The Annals of Probability
1974, Vol. 2, No. 2, 309-316

OPTIMAL STOPPING VARIABLES FOR STOCHASTIC PROCESSES WITH INDEPENDENT INCREMENTS

BY LEROY H. WALKER

Brigham Young University

Let $\{W(t): t$ a nonnegative real number$\}$ denote a stochastic process with right-continuous sample paths with probability one, independent increments which are statistically homogeneous, $E\{W(t)\} = 0$, and $E\{(W(t) - W(s))^2\} = \sigma|t - s|$ for some constant σ; let T denote the set of stopping variables with respect to W; and let c denote a non-increasing, right-continuous, square-integrable function on the nonnegative real line. Then $E\{\sup_{t \geqq 0} c(t)|W(t)|\}$ is shown to be finite which insures that $\sup_{\tau \in T} E\{c(\tau)W(\tau)\}$ is finite. Also, ε-optimal stopping variables are shown to exist with stopping points occurring only in discrete subsets of the nonnegative real line. These optimal stopping variables require observation of the process W only at the possible stopping points.

1. Introduction and summary of results. Most of the literature on the subject of optimal stopping concerns either discrete sequences [e.g. Y. S. Chow, H. Robbins, and D. Siegmund (1971), Y. S. Chow and H. Robbins (1967), A. Dvoretzky (1967), H. Teicher and J. Wolfowitz (1966), etc.] or continuous parameter stochastic processes [e.g. M. E. Thompson (1971), A. G. Fakeev (1970), H. M. Taylor (1968), etc.]. The main goal of this paper is to bridge the gap between the discrete parameter case and the continuous parameter case in a particular situation. The convergence of the discrete to the continuous has been used very profitably in other areas of probability theory and was used successfully by L. A. Shepp (1969), M. E. Thompson and W. L. Owen (1972), and the author (1968), (1969). The possibility of using such convergence is also contained in Section 6 of the paper by M. E. Thompson (1971).

The situation of interest, which unfortunately requires considerable notation for concise mathematical description, is as follows:

Let R denote the set of real numbers, R_+ the set of nonnegative real numbers, I the set of positive integers, and $(\Omega, \mathscr{F}, P)$ a probability space. Let $\{W(t): t \in R_+\}$ denote a stochastic process on $(\Omega, \mathscr{F}, P)$ having the following properties:

(a) $W(0) = 0$,

(b) W has independent increments which are statistically homogeneous in the parameter variable, i.e. for $s < t$, $W(t) - W(s)$ and $W(t - s)$ have the same distribution function,

(c) for every $t \in R_+$, $E\{W(t)\} = 0$,

Received February 5, 1973; revised July 3, 1973.

AMS 1970 *subject classifications*. Primary 62L15; Secondary 60G40, 60J30.

Key words and phrases. Optimal stopping variables (rules), stochastic processes with independent, statistically homogeneous, zero mean, bounded variance increments; right-continuous paths.

(d) for some $\sigma \in R_+$ and every pair $t, s \in R_+$,

$$E\{(W(t) - W(s))^2\} = \sigma|t - s| , \qquad \text{and}$$

(e) W has right-continuous sample paths with probability one.

Let c denote a right-continuous function of R_+ to R_+ which is non-increasing and square-integrable, i.e.

$$\int_0^\infty (c(x))^2 \, dx < \infty .$$

For each $k \in I$, put $r_i(k) = i2^{-k}$ for every $i \in I \cup \{0\}$. (The k in this and following notation will often be suppressed when the value of k is clear from the context.) Let $\mathscr{F}_j(k)$ denote the minimum σ-algebra of sets contained in $\mathscr{F}$ for which $W(r_i(k))$, $i = 1, \cdots, j$, are $\mathscr{F}$-measurable. Let $\mathscr{F}(t)$ denote the minimum σ-algebra of sets contained in $\mathscr{F}$ for which $W(s)$, $0 \leqq s \leqq t$, are $\mathscr{F}$-measurable. Let $N(k)$ denote the set of functions n on Ω to $I \cup \{\infty\}$ such that $\{n \leqq j\} \in \mathscr{F}_j(k)$ for every $j \in I$ and $P\{n < \infty\} = 1$. Let $\mathscr{N}(k)$ denote the set of functions n on Ω to $I \cup \{\infty\}$ such that $\{n \leqq j\} \in \mathscr{F}(r_j(k))$ for every $j \in I$ and $P\{n < \infty\} = 1$. Observe that $N(k) \subset \mathscr{N}(k)$ since $\mathscr{F}_j(k) \subset \mathscr{F}(r_j(k))$. Let T denote the set of functions τ on Ω to $R_+ \cup \{\infty\}$ such that $\{\tau \leqq t\} \in \mathscr{F}(t)$ for every $t \in R_+$ and $P\{\tau < \infty\} = 1$.

The main results of this paper are given in Theorem 1, Theorem 2, and its corollary.

THEOREM 1.

$$\sup_{\tau \in T} E\{c(\tau)W(\tau)\} < \infty$$

and, given $\varepsilon > 0$, *there exist* $k \in I$ *and* $q \in N(k)$ *such that*

$$E\{c(r_q)W(r_q)\} \geqq \sup_{\tau \in T} E\{c(\tau)W(\tau)\} - \varepsilon .$$

The new content of Theorem 1 is in the first part, namely the finiteness of $\sup_{\tau \in T} E\{c(\tau)W(\tau)\}$ since the second part is implied by the material in Section 6 of M. E. Thompson (1971), particularly Corollary 6.1, page 310. However, the corollary to Theorem 2 gives a somewhat stronger form for the second part of Theorem 1, and the method of proof does not require the sophisticated ideas contained in Thompson's work.

THEOREM 2. *Given* $\varepsilon > 0$, *there exist* $k \in I$ *and a function* $f_{\varepsilon,k}$ *on* R_+ *to* $R_+ \cup \{\infty\}$ *such that*

$$\begin{aligned} \sup_{n \in \mathscr{N}(k)} E\{c(t + r_n)(y + W(r_n))\} &> c(t)y - \varepsilon \qquad \text{when} \quad y < f_{\varepsilon,k}(t) , \\ &\leqq c(t)y - \varepsilon \qquad \text{when} \quad y \geqq f_{\varepsilon,k}(t) , \end{aligned}$$

and

$$q = \sup\{j : W(r_i) < f_{\varepsilon,k}(r_i),\ i = 1, \cdots, j - 1\}$$

belongs to $\mathscr{N}(k)$ *and is* ε-*optimal in the sense that*

$$E\{c(r_q)W(r_q)\} \geqq \sup_{n \in \mathscr{N}(k)} E\{c(r_n)W(r_n)\} - \varepsilon .$$

COROLLARY. *Given* $\varepsilon > 0$, *there exists* $k \in I$ *such that the* $f_{\varepsilon/2,k}$ *function of Theorem* 2 *provides an* ε*-optimal stopping variable*; *namely*,

$$q = \sup\{j : W(r_i) < f_{\varepsilon/2,k}(r_i),\ i = 1, \cdots, j-1\} \in N(k) \qquad \textit{and}$$
$$E\{c(r_q)W(r_q)\} \geqq \sup_{\tau \in T} E\{c(\tau)W(\tau)\} - \varepsilon .$$

The proof of Theorem 2 relies heavily on the theory of Y. S. Chow and H. Robbins (1967). However, the optimal stopping variable can be defined in a relatively simple form, the form which one would expect from the work of A. Dvoretzky (1967) and/or H. Teicher and J. Wolfowitz (1966).

2. Proof of the theorems. The proofs will be achieved by proving a number of lemmas. The first two lemmas are concerned with establishing the finiteness of the expected values of the random variables of interest.

LEMMA 1. *For every* $k \in I$,

$$E\{\sup_{j \in I} c(r_j)|W(r_j)|\} \leqq 2(\sigma \textstyle\int_0^\infty (c(x))^2\, dx)^{\frac{1}{2}} < \infty .$$

PROOF. That the sequence $\{(W(r_j))^2, \mathscr{F}_j : j \in I\}$ is a separable semimartingale is immediate from the right-continuity and independent increment properties of W. Hence, by Theorem 5.1 of Z. W. Birnbaum and A. W. Marshall [(1961) page 698],

$$P\{\sup_{i \in I} (c(r_i)W(r_i))^2 \geqq u^2\} \leqq P\{\sup_{t \in R_+} (c(t)W(t))^2 \geqq u^2\} \leqq \frac{M\sigma}{u^2}$$

where $M = \int_0^\infty (c(x))^2\, dx$. Therefore,

$$\begin{aligned} E\{\sup_{i \in I} c(r_i)|W(r_i)|\} &= -\textstyle\int_0^\infty u\, d_u P\{\sup_{i \in I} c(r_i)|W(r_i)| > u\} \\ &\leqq \lim_{u \to \infty} \frac{M\sigma}{u} + \textstyle\int_0^\infty P\{\sup_{i \in I} (c(r_i)W(r_i))^2 \geqq u^2\}\, du \\ &\leqq \textstyle\int_0^a du + \int_a^\infty \frac{M\sigma}{u^2}\, du = 2(M\sigma)^{\frac{1}{2}} \end{aligned}$$

where $a = (M\sigma)^{\frac{1}{2}}$. □

LEMMA 2. $E\{\sup_{t \in R_+} c(t)|W(t)|\} \leqq 2(\sigma \int_0^\infty (c(x))^2\, dx)^{\frac{1}{2}} < \infty$.

PROOF. The sequence $\{\sup_{j \in I} c(r_j(k))|W(r_j(k))| : k \in I\}$ is non-decreasing with upper bound $\sup_{t \in R_+} c(t)|W(t)|$. Therefore, by the right-continuity of c and W (with probability one) $\sup_{j \in I} c(r_j)|W(r_j)|$ converges monotonically upward to $\sup_{t \in R_+} c(t)|W(t)|$ as k tends to infinity. Hence, by the Lebesgue monotone convergence theorem,

$$\lim_{k \to \infty} E\{\sup_{j \in I} c(r_j)|W(r_j)|\} = E\{\sup_{t \in R_+} c(t)|W(t)|\} .$$

By Lemma 1,

$$\lim_{k \to \infty} E\{\sup_{j \in I} c(r_j)|W(r_j)|\} \leqq 2(\sigma \textstyle\int_0^\infty (c(x))^2\, dx)^{\frac{1}{2}} :$$

hence,

$$E\{\sup_{t \in R_+} c(t)|W(t)|\} \leqq 2(\sigma \textstyle\int_0^\infty (c(x))^2\, dx)^{\frac{1}{2}} . \qquad \square$$

Observe that the first statement of Theorem 1 is a corollary of Lemma 2 since

$$c(\tau)W(\tau) \leqq \sup_{t \in R_+} c(t)|W(t)|$$

for every $\tau \in T$. Also, since for every $k \in I$ and $n \in \mathscr{N}(k)$, $c(r_n)|W(r_n)| \leqq \sup_{t \in R_+} c(t)|W(t)|$,

$$E\{\sup_{n \in \mathscr{N}(k)} c(r_n)|W(r_n)|\} < \infty .$$

This result provides an integrable bounding function for any sequence $\{c(r_{n(i)})W(r_{n(i)}) : i \in I\}$ of variables; hence, the Lebesgue convergence theorem can be applied in appropriate situations such as the one which arises in the proof of Lemma 3.

LEMMA 3.

$$\lim_{k\to\infty} \sup_{n \in \mathscr{N}(k)} E\{c(r_n)W(r_n)\} = \sup_{\tau \in T} E\{c(\tau)W(\tau)\} .$$

PROOF. For each $n \in \mathscr{N}(k)$, $r_n \in T$ because $\{r_n \leqq t\} = \{n \leqq [t2^k]\} \in \mathscr{F}(r_{[t2^k]}) \subset \mathscr{F}(t)$ and $P\{r_n < \infty\} = P\{n < \infty\} = 1$. Hence,

$$\sup_{n \in \mathscr{N}(k)} E\{c(r_n)W(r_n)\} \leqq \sup_{\tau \in T} E\{c(\tau)W(\tau)\} .$$

Since $\sup_{\tau \in T} E\{c(\tau)W(\tau)\}$ is finite by the first part of Theorem 1, given $\delta > 0$, there exists $\lambda \in T$ such that

$$E\{c(\lambda)W(\lambda)\} \geqq \sup_{\tau \in T} E\{c(\tau)W(\tau)\} - \delta .$$

For each $k \in I$, define $n(k)$ by

$$n(k) = \inf\{j : r_{j-1}(k) \leqq \lambda < r_j(k)\} .$$

Observe that $\{n(k) \leqq j\} = \{\lambda < r_j(k)\} \in \mathscr{F}(r_j(k))$ and $P\{n(k) < \infty\} = P\{\lambda < \infty\} = 1$; hence, $n(k) \in \mathscr{N}(k)$. Since $r_{n(k)} \geqq \lambda$ and $\lim_{k\to\infty} r_{n(k)} = \lambda$ with probability one, the right-continuity of c and W (with probability one) and the Lebesgue convergence theorem insure that

$$\lim_{k\to\infty} E\{c(r_{n(k)})W(r_{n(k)})\} = E\{c(\lambda)W(\lambda)\} .$$

Therefore,

$$\lim_{k\to\infty} \sup_{n \in \mathscr{N}(k)} E\{c(r_n)W(r_n)\} \geqq \sup_{\tau \in T} E\{c(\tau)W(\tau)\} - \delta . \qquad \square$$

The condition $E\{\sup_{n \in \mathscr{N}(k)} c(r_n)|W(r_n)|\} < \infty$ established above insures that condition A^+ of Y. S. Chow and H. Robbins (1967) page 433, holds; hence, Theorem 2 will follow from their Corollary to Theorem 6 [Y. S. Chow and H. Robbins (1967) page 436] given Lemma 4 below and the comments which follow the proof of the lemma. Implied in this statement is the fact that the results of Chow and Robbins hold for stopping variables in $\mathscr{N}(k)$ as well as for stopping variables in $N(k)$. Checking this fact is a straightforward rewriting of their proofs in this somewhat more general context. In order to state and prove Lemma 4 reasonably, some more notation is needed. The probability space $(\Omega, \mathscr{F}, P)$ can be expressed in a cross product form when working with W due to the independent increment property of W; namely, for $j \in I$ and $k \in I$,

$(\Pi_j \times \Lambda_j, F_j \times G_j, H_j \times Q_j)$ where Π_j consist of the continuous functions on the interval $[0, r_j(k))$ and Λ_j consists of the continuous functions on the interval $[r_j(k), \infty)$, and F_j, G_j, H_j, and Q_j are the corresponding "probability space quantities" on the respective sample spaces Π_j and Λ_j. For $j \in I$ and $k \in I$, put $\mathscr{N}_j(k) = \{\max(j, n) : n \in \mathscr{N}(k)\}$ and $\nu_j(k) = \operatorname{ess\,sup}_{n \in \mathscr{N}_j(k)} E\{c(r_n)W(r_n) \mid F_j \times \{\Lambda_j\}\}$.

[The ν_j is the notation of Y. S. Chow and H. Robbins (1967) Equation 5, page 428. A definition of the essential supremum is given by Y. S. Chow, H. Robbins, and D. Siegmund (1971) page 8.]

LEMMA 4. *For $j \in I$ and $k \in I$,*

$$\nu_j(k) = \sup_{n \in \mathscr{N}_j(k)} \int_{\Lambda_j} c(r_n)W(r_n)\, dQ_j .$$

PROOF. For every $n \in \mathscr{N}_j(k)$, $\int_{\Lambda_j} c(r_n)W(r_n)\, dQ_j$ is $F_j \times \{\Lambda_j\}$-measurable because the definite integral has eliminated all dependence on points in Λ_j. Put

$$\bar{\nu}_j(k) = \sup_{n \in \mathscr{N}_j(k)} \int_{\Lambda_j} c(r_n)W(r_n)\, dQ_j .$$

Then $\bar{\nu}_j(k)$ is $F_j \times \{\Lambda_j\}$-measurable. Let $A \in F_j \times \{\Lambda_j\}$. Since for each $n \in \mathscr{N}_j(k)$,

$$\int_A c(r_n)W(r_n)\, d(H_j \times Q_j)$$

is finite, it may be expressed as an iterated integral; namely,

$$\int_{A/\Pi_j} \int_{\Lambda_j} c(r_n)W(r_n)\, dQ_j\, dH_j .$$

Hence, for every $n \in \mathscr{N}_j(k)$,

$$\begin{aligned} \int_A \bar{\nu}_j(k)\, d(H_j \times Q_j) &\geqq \int_A c(r_n)W(r_n)\, d(H_j \times Q_j) \\ &= \int_A E\{c(r_n)W(r_n) \mid F_j \times \{\Lambda_j\}\}\, d(H_j \times Q_j) , \end{aligned}$$

i.e.

$$\bar{\nu}_j(k) \geqq E\{c(r_n)W(r_n) \mid F_j \times \{\Lambda_j\}\}$$

with probability one.

Let Y be a $F_j \times \{\Lambda_j\}$-measurable random variable such that $Y \geqq E\{c(r_n)W(r_n) \mid F_j \times \{\Lambda_j\}\}$ for every $n \in \mathscr{N}_j(k)$. Put $A = \{Y < \bar{\nu}_j(k)\}$ and suppose that $(H_j \times Q_j)(A) > 0$, that is, suppose that the lemma is false. Then

$$\int_A Y d(H_j \times Q_j) < \int_A \bar{\nu}_j(k)\, d(H_j \times Q_j) ;$$

therefore, there exists $\varepsilon > 0$ such that

$$\int_A Y d(H_j \times Q_j) \leqq \int_A \bar{\nu}_j(k)\, d(H_j \times Q_j) - \varepsilon .$$

By the definition of a supremum, there exists $n \in \mathscr{N}_j(k)$ such that

$$\begin{aligned} \int_A \bar{\nu}_j(k)\, d(H_j \times Q_j) &\leqq \int_{A/\Pi_j} \int_{\Lambda_j} c(r_n)W(r_n)\, dQ_j\, dH_j + \frac{\varepsilon}{2} (H_j \times Q_j)(A) \\ &= \int_A c(r_n)W(r_n)\, d(H_j \times Q_j) + \frac{\varepsilon}{2} (H_j \times Q_j)(A) . \end{aligned}$$

Hence,

$$\int_A Y d(H_j \times Q_j) + \varepsilon \leqq \int_A c(r_n)W(r_n)\, d(H_j \times Q_j) + \frac{\varepsilon}{2} (H_j \times Q_j)(A) .$$

But

$$\int_A Y d(H_j \times Q_j) \geqq \int_A E\{c(r_n)W(r_n) \mid F_j \times \{\Lambda_j\}\} d(H_j \times Q_j)$$
$$= \int_A c(r_n)W(r_n)\, d(H_j \times Q_j)\,,$$

which contradicts the previous result. Therefore, $(H_j \times Q_j)(A) = 0$. □

According to the corollary to Theorem 6 of Y. S. Chow and H. Robbins (1967) page 436, an ε-optimal stopping variable is given by stopping at the smallest j such that

$$c(r_j)W(r_j) \geqq \nu_j(k) - \varepsilon\,.$$

By Lemma 4, this inequality can be written in the form

$$\varepsilon \geqq \sup_{n \in \mathscr{N}_j(k)} \int_{\Lambda_j} (c(r_n)W(r_n) - c(r_j)W(r_j))\, dQ_j$$
$$= \sup_{n \in \mathscr{N}_j(k)} \int_{\Lambda_j} (c(r_n)(W(r_n) - W(r_j)) + (c(r_n) - c(r_j))W(r_j))\, dQ_j\,.$$

Because of the independent increment and homogeneity properties of W, this inequality can be expressed in the more convenient form

$$\varepsilon \geqq \sup_{n \in \mathscr{N}(k)} \int_\Omega (c(t + r_n)W(r_n) + (c(t + r_n) - c(t))y)\, dP$$

where the symbols r_j and $W(r_j)$ have been replaced by t and y respectively to emphasize that they are constants relative to the variable of integration.

For every $n \in \mathscr{N}(k)$, the expression

$$E\{c(t + r_n)W(r_n) + (c(t + r_n) - c(t))y\}$$
$$= E\{c(t + r_n)W(r_n)\} + E\{c(t + r_n) - c(t)\}y$$

as a function of y is continuous and non-increasing because c is non-increasing. Hence, the supremum of the expression over all stopping variables $n \in \mathscr{N}(k)$ will also have these properties as a function of y. Therefore, the function $f_{\varepsilon,k}$ on R_+ to $R \cup \{-\infty, \infty\}$ defined by

$$f_{\varepsilon,k}(t) = \sup\{y : \sup_{n \in \mathscr{N}(k)} E\{c(t + r_n)(y + W(r_n))\} > c(t)y - \varepsilon\}$$

is characterized by the property

$$\begin{aligned} \sup_{n \in \mathscr{N}(k)} E\{c(t + r_n)(y + W(r_n))\} &> c(t)y - \varepsilon \qquad \text{when } \; y < f_{\varepsilon,k}(t)\,, \\ &\leqq c(t)y - \varepsilon \qquad \text{when } \; y \geqq f_{\varepsilon,k}(t)\,, \end{aligned}$$

with the interpretation that when $f_{\varepsilon,k}(t)$ is $+\infty$ or $-\infty$ the appropriate inequality is deleted from the above expression. Consideration of the stopping variable $n = 2^k$ shows that $f_{\varepsilon,k}(t) \geqq 0$ for all $t \in R_+$, i.e.

$$E\{c(t + 1)(y + W(1))\} = c(t + 1)y > c(t)y - \varepsilon$$

for all $y < 0$. The ε-optimal stopping variable q can now be expressed in terms of $f_{\varepsilon,k}$ as

$$q = \sup\{j : W(r_i) < f_{\varepsilon,k}(r_i), i = 1, \cdots, j - 1\}\,.$$

Theorem 2 is proved.

Observe that q depends only on the values of W at parameter points $\{r_i : i \in I\}$; hence, $q \in N(k)$ and

$$\begin{aligned}\sup_{n \in N(k)} E\{c(r_n)W(r_n)\} &\geqq E\{c(r_q)W(r_q)\} \\ &\geqq \sup_{n \in \mathscr{N}(k)} E\{c(r_n)W(r_n)\} - \varepsilon .\end{aligned}$$

Consequently,

$$\sup_{n \in N(k)} E\{c(r_n)W(r_n)\} = \sup_{n \in \mathscr{N}(k)} E\{c(r_n)W(r_n)\} .$$

This result coupled with Lemma 3 can be used to complete the proof of Theorem 1 as follows. For each $k \in I$, select a sequence $\{n(i) : i \in I\}$, each $n(i) \in N(k)$, such that

$$\lim_{i\to\infty} E\{c(r_{n(i)})W(r_{n(i)})\} = \sup_{n \in N(k)} E\{c(r_n)W(r_n)\} .$$

Then a Cantor diagonalization gives

$$\lim_{k\to\infty} E\{c(r_{n(k)})W(r_{n(k)})\} = \sup_{\tau \in T} E\{c(\tau)W(\tau)\} ;$$

hence, given $\varepsilon > 0$, there exist $k \in I$ and $q \in N(k)$ such that

$$E\{c(r_q)W(r_q)\} \geqq \sup_{\tau \in T} E\{c(\tau)W(\tau)\} - \varepsilon .$$

Theorem 1 is proved. However, by using Theorem 2 more can be said concerning a possible form for q. Take $\varepsilon > 0$. Select $k \in I$ such that

$$\sup_{n \in \mathscr{N}(k)} E\{c(r_n)W(r_n)\} \geqq \sup_{\tau \in T} E\{c(\tau)W(\tau)\} - \frac{\varepsilon}{2} .$$

Then q defined by

$$q = \sup\{j : W(r_i) < f_{\varepsilon/2,k}(r_i),\ i = 1, \cdots, j-1\}$$

has the desired property; namely,

$$\begin{aligned}E\{c(r_q)W(r_q)\} &\geqq \sup_{n \in \mathscr{N}(k)} E\{c(r_n)W(r_n)\} - \frac{\varepsilon}{2} \\ &\geqq \sup_{\tau \in T} E\{c(\tau)W(\tau)\} - \varepsilon\end{aligned}$$

by Theorem 2. The corollary to Theorem 2 is proved.

REFERENCES

[1] Birnbaum, Z. W. and Marshall, A. W. (1961). Some multivariate Chebyshev inequalities with extensions to continuous parameter processes. *Ann. Math. Statist.* **32** 687–703.

[2] Chow, Y. S. and Robbins, H. (1967). On values associated with a stochastic sequence. *Proc. Fifth Berkeley Symp. Math. Statist. Prob.* **1** 427–440. Univ. of California Press, Berkeley.

[3] Chow, Y. S., Robbins, H., and Siegmund, D. (1971). *Great Expectations: The Theory of Optimal Stopping.* Houghton Mifflin Co., Boston.

[4] Dvoretzky, A. (1967). Existence and properties of certain optimal stopping rules. *Proc. Fifth Berkeley Symp. Math. Statist. Prob.* **1** 441–452. Univ. of California Press, Berkeley.

[5] Fakeev, A. G. (1970). Optimal stopping rules for stochastic processes with continuous parameter. *Theor. Probability Appl.* **15** 324–331.

[6] Shepp, L. A. (1969). Explicit solutions to some problems of optimal stopping. *Ann. Math. Statist.* **40** 993–1010.

[7] TAYLOR, H. M. (1968). Optimal stopping in a Markov process. *Ann. Math. Statist.* **39** 1333–1344.

[8] TEICHER, H. and WOLFOWITZ, J. (1966). Existence of optimal stopping rules for linear and quadratic rewards. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **5** 361–368.

[9] THOMPSON, M. E. (1971). Continuous parameter optimal stopping problems. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **19** 302–318.

[10] THOMPSON, M. E. and OWEN, W. L. (1972). The structure of the optimal stopping rule in the S_n/n problem. *Ann. Math. Statist.* **43** 1110–1121.

[11] WALKER, L. H. (1968). Stopping rules for Brownian motion and random walks. Ph. D. dissertation, University of California, Los Angeles.

[12] WALKER, L. H. (1969). Regarding stopping rules for Brownian motion and random walks. *Bull. Amer. Math. Soc.* **75** 46–50.

MATHEMATICS DEPARTMENT
292 MSCB
BRIGHAM YOUNG UNIVERSITY
PROVO, UTAH 84601