{aligned}$$

LEMMA 4.1. (1) *Take $M\in\mathcal{M}_{\mathrm{loc}}$ with its jump function $\varphi\in\mathcal{J}$. Then for $g\in\dot{\mathcal{F}}_{\mathrm{loc}}\cap L^2(E;\mu_{\langle M^d\rangle})$ with $g(\partial)=0$,*

$$\begin{aligned}\Gamma(g*M^{d,\ell})_t&=\frac{1}{2}\int_0^t N\big(\mathbf{1}_{E\times E}(g\varphi_\ell-\overline{g\varphi_\ell})\big)(X_s)\,dH_s\\&\quad+\int_0^t g(X_s)\varphi_\ell(X_s,\partial)N(X_s,\{\partial\})\,dH_s\end{aligned}\tag{4.1}$$

holds for all $t \in [0, \infty[$ $\mathbb{P}_x$*-a.s. for q.e.* $x \in E$. *Moreover, for* $f \in \mathcal{F}_e \cap L^2(E; \mu_{\langle g*M^{d,\ell}\rangle})$,

$$\begin{aligned}\int_0^t f(X_s)\,d\Gamma(g*M^{d,\ell})_t &= \frac{1}{2}\int_0^t f(X_s)N\big(\mathbf{1}_{E\times E}(g\varphi_\ell - \overline{g\varphi_\ell})\big)(X_s)\,dH_s \\ &\quad + \int_0^t f(X_s)g(X_s)\varphi_\ell(X_s,\partial)N(X_s,\{\partial\})\,dH_s\end{aligned} \tag{4.2}$$

holds for all $t \in [0, \infty[$ $\mathbb{P}_x$*-a.s. for q.e.* $x \in E$. *More generally if* $M \in \mathcal{M}_{\mathrm{loc}}^{[\![0,\zeta[\![}$ *with its jump function* $\varphi \in \widehat{\mathcal{J}}$, *then for* $M^{j,\ell} \in \mathcal{M}_{\mathrm{loc}}^{j}$ *with its jump function* $\mathbf{1}_{E\times E}\varphi_\ell \in \mathcal{J}$ *and* $g \in \dot{\mathcal{F}}_{\mathrm{loc}} \cap L^2(E; \mu_{\langle M^{j,\ell}\rangle})$ *with* $g(\partial) = 0$,

$$\Gamma(g*M^{j,\ell})_t = \frac{1}{2}\int_0^t N\big(\mathbf{1}_{E\times E}(g\varphi_\ell - \overline{g\varphi_\ell})\big)(X_s)\,dH_s \tag{4.3}$$

holds for all $t \in [0, \infty[$ $\mathbb{P}_x$*-a.s. for q.e.* $x \in E$, *and for* $f \in \mathcal{F}_e \cap L^2(E; \mu_{\langle g*M^{j,\ell}\rangle})$,

$$\int_0^t f(X_s)\,d\Gamma(g*M^{j,\ell})_t = \frac{1}{2}\int_0^t f(X_s)N\big(\mathbf{1}_{E\times E}(g\varphi_\ell - \overline{g\varphi_\ell})\big)(X_s)\,dH_s \tag{4.4}$$

holds for all $t \in [0, \infty[$ $\mathbb{P}_x$*-a.s. for q.e.* $x \in E$.

(2) *Take* $M \in (\mathcal{M}_{\mathrm{loc}}^{[\![0,\zeta[\![})_*$ *and* $K^\ell \in \mathcal{M}_{\mathrm{loc}}^{j}$ *associated with* $-\mathbf{1}_{E\times E}(\varphi_\ell + \overline{\varphi_\ell}) \in \mathcal{J}_*$. *Then for* $g \in \dot{\mathcal{F}}_{\mathrm{loc}} \cap L^2(E; \mu_{\langle K^\ell\rangle})$ *with* $g(\partial) = 0$, *we have*

$$\Gamma(g*K^\ell)_t = \frac{1}{2}\int_0^t N\big(\mathbf{1}_{E\times E}(\bar{g} - g)(\varphi_\ell + \overline{\varphi_\ell})\big)(X_s)\,dH_s \tag{4.5}$$

holds for all $t \in [0, \infty[$ $\mathbb{P}_x$*-a.s. for q.e.* $x \in E$. *Moreover, for* $f \in \mathcal{F}_e \cap L^2(E; \mu_{\langle g*K^\ell\rangle})$,

$$\begin{aligned}\int_0^t f(X_s)\,d\Gamma(g*K^\ell)_t &= \frac{1}{2}\int_0^t f(X_s)N\big(\mathbf{1}_{E\times E}(\bar{g} - g)(\varphi_\ell + \overline{\varphi_\ell})\big)(X_s)\,dH_s\end{aligned} \tag{4.6}$$

holds for all $t \in [0, \infty[$ $\mathbb{P}_x$*-a.s. for q.e.* $x \in E$.

COROLLARY 4.1. *Take* $M \in (\mathcal{M}_{\mathrm{loc}}^{[\![0,\zeta[\![})_*$ *and* $f \in \dot{\mathcal{F}}_{\mathrm{loc}}$. *Let* K *be an element in* $(\mathcal{M}_{\mathrm{loc}}^{d})_*$ *associated with* $-\mathbf{1}_{E\times E}(\varphi + \overline{\varphi})$ *constructed in Corollary* 2.3. *Then we have that*

$$\int_0^t f(X_s)\,d\Gamma(K)_s = 0$$

holds for all $t \in [0, \infty[$ $\mathbb{P}_x$*-a.s. for q.e.* $x \in E$. *In particular,* $\Gamma(K)_t = 0$ *for all* $t \in [0, \infty[$ $\mathbb{P}_x$*-a.s. for q.e.* $x \in E$.

COROLLARY 4.2. *Take $M \in \mathring{\mathcal{M}}$ with its jump function $\varphi \in \mathring{\mathcal{J}}$. Let K^ℓ be an element in $\mathring{\mathcal{M}}^j$ associated with $-\mathbf{1}_{E\times E}(\varphi_\ell + \overline{\varphi_\ell})$. Set $\overline{A}^{d,\ell} := A^{d,\ell} + \frac{1}{2}K^\ell$. Then we have*

$$\overline{A}_t^{d,\ell} = \frac{1}{2}\sum_{0<s\le t}(\varphi_\ell - \overline{\varphi_\ell})(X_{s-}, X_s)\mathbf{1}_{\{s<\zeta\}} + \varphi_\ell(X_{\zeta-}, \partial)\mathbf{1}_{\{t\ge\zeta\}}$$

holds for all $t \in [0,\infty[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$. Moreover, for $f \in \mathcal{F}_e \cap L^2(E; \mu_{\langle M^d\rangle})$,

$$\int_0^t f(X_{s-})\,d\overline{A}_s^{d,\ell} = \sum_{0<s\le t} f(X_{s-})\frac{\varphi_\ell - \overline{\varphi_\ell}}{2}(X_{s-}, X_s)\mathbf{1}_{\{s<\zeta\}} + f(X_{\zeta-})\varphi_\ell(X_{\zeta-}, \partial)\mathbf{1}_{\{t\ge\zeta\}}$$

and

$$\int_0^t f(X_{s-})\circ d\overline{A}_s^{d,\ell} = \sum_{0<s\le t}\frac{f(X_s)+f(X_{s-})}{2}\frac{\varphi_\ell - \overline{\varphi_\ell}}{2}(X_{s-}, X_s)\mathbf{1}_{\{s<\zeta\}} + f(X_{\zeta-})\varphi_\ell(X_{\zeta-}, \partial)\mathbf{1}_{\{t\ge\zeta\}}$$

hold for all $t \in [0,\infty[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$.

PROOF OF LEMMA 4.1. We only prove (4.1). The proofs of (4.2), (4.5) and (4.6) are similar. Equation (4.3) [resp., (4.4)] is clear from (4.1) [resp., (4.2)]. By (3.3), for $h \in \mathcal{F}_b$

$$\begin{aligned}
&\lim_{t\downarrow 0}\frac{1}{t}\mathbb{E}_{h\cdot m}[\Gamma(g * M^{d,\ell})_t] \\
&\quad= -\frac{1}{2}\int_E g\,d\mu_{\langle M^h + M^{h,\kappa}, M^{d,\ell}\rangle} \\
&\quad= -\frac{1}{2}\int_E g\,d\mu_{\langle M^{h,j}, M^{j,\ell}\rangle} - \int_E g\,d\mu_{\langle M^{h,\kappa}, M^{\kappa,\ell}\rangle} \\
&\quad= \int_{E\times E} h(x)(g\varphi_\ell - \overline{g\varphi_\ell})(x,y)J(dx\,dy) + \int_E h(x)g(x)\varphi_\ell(x,\partial)\kappa(dx) \\
&\quad= \frac{1}{2}\int_E hN\bigl(\mathbf{1}_{E\times E}(g\varphi_\ell - \overline{g\varphi_\ell})\bigr)\,d\mu_H + \int_E hgN\bigl(\mathbf{1}_{E\times\{\partial\}}\varphi_\ell\bigr)\,d\mu_H.
\end{aligned}$$

Therefore, by Theorem 2.2 in [22], we have the desired assertion. □

PROOF OF COROLLARY 4.1. Let $\{G_k\} \in \Theta$ be a common nest such that $f|_{G_k} \in \mathcal{F}_b|_{G_k}$ and $\mathbf{1}_{G_k\times E}\varphi$, $\mathbf{1}_{G_k\times E}\overline{\varphi} \in \mathring{\mathcal{J}}$ for each $k \in \mathbb{N}$. Let $\{E_k\} \in \Theta$ be the nest and let e_k be the function constructed through $\{G_k\}$ as in Definition 3.1. Replacing f with fe_k, it suffices to prove the assertion for the case $f \in (\mathcal{F}_{G_k})_b$ in view of Definition 3.1. For $f \in (\mathcal{F}_{G_k})_b$, we have that

$$\int_0^t f(X_s)\,d\Gamma(K)_s = \Gamma(f * K)_t + \frac{1}{2}\langle M^{f,j}, K\rangle_t$$

holds for $t \in [0,\infty[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$. From Lemma 4.1, we have $\Gamma(f * K^\ell)_t + \frac{1}{2}\langle M^{f,j}, K^\ell\rangle_t = 0$ holds for $t \in [0,\infty[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$ in view of Theorem 2.2 in [22]. On the other hand, we see $f * K, f * K^\ell, \mathbf{1}_{G_j} * K, \mathbf{1}_{G_j} * K^\ell \in \mathring{\mathcal{M}}$ for $j > k$ with

$$\mathbf{e}\big(f * (K - K^\ell)\big) \le \|f\|_\infty^2 \mathbf{e}\big(\mathbf{1}_{G_j} * (K - K^\ell)\big) \to 0, \qquad \ell \to \infty.$$

Hence we obtain the assertion in view of Theorem 5.2.1 in [11] and Theorem 3.2 in [22]. $\square$

PROOF OF COROLLARY 4.2. Since $K^\ell \in \mathring{\mathcal{M}}$, we have from Corollary 4.1 that $\Gamma(K^\ell)_t = 0$ holds for $t \in [0,\infty[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$. Note that $M^{d,\ell} + \frac{1}{2}K^\ell$ is given by

$$\sum_{0<s\le t} \frac{\varphi_\ell - \overline{\varphi_\ell}}{2}(X_{s-}, X_s)\mathbf{1}_{\{s<\zeta\}} + \varphi_\ell(X_{\zeta-}, \partial)\mathbf{1}_{\{t\ge\zeta\}}$$
$$- \int_0^t N\left(\mathbf{1}_{E\times E}\frac{\varphi_\ell - \overline{\varphi_\ell}}{2}\right)(X_s)\,dH_s - \int_0^t \varphi_\ell(X_s, \partial) N(X_s, \{\partial\})\,dH_s.$$

Then we obtain the assertion in view of Lemma 4.1. $\square$

DEFINITION 4.3 (Extensions of stochastic integrals by Dirichlet processes). For $M \in (\mathcal{M}_{\mathrm{loc}})_*$ with its jump function $\varphi \in \mathcal{J}_*$ and $f \in L^2(G_n; \mu_{\langle M^c\rangle})$ for each $n \in \mathbb{N}$ and some $\{G_n\} \in \Theta$, we define

$$\int_0^t f(X_s) \circ dA_s^c := (f * M^c)_t + \Gamma(f * M^c)_t$$

for all $t \in [0,\infty[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$. Moreover, if $f \in \dot{\mathcal{F}}_{\mathrm{loc}}$ with $f(\partial) = 0$, we define

$$\int_0^t f(X_{s-}) \circ dM_s := \int_0^t f(X_{s-})\,dM_s + \frac{1}{2}[f(X), M]_t,$$
$$\int_0^t f(X_{s-})\,dA_s := \int_0^t f(X_{s-})\,dM_s + \int_0^t f(X_s)\,d\Gamma(M)_s,$$
$$\int_0^t f(X_s) \circ dA_s := \int_0^t f(X_{s-}) \circ dM_s + \int_0^t f(X_s)\,d\Gamma(M)_s$$

for all $t \in [0,\infty[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$, where $[f(X), M]_t := \langle M^{f,c}, M^c\rangle_t + \sum_{0<s\le t}(f(X_s) - f(X_{s-}))(M_s - M_{s-})$. For $M \in (\mathcal{M}_{\mathrm{loc}}^{[\![0,\zeta[\![})_*$ with its jump function

$\varphi \in \widehat{\mathcal{J}}_*$ and f as above, these are defined for all $t \in [0, \zeta[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$. We can define $\int_0^t f(X_{s-})\, d\overline{A}_s$, $\int_0^t f(X_s) \circ d\overline{A}_s$ for $\overline{A} := A + \frac{1}{2}K$ by replacing M with $M + \frac{1}{2}K$. Note that $\Gamma(K) \equiv 0$.

Hereafter we use the following convention: let $f \in \dot{\mathcal{F}}_{\mathrm{loc}}$ with $f(\partial) = 0$ and take $\phi, \psi : E_\partial \times E_\partial \to \mathbb{R}$ vanishing on the diagonal such that $|\phi| \le M|\psi|$ on $E \times E_\partial$ for some $M > 0$ and $\sum_{0<s\le t} \psi^2(X_{s-}, X_s) < \infty$ for all $t \in [0, \infty[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$. If there exists a nest $\{G_n\} \in \Theta$ such that $f|_{G_n} \in \mathcal{F}_b|_{G_n}$ for each $n \in \mathbb{N}$ and a subsequence $\{\ell_k\}$ depending only on $\{G_n\}$, f, ϕ and ψ such that $t \mapsto \sum_{0<s\le t} f(X_{s-})\phi(X_{s-}, X_s)\mathbf{1}_{\{|\psi(X_{s-},X_s)|>1/\ell_k\}}$ converges uniformly on each compact subinterval of $[0, \sigma_{E\setminus G_n}[$ for each $n \in \mathbb{N}$ as $k \to \infty$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$, then we shall denote its limit by

$$\sum_{0<s\le t}^{*} f(X_{s-})\phi(X_{s-}, X_s).$$

Note that if $t \mapsto \sum_{s\le t} f(X_{s-})\phi(X_{s-}, X_s)$ absolutely converges uniformly on each compact subinterval of $[0, \infty[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$, then we can eliminate the symbol $*$ from the above sum. We shall use $\sum_{s\le t}^{*}(f(X_s) + f(X_{s-}))\phi(X_{s-}, X_s)$ and $\sum_{s\le t}^{*} \phi(X_{s-}, X_s)$ in a similar fashion.

We then have the following:

THEOREM 4.1. *Let $f \in \dot{\mathcal{F}}_{\mathrm{loc}}$ and suppose that f is extended to be a real-valued function on E_∂ with $f(\partial) = 0$. Take $M \in (\mathcal{M}_{\mathrm{loc}})_*$ with its jump function $\varphi \in \mathcal{J}_*$ and set $\overline{A} := A + \frac{1}{2}K = M + \Gamma(M) + \frac{1}{2}K$, where $K \in (\mathcal{M}_{\mathrm{loc}})_*$ associated with $-\mathbf{1}_{E\times E}(\varphi + \overline{\varphi}) \in \mathcal{J}_*$. Then*

$$\overline{A}_t = A_t^c + \sum_{0<s\le t}^{*} \mathbf{1}_{E\times E} \frac{\varphi - \overline{\varphi}}{2}(X_{s-}, X_s) + \varphi(X_{\zeta-}, \partial)\mathbf{1}_{\{t\ge\zeta\}},$$

$$\begin{aligned}\int_0^t f(X_{s-})\, d\overline{A}_s = \int_0^t f(X_{s-})\, dA_s^c \\ &+ \sum_{0<s\le t}^{*} f(X_{s-})\mathbf{1}_{E\times E} \frac{\varphi - \overline{\varphi}}{2}(X_{s-}, X_s) \\ &+ f(X_{\zeta-})\varphi(X_{\zeta-}, \partial)\mathbf{1}_{\{t\ge\zeta\}}\end{aligned}$$

and

$$\begin{aligned}\int_0^t f(X_s) \circ d\overline{A}_s &= \int_0^t f(X_s) \circ dA_s^c \\ &+ \sum_{0<s\le t}^{*} \frac{f(X_s) + f(X_{s-})}{2} \mathbf{1}_{E\times E} \frac{\varphi - \overline{\varphi}}{2}(X_{s-}, X_s) \\ &+ f(X_{\zeta-})\varphi(X_{\zeta-}, \partial)\mathbf{1}_{\{t\ge\zeta\}}\end{aligned}$$

hold for all $t \in [0,\infty[$ $\mathbb{P}_x$*-a.s. for q.e.* $x \in E$. *More generally, for* $M \in (\mathcal{M}_{\mathrm{loc}}^{[\![0,\zeta[\![})_*$ *with its jump function* $\varphi \in \widehat{\mathcal{J}}_*$, *these expressions hold for* $t \in [0,\zeta[$ $\mathbb{P}_x$*-a.s. for q.e.* $x \in E$.

PROOF. First we assume $M \in \overset{\circ}{\mathcal{M}}$ and $f \in \mathcal{F}_e \cap L^2(E;\mu_{\langle M\rangle})$. Since $f * M^{d,\ell}$ converges $f * M^d$ in $(\overset{\circ}{\mathcal{M}},\mathbf{e})$ as $\ell \to \infty$, there exists a common subsequence $\{\ell_k\}$ such that $f * M^{d,\ell_k}$ [resp., $\Gamma(f * M^{d,\ell_k})$] uniformly converges to $f * M^d$ [resp., $\Gamma(f * M^d)$] on each compact subinterval of $[0,\infty[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$ by Theorem 5.2.1 in [11] and Theorem 3.2 in [22]. On the other hand, $M^{d,\ell}$ also converges M^d in $(\overset{\circ}{\mathcal{M}},\mathbf{e})$, which yields that there exists a subsequence $\{\ell_k\}$ such that $[M^{f,d}, M^{d,\ell_k}]$ (resp., $\langle M^{f,d}, M^{d,\ell_k}\rangle$) uniformly converges to $[M^{f,d}, M^d]$ (resp., $\langle M^{f,d}, M^d\rangle$) on each compact subinterval of $[0,\infty[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$. Therefore, for such subsequence, $\int_0^t f(X_{s-})\,d\overline{A}_s^{d,\ell_k}$ [resp., $\int_0^t f(X_{s-})\,d\overline{A}_s^{d,\ell_k}$] uniformly converges to $\int_0^t f(X_s)\circ d\overline{A}_s^d$ [resp., $\int_0^t f(X_s)\circ d\overline{A}_s^d$] on each compact subinterval of $[0,\infty[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$. So the conclusion holds by Corollary 4.2. For general $M \in (\mathcal{M}_{\mathrm{loc}})_*$ [resp., $M \in (\mathcal{M}_{\mathrm{loc}}^{[\![0,\zeta[\![})_*$] with its jump function $\varphi \in \mathcal{J}_*$ (resp., $\varphi \in \widehat{\mathcal{J}}_*$), the assertion is clear from Lemma 3.4. □

Recalling Theorem 2.1 and the last description of Section 2, for $u \in \dot{\mathcal{F}}_{\mathrm{loc}}^{\ddagger}$ (resp., $u \in \dot{\mathcal{F}}_{\mathrm{loc}}^{\dagger}$), there exists an $M^{u,d} \in \mathcal{M}_{\mathrm{loc}}^d (\subset \overset{\circ}{\mathcal{M}}{}_{f\text{-loc}}^d)$ [resp., $M^{u,d} \in \mathcal{M}_{\mathrm{loc}}^{d,[\![0,\zeta[\![} (\subset \overset{\circ}{\mathcal{M}}{}_{\mathrm{loc}}^d)$] such that $M_t^{u,d} - M_{t-}^{u,d} = u(X_t) - u(X_{t-})$ for all $t \in [0,\infty[$ (resp., $[0,\zeta[$) $\mathbb{P}_x$-a.s. for q.e. $x \in E$. By Lemma 2.4(1), we can define $M^{u,c} \in \mathcal{M}_{\mathrm{loc}}^c = \overset{\circ}{\mathcal{M}}{}_{f\text{-loc}}^c$ for $u \in \dot{\mathcal{F}}_{\mathrm{loc}}$; $M_t^{u,c} := M_t^{u_n,c}$ for $t < \sigma_{E\setminus G_n}$ for some $\{G_n\} \in \Theta$ and $u_n \in \mathcal{F}$ such that $u = u_n$ m-a.e. on G_n for each $n \in \mathbb{N}$ (see Remark 2.3). Put $A^{u,c} := M^{u,c} + \Gamma(M^{u,c})$ for $M^{u,c}$, which can be defined by way of Lemma 3.2, and $A^{u,d} := M^{u,d} + \Gamma(M^{u,d})$ for $M^{u,d}$, which is defined by Definition 3.1.

COROLLARY 4.3. *Take* $f \in \dot{\mathcal{F}}_{\mathrm{loc}}$ *and* $u \in \dot{\mathcal{F}}_{\mathrm{loc}}^{\ddagger}$. *Suppose that* f *is extended to be a real-valued function on* E_∂ *with* $f(\partial) := 0$. *Then*

$$\int_0^t f(X_{s-})\,dA_s^{u,d} = \sum_{0<s\le t}^{*} f(X_{s-})\big(u(X_s) - u(X_{s-})\big), \tag{4.7}$$

$$\int_0^t f(X_s)\circ dA_s^{u,d} = \sum_{0<s\le t}^{*} \frac{f(X_s) + f(X_{s-})}{2}\big(u(X_s) - u(X_{s-})\big) \tag{4.8}$$

hold for all $t \in [0,\infty[$ $\mathbb{P}_x$*-a.s. for q.e.* $x \in E$. *Similarly*

$$A_t^{u,d} = \sum_{0<s\le t}^{*}\big(u(X_s) - u(X_{s-})\big) \tag{4.9}$$

hold for all $t \in [0, \infty[$ $\mathbb{P}_x$*-a.s. for q.e.* $x \in E$. *More generally, if* $u \in \dot{\mathcal{F}}^{\dagger}_{\mathrm{loc}}$ *and* $f \in \dot{\mathcal{F}}_{\mathrm{loc}}$ *is only defined on* E, *then all assertions above hold for all* $t \in [0, \zeta[$ $\mathbb{P}_x$*-a.s. for q.e.* $x \in E$.

Owing to (4.9), we can obtain a generalized Fukushima decomposition for $u \in \dot{\mathcal{F}}^{\dagger}_{\mathrm{loc}}$:

THEOREM 4.2 (Generalized Fukushima decomposition). *For* $u \in \dot{\mathcal{F}}^{\ddagger}_{\mathrm{loc}}$, *the additive functional* A^u *defined by* $A^u_t := u(X_t) - u(X_0)$ *can be decomposed as*

$$A^u = M^u + N^u, \qquad M^u \in \mathcal{M}_{\mathrm{loc}}, \qquad N^u \in \mathcal{N}_{c,f\text{-loc}}$$

in the sense that $A^u_t = M^u_t + N^u_t$, $t \in [0, \infty[$ $\mathbb{P}_x$*-a.s. for q.e.* $x \in E$. *More generally, if* $u \in \dot{\mathcal{F}}^{\dagger}_{\mathrm{loc}}$, *then* A^u *is decomposed as*

$$A^u = M^u + N^u, \qquad M^u \in \mathcal{M}^{[\![0,\zeta[\![}_{\mathrm{loc}}, \qquad N^u \in \mathcal{N}_{c,\mathrm{loc}}$$

in the sense that $A^u_t = M^u_t + N^u_t$, $t \in [0, \zeta[$ $\mathbb{P}_x$*-a.s. for q.e.* $x \in E$. *Such decompositions are unique up to the equivalence of* (*local*) *additive functionals.*

PROOF. The uniqueness is proved in Corollary 2.4. We shall only prove the existence in the first assertion. We set $M^u := M^{u,c} + M^{u,d} \in (\mathcal{M}_{\mathrm{loc}})_{\mathrm{as}}$ and $N^u := \Gamma(M^u) \in \mathcal{N}_{c,f\text{-loc}}$, where $M^{u,c}$ and $M^{u,d}$ are defined above. Take $\{G_n\} \in \Theta$ and $\{u_n\} \subset \mathcal{F}_b$ such that $u - u(\partial) = u_n$ m-a.e. on G_n. Then (4.9) implies that for $t \in [0, \sigma_{E \setminus G_n}[$

$$\begin{aligned}
u(X_t) - u(X_0) &= u_n(X_t) - u_n(X_0) = A^{u_n,c}_t + A^{u_n,d}_t \\
&= A^{u_n,c}_t + \sum_{0<s\le t}{}^{*}\big(u_n(X_s) - u_n(X_{s-})\big) \\
&= A^{u,c}_t + \sum_{0<s\le t}{}^{*}\big(u(X_s) - u(X_{s-})\big) \overset{(4.9)}{=} A^{u,c}_t + A^{u,d}_t \\
&= M^u_t + \Gamma(M^u)_t
\end{aligned}$$

$\mathbb{P}_x$-a.s. for q.e. $x \in E$. $\square$

REMARK 4.2. (1) We emphasize that $\mathbf{1}_{E_\partial}$ does not satisfy $\mathbf{1}_{E_\partial}(\partial) = 0$. So we cannot deduce (4.9) from (4.7), (4.8).

(2) For $f \in \dot{\mathcal{F}}_{\mathrm{loc}}$ with $f(\partial) = 0$ and $u \notin \dot{\mathcal{F}}^{\dagger}_{\mathrm{loc}}$, we have no way to define $M^{u,d}$, $\Gamma(M^{u,d})$ and stochastic integrals with respect to them. However, we can define the left-hand sides of (4.7) and (4.8) keeping the same expressions as they have.

(3) In Theorem 4.2, M^u for $u \in \dot{\mathcal{F}}^{\ddagger}_{\mathrm{loc}}$ (resp., $u \in \dot{\mathcal{F}}^{\dagger}_{\mathrm{loc}}$) can be decomposed to $M^u_t = M^{u,c}_t + M^{u,j}_t + M^{u,\kappa}_t$ $t \in [0, \infty[$ (resp., $M^u_t = M^{u,c}_t + M^{u,d}_t$ $t \in [0, \zeta[$) $\mathbb{P}_x$-

a.s. for q.e. $x \in E$, where $M^{u,i} \in \mathcal{M}^i_{\mathrm{loc}}$, $i = c, j, \kappa$ (resp., $M^{u,c} \in \mathcal{M}^{c,[\![0,\zeta[\![}_{\mathrm{loc}}$, $M^{u,d} \in \mathcal{M}^{d,[\![0,\zeta[\![}_{\mathrm{loc}}$).

Now we expose a generalized Itô formula in terms of our stochastic integrals.

THEOREM 4.3 (Generalized Itô formula). *Suppose that $\Phi \in C^1(\mathbb{R}^N)$ and take $u = (u_1, \ldots, u_N) \in (\dot{\mathcal{F}}_{\mathrm{loc}})^N$ having an $\mathbb{R}^N$-valued extension on E_∂. Then*:

(1) *$\Phi(u) \in \dot{\mathcal{F}}_{\mathrm{loc}}$ and for each $k = 1, 2, \ldots, N$, $\Phi_k(u) \in L^2_{\mathrm{loc}}(\{G_n\}; \mu_{\langle M^{u_k,c}\rangle})$ for some $\{G_n\} \in \Theta$, where $\Phi_k := \frac{\partial \Phi}{\partial x_k}$ and*

$$A^{\Phi(u),c}_t = \sum_{k=1}^N \int_0^t \Phi_k(u(X_s)) \circ dA^{u_k,c}_s \tag{4.10}$$

holds for all $t \in [0, \infty[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$. If we further assume $\Phi \in C^2(\mathbb{R}^N)$, then for each $k, \ell = 1, 2, \ldots, N$, $\Phi_k(u) \in \dot{\mathcal{F}}_{\mathrm{loc}}$, $\Phi_{k\ell}(u) \in L^2_{\mathrm{loc}}(\{G_n\}; \mu_{\langle M^{u_k,c}\rangle})$ for some $\{G_n\} \in \Theta$, where $\Phi_{k\ell} := \frac{\partial^2 \Phi}{\partial x_k \partial x_\ell}$, and

$$\begin{aligned} A^{\Phi(u),c}_t &= \sum_{k=1}^N \int_0^t \Phi_k(u(X_{s-}))\, dA^{u_k,c}_s \\ &\quad + \frac{1}{2} \sum_{k,\ell=1}^N \int_0^t \Phi_{k\ell}(u(X_s))\, d\langle M^{u_k,c}, M^{u_\ell,c}\rangle_s \end{aligned} \tag{4.11}$$

holds for all $t \in [0, \infty[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$.

(2) *Suppose $u \in (\dot{\mathcal{F}}^{\ddagger}_{\mathrm{loc}})^N$ and $\Phi \in C^2(\mathbb{R}^N)$. Then $\Phi_k(u) \in \dot{\mathcal{F}}_{\mathrm{loc}}$ for each $k = 1, 2, \ldots, N$. Moreover, if we assume $\Phi(u) \in \dot{\mathcal{F}}^{\ddagger}_{\mathrm{loc}}$, then*

$$\begin{aligned} A^{\Phi(u),d}_t &= \sum_{k=1}^N \int_0^t \Phi_k(u(X_{s-}))\, dA^{u_k,d}_s \\ &\quad + \sum_{s \le t} \left(\Delta\Phi(u(X_s)) - \sum_{k=1}^N \Phi_k(u(X_{s-}))\Delta u_k(X_s) \right) \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} A^{\Phi(u),d}_t &= \sum_{k=1}^N \int_0^t \Phi_k(u(X_s)) \circ dA^{u_k,d}_s \\ &\quad + \sum_{s \le t} \left(\Delta\Phi(u(X_s)) - \sum_{k=1}^N \frac{\Phi_k(u(X_s)) + \Phi_k(u(X_{s-}))}{2} \Delta u_k(X_s) \right) \end{aligned} \tag{4.13}$$

hold for all $t \in [0, \infty[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$, where the last terms in the right-hand sides are absolutely convergent uniformly on each compact interval of $[0, \infty[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$. If we replace $\dot{\mathcal{F}}^{\ddagger}_{\mathrm{loc}}$ with $\dot{\mathcal{F}}^{\dagger}_{\mathrm{loc}}$ in the above condi-

tions, then formulas (4.12) *and* (4.13) *hold only on* $[0,\zeta[$ $\mathbb{P}_x$*-a.s. for q.e.* $x \in E$ *without assuming the* $\mathbb{R}^N$*-valued extension of* u *on* E_∂.

(3) *Under the same conditions in* (2), *we have* $\Phi_k(u) \in \dot{\mathcal{F}}_{\mathrm{loc}}$ *for each* $k = 1, 2, \dots, N$,

$$
\begin{aligned}
A_t^{\Phi(u)} &= \sum_{k=1}^N \int_0^t \Phi_k(u(X_{s-}))\,dA_s^{u_k} \\
&\quad + \frac{1}{2}\sum_{k,\ell=1}^N \int_0^t \Phi_{k\ell}(u(X_s))\,d\langle M^{u_k,c}, M^{u_\ell,c}\rangle_s \\
&\quad + \sum_{s\le t}\left(\Delta\Phi(u(X_s)) - \sum_{k=1}^N \Phi_k(u(X_{s-}))\Delta u_k(X_s)\right)
\end{aligned}
\tag{4.14}
$$

and

$$
\begin{aligned}
A_t^{\Phi(u)} &= \sum_{k=1}^N \int_0^t \Phi_k(u(X_s))\circ dA_s^{u_k} \\
&\quad + \sum_{s\le t}\left(\Delta\Phi(u(X_s)) - \sum_{k=1}^N \frac{\Phi_k(u(X_s)) + \Phi_k(u(X_{s-}))}{2}\Delta u_k(X_s)\right)
\end{aligned}
\tag{4.15}
$$

hold for all $t \in [0,\infty[$ $\mathbb{P}_x$*-a.s. for q.e.* $x \in E$, *where the last terms in the right-hand sides are absolutely convergent uniformly on each compact interval of* $[0,\infty[$ $\mathbb{P}_x$*-a.s. for q.e.* $x \in E$. *If we replace* $\dot{\mathcal{F}}^{\ddagger}_{\mathrm{loc}}$ *with* $\dot{\mathcal{F}}^{\dagger}_{\mathrm{loc}}$ *in the above conditions, then formulas* (4.14) *and* (4.15) *hold only on* $[0,\zeta[$ $\mathbb{P}_x$*-a.s. for q.e.* $x \in E$ *without assuming the* $\mathbb{R}^N$*-valued extension of* u *on* E_∂.

We call (4.14) the *Itô formula for Itô-type integrals* and (4.15) the *Itô formula for Fisk–Stratonovich-type integrals*.

COROLLARY 4.4 (Chain and Leibniz rules for purely discontinuous part). *Under the same conditions as in Theorem* 4.3(2), *we have that*

$$
\begin{aligned}
M_t^{\Phi(u),d} &= \sum_{k=1}^N \int_0^t \Phi_k(u(X_{s-}))\,dM_s^{u_k,d} \\
&\quad + \sum_{s\le t}\left(\Delta\Phi(u(X_s)) - \sum_{k=1}^N \Phi_k(u(X_{s-}))\Delta u_k(X_s)\right) \\
&\quad - \left\{\sum_{s\le t}\left(\Delta\Phi(u(X_s)) - \sum_{k=1}^N \Phi_k(u(X_{s-}))\Delta u_k(X_s)\right)\right\}^p,
\end{aligned}
\tag{4.16}
$$

$$
\begin{aligned}
M_t^{\Phi(u),d} &= \sum_{k=1}^{N} \int_0^t \Phi_k(u(X_s)) \circ dM_s^{u_k,d} \\
&\quad + \sum_{s\le t}\biggl(\Delta\Phi(u(X_s)) \\
&\qquad - \sum_{k=1}^{N} \frac{\Phi_k(u(X_s)) + \Phi_k(u(X_{s-}))}{2}\Delta u_k(X_s)\biggr) \\
&\quad - \biggl\{\sum_{s\le t}\biggl(\Delta\Phi(u(X_s)) - \sum_{k=1}^{N}\Phi_k(u(X_{s-}))\Delta u_k(X_s)\biggr)\biggr\}^p
\end{aligned}
\tag{4.17}
$$

and

$$
\begin{aligned}
\Gamma\bigl(M^{\Phi(u),d}\bigr)_t &= \sum_{k=1}^{N} \int_0^t \Phi_k(u(X_s))\, d\Gamma(M^{u_k,d})_s \\
&\quad + \biggl\{\sum_{s\le t}\biggl(\Delta\Phi(u(X_s)) - \sum_{k=1}^{N}\Phi_k(u(X_{s-}))\Delta u_k(X_s)\biggr)\biggr\}^p
\end{aligned}
\tag{4.18}
$$

hold for all $t \in [0,\infty[$ $\mathbb{P}_x$*-a.s. for q.e.* $x \in E$, *where* B_t^p *denotes the dual predictable projection of* B_t *for an AF* B. *If we replace* $\dot{\mathcal{F}}^{\ddagger}_{\mathrm{loc}}$ *with* $\dot{\mathcal{F}}^{\dagger}_{\mathrm{loc}}$ *in the conditions*, (4.16) *and* (4.17) *hold only on* $[0,\zeta[$ $\mathbb{P}_x$*-a.s. for q.e.* $x \in E$ *without assuming the* $\mathbb{R}^N$*-valued extension of* u *on* E_∂. *In particular, for* $u, v \in \dot{\mathcal{F}}^{\dagger}_{\mathrm{loc}}$ (*resp.*, $u, v \in \dot{\mathcal{F}}^{\ddagger}_{\mathrm{loc}}$) *with* $uv \in \dot{\mathcal{F}}^{\dagger}_{\mathrm{loc}}$ [*resp.*, $uv \in \dot{\mathcal{F}}^{\ddagger}_{\mathrm{loc}}$ *with* $u(\partial) = v(\partial) = 0$],

$$
\begin{aligned}
M_t^{uv,d} &= \int_0^t u(X_{s-})\, dM_s^{v,d} + \int_0^t v(X_{s-})\, dM_s^{u,d} \\
&\quad + [M^{u,d}, M^{v,d}]_t - \langle M^{u,d}, M^{v,d}\rangle_t
\end{aligned}
\tag{4.19}
$$

$$
\begin{aligned}
&= \int_0^t u(X_{s-}) \circ dM_s^{v,d} + \int_0^t v(X_{s-}) \circ dM_s^{u,d} \\
&\quad - \langle M^{u,d}, M^{v,d}\rangle_t,
\end{aligned}
\tag{4.20}
$$

$$
\begin{aligned}
\Gamma(M^{uv,d})_t &= \int_0^t u(X_s)\, d\Gamma(M^{v,d})_s + \int_0^t v(X_s)\, d\Gamma(M^{u,d})_s \\
&\quad + \langle M^{u,d}, M^{v,d}\rangle_t
\end{aligned}
\tag{4.21}
$$

hold for all $t \in [0,\zeta[$ (*resp.*, $t \in [0,\infty[$) $\mathbb{P}_x$*-a.s. for q.e.* $x \in E$.

COROLLARY 4.5 (Fisk–Stratonovich integration by parts formula). *For* $u, v \in \dot{\mathcal{F}}^{\dagger}_{\mathrm{loc}}$ (*resp.*, $u, v \in \dot{\mathcal{F}}^{\ddagger}_{\mathrm{loc}}$) *with* $uv \in \dot{\mathcal{F}}^{\dagger}_{\mathrm{loc}}$ [*resp.*, $uv \in \dot{\mathcal{F}}^{\ddagger}_{\mathrm{loc}}$ *with* $u(\partial) = v(\partial) = 0$],

$$
u(X_t)v(X_t) - u(X_0)v(X_0) = \int_0^t u(X_s) \circ dA_s^v + \int_0^t v(X_s) \circ dA_s^u
\tag{4.22}
$$

holds for all $t \in [0, \zeta[$ (*resp*., $t \in [0, \infty[$) $\mathbb{P}_x$*-a.s. for q.e.* $x \in E$.

REMARK 4.3. (1) In [3], we prove a generalized Itô formula for $u \in (\mathcal{F})^N$ under the law for quasi everywhere starting points, extending the early result by Nakao [22]. Our Itô formula can be applied to a wider class of integrators than that in [3].

(2) In Theorem 4.3(2), if $u \in ((\dot{\mathcal{F}}_{\rm loc})_b)^N$ with $u(\partial) \in \mathbb{R}^N$, $\Phi \in C^2(\mathbb{R}^N)$, then $\Phi(u) \in \dot{\mathcal{F}}^{\ddagger}_{\rm loc}$.

(3) $\Phi_k(u) \in \dot{\mathcal{F}}_{\rm loc}$ does not necessarily satisfy $\Phi_k(u)(\partial) = 0$. So the integrals appeared in the first terms of the right-hand sides of (4.11)–(4.14) should be understood to be modified, for example, $\int_0^t \Phi_k(u(X_s))\, dA_s^{u_k}$ should be understood as $\int_0^t (\Phi_k(u(X_s)) - \Phi_k(u(\partial)))\, dA_s^{u_k} + \Phi_k(u(\partial))A_t^{u_k}$.

(4) Comparing with (4.10), the case for diffusion part, our Itô formulas, (4.13) and (4.15), for Fisk–Stratonovich integrals are not so simple. This phenomenon can be found in the Itô formula for Fisk–Stratonovich integral exposed in II. Theorem 34 and V. Theorem 21 of [23] in the framework of semi-martingales. We emphasize that the expression of the second term (denoted by C_t) of the right-hand side in (4.15) is different from theirs [i.e., the third term of the right-hand side in (4.14), which is the usual expression of the Itô formula for purely discontinuous part]. Note that C_t is an *odd additive functional*, that is, for each $t > 0$, $C_t \circ r_t + C_t = 0$ $\mathbb{P}_m$-a.e. on $\{t < \zeta\}$, where r_t is the time reverse operator. Hence, both sides in our formula (4.15) possess this property, which is not yielded by the Itô formula in [23].

(5) In Theorems 4.1, 4.3 and Corollaries 4.3–4.5, we do not require the strict $\mathcal{E}$-quasi-continuities of f, u and v. If we do not impose the condition that such functions are extended on E_∂ and vanish on $\{\partial\}$, or if we only assume that $(\mathcal{E}, \mathcal{F})$ is not necessarily regular (i.e., quasi-regularity only holds), then all assertions are restricted to "for all $t \in [0, \zeta[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$" and each convergence of the right-hand side is uniform on compact subinterval of $[0, \zeta[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$.

(6) In [10], an Itô formula for a general multi-dimensional process with finite quadratic variation is presented, but the formula like (4.15) is not exposed in [10].

(7) As noted in Remark 4.2, even for $u \notin (\dot{\mathcal{F}}^{\dagger}_{\rm loc})^N$, we can define the first terms in the right-hand sides of (4.12), (4.13), (4.14) and (4.15). So the formulas hold in this setting without using stochastic integrals with respect to $M^{u,d}$, $\Gamma(M^{u,d})$. So the conclusion of Corollary 4.5 also holds for $u, v \in \dot{\mathcal{F}}_{\rm loc}$ with $u(\partial) = v(\partial) = 0$ in this context.

PROOF OF THEOREM 4.3. (2) is a consequence of (1) and (3). We first prove (1). The former assertion of (1) follows from Theorems 6.1 and 7.2 in [16]. Note that $M^{\Phi(u),c}, M^{u_k,c} \in \mathcal{M}^c_{\rm loc} = \mathring{\mathcal{M}}^c_{f\text{-loc}}$ and $\Gamma(M^{\Phi(u),c}), \Gamma(M^{u_k,c}) \in \mathcal{N}^c_{f\text{-loc}}$, which are defined on $[0,\infty[$ under $\mathbb{P}_x$ for q.e. $x \in E$. Formula (4.10) can be obtained from the chain rule for continuous part of MAF

$$M_t^{\Phi(u),c} = \sum_{k=1}^{N} \int_0^t \Phi_k(u(X_{s-}))\, dM_s^{u_k,c} \tag{4.23}$$

for all $t \in [0,\infty[\mathbb{P}_x$-a.s. for q.e. $x \in E$ (see Theorem 7.2 in [16]). The latter assertion of (1) also follows from Theorem 7.2 in [16] and (3.7).

Next we prove (3). Applying Theorem 6.1 in [16] to $\Phi_k \in C^1(\mathbb{R}^N)$ again, we have $\Phi_k(u) \in \dot{\mathcal{F}}_{\mathrm{loc}}$ for $u \in (\dot{\mathcal{F}}^{\dagger}_{\mathrm{loc}})^N$. Equation (4.14) is proved by Nakao [22] for the case $u \in (\mathcal{F}_b)^N$. (4.15) for $u \in (\mathcal{F}_b)^N$ also follows from (4.14) for $u \in (\mathcal{F}_b)^N$ and that for each $k = 1, 2, \dots, N$

$$\int_0^t \Phi_k(u(X_s)) \circ dM_s^{u_k,d} = \int_0^t \Phi_k(u(X_{s-}))\, dM_s^{u_k,d} + \frac{1}{2}\big[M^{\Phi_k(u),d}, M^{u_k,d}\big]_t$$

for all $t \in [0,\infty[$ $\mathbb{P}_x$-a.s. for q.e. $x \in E$. Equations (4.14) and (4.15) for general $u \in \dot{\mathcal{F}}^{\ddagger}_{\mathrm{loc}}$ (or $u \in \dot{\mathcal{F}}^{\dagger}_{\mathrm{loc}}$) hold for all $t \in [0,\infty[$ (resp., $t \in [0,\zeta[$) $\mathbb{P}_x$-a.s. for q.e. $x \in E$ in view of the both of local properties of stochastic integrals by $M^{u_k,d}$ and $\Gamma(M^{u_k,d})$ (see Lemma 3.4). □

EXAMPLE 4.1 (Symmetric Lévy process on $\mathbb{R}^N$). Let $\mathbf{M} = (\Omega, X_t, \zeta, \mathbb{P}_x)_{x\in\mathbb{R}^N}$ be the symmetric Lévy process. That is, $\mathbf{M}$ is a time homogeneous additive process determined by a family $\{\nu_t\}$ of probability measures on $\mathbb{R}^N$ satisfying (4.17), (4.18) and (4.19) in [11]. Let $(\mathcal{E},\mathcal{F})$ be the corresponding Dirichlet form on $L^2(\mathbb{R}^N)$. Then $(\mathcal{E},\mathcal{F})$ is given by

$$\begin{cases} \mathcal{F} = \left\{u \in L^2(\mathbb{R}^N) \;\Big|\; \displaystyle\int_{\mathbb{R}^N} |\hat{u}(\xi)|^2 \psi(\xi)\, d\xi < \infty\right\}, \\ \mathcal{E}(u,v) = \displaystyle\int_{\mathbb{R}^N} \hat{u}(\xi)\overline{\hat{v}}(\xi)\psi(\xi)\, d\xi, \qquad u,v \in \mathcal{F}, \end{cases}$$

where $\hat{u}(\xi) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^N} e^{i\langle \xi,x\rangle} u(x)\, dx$ and $\psi(x)$ is the function determined by $\mathbb{E}_0[e^{i\langle \xi, X_t\rangle}] = e^{-t\psi(\xi)}$. We assume that $\mathbf{M}$ is purely discontinuous; namely ψ has the following expression:

$$\psi(\xi) = \int_{\mathbb{R}^N} (1 - \cos\langle \xi,\eta\rangle)\nu(d\eta),$$

where ν is a symmetric measure on $\mathbb{R}^N \setminus \{0\}$ such that $\int_{\mathbb{R}^N\setminus\{0\}} (|\xi|^2 \wedge 1)\nu(d\xi) < \infty$, which is called the *Lévy measure* of $\mathbf{M}$. We see $C_0^{\mathrm{Lip}}(\mathbb{R}^N) \subset \mathcal{F}$, hence $C_{\mathrm{loc}}^{\mathrm{Lip}}(\mathbb{R}^N) \subset \mathcal{F}_{\mathrm{loc}}$, because, in view of Corollary 7.16 in [1],

$$1 + \psi(\xi) \le c(1 + |\xi|^2) \qquad \forall \xi \in \mathbb{R}^N$$

for some constant $c > 0$. Here $C_0^{\mathrm{Lip}}(\mathbb{R}^N)$ [resp., $C_{\mathrm{loc}}^{\mathrm{Lip}}(\mathbb{R}^N)$] is the family of Lipschitz continuous functions with compact support (resp., locally Lipschitz continuous functions) and $\mathcal{F}_{\mathrm{loc}}$ is the space of functions locally in $\mathcal{F}$ in the ordinary sense (see [11]). Further $(\mathcal{E},\mathcal{F})$ is a regular Dirichlet form having $C_0^\infty(\mathbb{R}^N)$ as its core (see [26]). Define $N(x,A):=\nu(A-x)$, $N(x,\{\partial\})=0$ for $A\in\mathcal{B}(\mathbb{R}^N)$, $x\in\mathbb{R}^N$ and $H_t=t$. By Theorem 19.2(i) in [24], we have

$$N(x,A)=\mathbb{E}_x\biggl[\sum_{0<s\le 1}\mathbf{1}_A(X_s-X_{s-})\biggr],\qquad A\in\mathcal{B}(\mathbb{R}^N),$$

and hence (N,H) becomes a Lévy system of $\mathbf{M}$ (see also Section 7 in [14]). By Corollary 4.3, we have that for any $u\in C_{\rm loc}^{\rm Lip}(\mathbb{R}^N)$

$$u(X_t)-u(X_0)=\sum_{0<s\le t}^{*}\bigl(u(X_s)-u(X_{s-})\bigr)$$

holds for all $t\in[0,\zeta[$ $\mathbb{P}_x$-a.s. for q.e. $x\in\mathbb{R}^N$. Further we assume $\nu(dy)=f(|y|)\,dy$, where f is a Borel function satisfying (2.8). Let $u\in C^{\rm Lip}(\mathbb{R}^N)$. Then

$$\sup_{x\in K}\int_{\mathbb{R}^N}\bigl(u(x+y)-u(x)\bigr)^2\nu(dy)<\infty\qquad\text{for any compact set } K,$$

hence $u\in\mathcal{F}_{\rm loc}^{\dagger}$. Therefore u admits the generalized Fukushima decomposition.

EXAMPLE 4.2 (Symmetric stable process on $\mathbb{R}^N$). We fix $\alpha\in\,]0,2[$. Let $\mathbf{M}=(\Omega,X_t,\mathbb{P}_x)_{x\in\mathbb{R}^N}$ be a Lévy process on $\mathbb{R}^N$ with

$$\mathbb{E}_0\bigl[e^{\sqrt{-1}\langle\xi,X_t\rangle}\bigr]=e^{-t|\xi|^\alpha}.$$

$\mathbf{M}$ is called the *symmetric α-stable process*. It is known that $\mathbf{M}$ is conservative. Let $(\mathcal{E},\mathcal{F})$ be the associated Dirichlet form on $L^2(\mathbb{R}^N)$ with $\mathbf{M}$, which is given by

$$\begin{cases}\mathcal{F}=\biggl\{u\in L^2(\mathbb{R}^N)\,\Big|\,\displaystyle\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{(u(x)-u(y))^2}{|x-y|^{N+\alpha}}\,dx\,dy<\infty\biggr\},\\ \mathcal{E}(u,v)=\dfrac{A(N,-\alpha)}{2}\displaystyle\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+\alpha}}\,dx\,dy,\\ \qquad u,v\in\mathcal{F},\end{cases}\tag{4.24}$$

where $A(N,\gamma):=\frac{|\gamma|\Gamma((N-\gamma)/2)}{2^{1+\gamma}\pi^{N/2}\Gamma(1+\gamma/2)}$, $\gamma<N$. The Lévy system (N,H) of $\mathbf{M}$ is given by $N(x,dy):=A(N,-\alpha)|x-y|^{-(N+\alpha)}\,dy$ and $H_t=t$. So $\mu_H(dx)=dx$. Hence $J(dx\,dy)=f(|x-y|)\,dx\,dy$ for $f(r):=A(N,-\alpha)r^{-N-\alpha}$, $r>0$. Note that f does not satisfy (2.8). Take $\beta\in[0,\alpha[$. Assume that $N\ge\alpha$, hence $\{0\}$ is polar, and take $u\in C^1(\mathbb{R}^N\setminus\{0\})\cap C^{0,\beta/2}(\mathbb{R}^N)$. Here $C^{0,\beta/2}(\mathbb{R}^N)$ is the family of $\beta/2$-Hölder continuous functions on $\mathbb{R}^N$. For example, for a function $F\in C^1([0,\infty[)$ with bounded derivative F', $u(x):=F(|x|^{\beta/2})$ is a function in $C^1(\mathbb{R}^N\setminus\{0\})\cap C^{0,\beta/2}(\mathbb{R}^N)$. Then $u\in\dot{\mathcal{F}}_{\rm loc}^{\dagger}=\dot{\mathcal{F}}_{\rm loc}^{\ddagger}$. Indeed, the polarity of $\{0\}$ implies $C^1(\mathbb{R}^N\setminus\{0\})\subset\dot{\mathcal{F}}_{\rm loc}$ and we have that for any compact set

$K(\subset \mathbb{R}^N \setminus \{0\})$ with its relatively compact neighborhood $G(\subset \mathbb{R}^N \setminus \{0\})$ satisfying $d(K, G^c) := \inf_{x\in K, y\in G^c}|x-y| > 0$

$$\iint_{K\times G^c} \frac{(u(x)-u(y))^2}{|x-y|^{N+\alpha}}\,dx\,dy \le \frac{|K|\|u\|^2_{C^{0,\beta/2}}\sigma(\mathbb{S}^{N-1})}{(\alpha-\beta)d(K,G^c)^{\alpha-\beta}} < \infty,$$

equivalently,

$$\iint_{K\times\mathbb{R}^N} \frac{(u(x)-u(y))^2}{|x-y|^{N+\alpha}}\,dx\,dy < \infty,$$

where $|K|$ is the volume of K and $\|u\|_{C^{0,\beta/2}} := \sup_{x\neq y}\frac{|u(x)-u(y)|}{|x-y|^{\beta/2}}$. Therefore u admits the generalized Fukushima decomposition.

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