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ON STABILITY FOR OPTIMIZATION PROBLEMS[1]

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Small perturbations of certain species of optimization problems,—in particular of so-called casino problems with a fixed goal—cause only small perturbations of the optimal strategies.

0. Introduction. If a stochastic control, dynamic programming, or, more briefly, a gambling problem, is modified by a small amount, is the optimal solution also modified by only a small amount? That is, is there some kind of stability, or continuity, for such problems? Precise formulations and solutions to this general problem are not yet in sight. So perhaps a study of a special case can be suggestive. As pointed out in [6], where this question was raised, even in the special case of casino problems, a species of optimization problem which, as a mathematical object, was there introduced and studied, continuity does not always prevail; for arbitrarily small perturbations can transform a fair casino into a superfair casino. Once superfair and fair casinos are set aside, however, there is indeed a kind of stability for casino problems, as this note explains.

A solution to a gambling problem involves finding two distinct, though related, entities. The first is U, the best that can be achieved, and the second is one or more optimal, or, if none such exist, one or more nearly optimal, strategies. Correspondingly, one might say that *continuity* prevails for a house Γ if a small change in Γ causes at most a small change in U. A closely related notion is that of *semi-stability*, that is, with every sufficiently small change in Γ, there is at least one optimal or nearly optimal strategy for Γ that can be closely imitated in the perturbed house Γ'. But to say that *stability* prevails is a stronger claim, for it says that *every* strategy in Γ can be closely imitated in Γ'.

A main purpose of this note is to find a notion of nearness for a pair of subfair casinos Γ and Γ' that will guarantee a kind of continuity property for the mapping $\Gamma \to U$. Section 1 is devoted to the formulation of the not completely satisfying result (Theorem 1) and, except for the last section, the remainder of the paper is devoted to its proof.

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A main step intermediary to showing that U and U' are close is to establish a kind of stability for subfair casinos (Theorems 2 and 2*) to which Section 2 and Section 3 are devoted. What these Sections are about can be imparted quickly here once these definitions are given.

If θ is the distribution of a random variable X and $t > 0$, then the distribution $t\theta$ of tX is a *dilitation*. A stochastic process $f_0, f_1, \cdots$ is a *θ-process* if every conditional distribution of every increment $f_{n+1} - f_n$ given the past is a dilitation of θ. Two processes are *equivalent* if all their finite dimensional distributions are the same.

An idea of the contents of this paper can be gleaned from the following which is a corollary to the Stability Theorem 2.

COROLLARY TO THEOREM 2. *Suppose that θ and θ' are probability measures of bounded support and with negative means which satisfy*

$$\int |\theta(-\infty, x] - \theta'(-\infty, x]|\, dx \leqq \varepsilon\delta \left|\int x\, d\theta(x) + \int x\, d\theta'(x)\right| , \tag{0.1}$$

for some positive ε and δ. Then, for every θ-process with values in the unit interval, there is an equivalent θ-process $f_0, f_1, \cdots$ and a θ'-process $f_0', f_1', \cdots$ such that

$$\mathrm{Prob}\,(\exists i\colon |f_i - f_i'| \geqq 2\delta) \leqq 2\varepsilon . \tag{0.2}$$

1. A formulation of continuity (Theorem 1).

1.1. *When are two optimization problems ε-close?* The gambling houses of primary interest for this note are based on the unit interval, that is, for each f and each $\gamma \in \Gamma(f)$, γ is a probability measure supported by the closed unit interval. The *lottery θ* that corresponds to γ at f is simply γ translated to the left by f, so θ is supported by the closed interval $[-f, 1 - f]$.

Of course, Γ determines, and is determined by, Θ, where $\Theta(f)$ is the set of all lotteries θ available at f.

In order to reduce the number of different symbols, the same symbol is sometimes used to refer to related but distinct objects. In particular, the symbol "θ" designates a lottery, which is a probability measure, as well as its distribution function. Therefore, by convention,

$$\theta(x) = \theta(-\infty, x] \qquad \text{for all } x . \tag{1.1.1}$$

As a first orientation, two lotteries θ and θ' can be considered close if $\int |\theta'(x) - \theta(x)|\, dx$ is small. However, since one gets somewhat sharper inequalities by considering the implications of the smallness of $\int (\theta'(x) - \theta(x))^+\, dx$ and of $\int (\theta'(x) - \theta(x))^-\, dx$ separately, we shall do so. Moreover, the smallness of $\int (\theta'(x) - \theta(x))^+\, dx$, or even of $\int |\theta'(x) - \theta(x)|\, dx$, is inadequate for our purposes. It will be important that $\int (\theta'(x) - \theta(x))^+\, dx$ be small even relative to a function v of θ that may itself be small. Two v's will be exploited, the first and most important of which is given by setting $v(\theta)$ equal to the negative of the mean of θ, and the second is given by setting $v(\theta)$ equal to one-half of the variance of θ.

Of principal interest are *subfair* θ's, that is, θ's of nonpositive mean.

Since the arguments below yield different inequalities for different v's, definitions and proofs will be formulated so as to be meaningful for a general nonnegative v.

For each fixed v and $\varepsilon > 0$, a binary relation $\theta \leqq \theta' \bmod (v, \varepsilon)$ will now be defined. For simplicity, the notation will not include the dependence on v. Rather, say that θ is *at most ε-better than θ'*, or in symbols, $\theta \leqq \theta' + \varepsilon$, if

$$(1.1.2) \qquad \int (\theta'(x) - \theta(x))^+ \, dx \leqq \varepsilon v(\theta) .$$

For sets of lotteries Δ and Δ', define $\Delta \leqq \Delta' + \varepsilon$ if, for each $\theta \in \Delta$, there is a $\theta' \in \Delta'$ for which $\theta \leqq \theta' + \varepsilon$.

How should $\Gamma \leqq \Gamma' + \varepsilon$ be defined? Plainly, it shall be necessary that, for each f,

$$(1.1.3) \qquad \Theta(f) \leqq \Theta'(f) + \varepsilon ,$$

where $\Theta(f)$ is the set of lotteries available in Γ at f.

Even for moderately general houses Γ and Γ', as simple examples show, (1.3) is not a sufficiently strong condition to imply that U is not much larger than U'. The kind of condition on Γ and Γ' that may prove to be sufficient for more general pairs of houses than are the center of attention in this note is:

$$(1.1.4) \qquad \Theta(f) \leqq \Theta'(f') + \varepsilon$$

for all f, and all f' in a suitable neighborhood of f.

For the special purposes of this note, the simpler, and weaker, condition (1.3) is adequate and is, therefore, here adopted as the formal definition of $\Gamma \leqq \Gamma' + \varepsilon$.

1.2. *A formulation of continuity* (*Theorem* 1). In Theorem 1, and throughout this paper, unless plainly indicated otherwise, these notations are in effect:

(i) Γ is a subfair house based on the unit interval F;

(ii) $\Gamma' = \Gamma^c$ is a nontrivial casino based on the unit interval F';

(iii) ε and δ with, as well as without, subscripts and superscripts are positive numbers;

(iv) The *usual* u is defined by $u(f) = 0$ for $f < 1$ and $u(1) = 1$;

(v) $U = \Gamma u$ is the U of Γ; $U' = U^c = \Gamma^c u$ is the U of Γ^c.

THEOREM 1. *For each Γ^c and ε there is an ε' such that, for the usual u and for any subfair Γ for which $\Gamma \leqq \Gamma^c + \varepsilon'$, $U \leqq U^c + \varepsilon$.*

2. A formulation of stability (Theorem 2). Four sub-sections are devoted to the formulation.

2.1. *The problem.* The main problem is to give a precise meaning to the intuitive idea that a gambler in one house may be able to imitate closely the strategy σ of a gambler in another house. If σ employs but one gamble and then ceases to gamble, or stagnates, there is no problem of formulation. For let the imitation strategy σ' employ a single gamble σ_0' that is close to σ_0 and then let σ' stagnate, too. But if the gambler, after having his initial fortune f changed under σ_0 to a fortune f_1 employs a second gamble $\sigma_1(f_1)$ before stagnating, a problem presents itself. For, call f_1' the fortune that the imitator-gambler has as the result of his initial gamble σ_0'. What is desired is that if f_1' is close to f_1, then there should be available in Γ' at f_1' a lottery $\sigma_1'(f_1')$ that is close to $\sigma_1(f_1)$. But, unless care is exercised, there is no reason for f_1' to be close to f_1, for even if $\sigma_0 = \sigma_0'$, f_1 could be independent of f_1'. Thus, to return to the initial gambles σ_0 and σ_0', it is not enough that f_1' and f_1 be similarly distributed, but, rather, what is needed is that the pair $f^* = (f_1, f_1')$ have a distribution $\sigma_0{}^*$ under which not only are its marginals σ_0 and σ_0' close to each other, but also, under $\sigma_0{}^*$, it must be very probable that f_1' is close to f_1. This leads to the definition of the product of two houses.

2.2. *Product houses.* The product $\Gamma^* = \Gamma \times \Gamma'$ of Γ and Γ' is a house based on $F^* = F \times F'$, and, for $f^* = (f, f')$, $\gamma^* \in \Gamma^*(f^*)$ if, and only if, the first marginal of γ^* is an element of $\Gamma(f)$ and the second is an element of $\Gamma'(f')$. The definiton is plainly meaningful for abstract houses Γ and Γ'. However, when F and F' are intervals, and gambles defined on all subsets are considered, it will be of technical convenience to let Γ^* have an enlargement that, for the purposes of this paper, is innocuous.

Call two gambles on the real line *equivalent* if they assign the same probability to all intervals, and put also in $\Gamma^*(f^*)$ every gamble whose two marginals are equivalent to gambles in $\Gamma(f)$ and $\Gamma(f')$.

2.3. *Extensions and imitators of strategies.* A *partial history* p, unless vacuous, is of the form $p = (f_1, \cdots, f_n)$ with $f_i \in F$. Similarly, a *partial history* p^*, unless vacuous, is of the form $p^* = (f_1{}^*, \cdots, f_n{}^*)$ with $f_i{}^* = (f_i, f_i')$, $f_i \in F$, $f_i' \in F'$. If $p' = (f_1', \cdots, f_n')$, then p and p', the *components* of p^*, obviously determine, and are determined, by p^*. A *strategy* σ assigns to every p a gamble $\sigma(P)$ based on F, and, of course, a strategy σ^* assigns to every p^* a gamble $\sigma^*(p^*)$ based on F^*. Just as $\sigma(p)$ is the conditional distribution of f_{n+1} given p, so $\sigma^*(p^*)$ is the conditional distribution of f_{n+1}^*, that is, of (f_{n+1}, f_{n+1}'), given p^*. Plainly, then, given p^*, $\sigma^*(p^*)$ has two marginal distributions, $\sigma_1{}^*(p^*)$ and $\sigma_2{}^*(p)$, the conditional distribution respectively of f_{n+1} and of f_{n+1}' given p^*.

If, for all p^*, $\sigma_1{}^*(p^*)$ and $\sigma(p)$ are the same, then σ^* is an *extension* of σ. When, as in this paper, F and F' are intervals, also to be deemed as extensions of σ are those σ^* for which $\sigma_1{}^*(p^*)$ is equivalent to $\sigma(p)$.

If $A(\delta)$ is the event that for some i, $f_i \geqq f_i' + 2\delta$, and σ^* is an extension of σ for which the σ^*-probability of $A(\delta)$ is at most ε, then σ^*, as well as $\sigma_2{}^*$, is an (ε, δ)-*imitator* of σ.

The σ^*-probability of $A(\delta)$ is defined to be the supremum over all stop rules t of the σ^*-probability of the event

$$(\exists i \leqq t : f_i \geqq f_i' + 2\delta)\,, \tag{2.3.1}$$

which, incidentally, is a definition in concordance with the fact that open sets of histories can always consistently be assigned a probability equal to the supremum of the probabilities of its closed-open, or finitary, subsets as is shown in [4].

2.4. *A formulation of stability realized* (*Theorem* 2). With the help of the definitions of the preceding sections, the theorem is easily formulated thus.

THEOREM 2. *Suppose that* $\Gamma \leqq \Gamma^c + \varepsilon\delta$. *Then, if* σ *is available in* Γ *at* f, *there is an* (ε, δ)-*imitator of* σ *available in* $\Gamma \times \Gamma^c$ *at* (f, f).

When Theorem 2 is applied to one-game casinos, the corollary stated in the introduction is obtained.

Section 3 will now be devoted to the proof of Theorem 2.

3. The proof of stability (Theorem 2).

3.1. *The probability that the difference of two sequences never exceeds* δ. Plainly, for any sequences of real numbers, $f_0, f_1, f_2, \cdots$, and $f_0', f_1', f_2', \cdots$ with $f_0 = f_0'$, and any positive integer t,

$$\sup_{1 \leqq j \leqq t} (f_j - f_j') \leqq \sum_{j=1}^{t} (x_j - x_j')^+ , \tag{3.1.1}$$

where:

$$x_j = f_j - f_{j-1} ; \qquad x_j' = f_j' - f_{j-1}' . \tag{3.1.2}$$

In view of (3.1.1), the following lemma is immediate.

LEMMA 3.1.1. *Let* x_i, x_i', f_i *and* f_i' *be stochastic processes with* $f_i = f + \sum_1^i x_j$; $f_i' = f + \sum_1^i x_j'$; *let* $\delta > 0$, *and let* t *be the least* i, *if any, such that* $f_i - f_i' \geqq \delta$. *Then*

$$\text{Prob}\,(t < \infty) \leqq \frac{1}{\delta} E(\textstyle\sum_{j \leqq t} (x_j - x_j')^+) . \tag{3.1.3}$$

3.2. *The infimum of two lotteries.*[1] For distribution functions θ and θ', define, possibly as Fréchet first did, a two-dimensional distribution function $\theta \wedge \theta'$, thus;

$$(\theta \wedge \theta')(x, x') = \theta(x) \wedge \theta'(x') , \tag{3.2.1}$$

where the right-hand side of (3.2.1) is the minimum of the two numbers $\theta(x)$ and $\theta'(x')$.

Also, if θ and θ' are lotteries, any two-dimensional lottery θ^* whose distribution function is the infimum of the distribution functions of θ and of θ' will be called the *infimum* of θ and θ', and will be designated by $\theta \wedge \theta'$. The following lemma is no doubt well known.

LEMMA 3.2.1. *For any distribution functions* θ *and* θ' *on the real line, there exists a two-dimensional distribution function* θ^* *whose marginals are* θ *and* θ', *and under which*

$$E(X - X')^+ = \textstyle\int_{-\infty}^{\infty} (\theta'(x) - \theta(x))^+ \, dx , \tag{3.2.2}$$

[1] One of the authors, Dubins, is appreciative of stimulating verbal comments by Giorgio Dall' Aglio or this topic.

where X and X' are the coordinate variables:

$$X(x, x') = x \qquad \textit{and} \qquad X'(x, x') = x' .$$

PROOF. Let $\theta^* = \theta \wedge \theta'$. Under θ^*,

(i) $P(\max(X, X') > x) = \max(P(X > x), P(X' > x))$.

Moreover, as is well known,

(ii) If $P(Y \geqq 0) = 1$, then $E(Y) = \int_0^\infty P(Y > x)\,dx$,

and

(iii) $(X - X')^+ = \max(X, X') - X'$.

Therefore,

$$\begin{aligned} E(X - X')^+ &= E[\max(X, X') - X'] \\ &= \textstyle\int_{-\infty}^{\infty} [P(\max(X, X') > x) - P(X' > x)]\,dx \\ &= \textstyle\int_{-\infty}^{\infty} [\max(P(X > x), P(X' > x)) - P(X' > x)]\,dx \\ &= \textstyle\int_{-\infty}^{\infty} (P(X > x) - P(X' > x))^+\,dx \\ &= \textstyle\int_{-\infty}^{\infty} (\theta'(x) - \theta(x))^+\,dx . \end{aligned} \tag{3.2.2'}$$

LEMMA 3.2.2. *If $\theta \leqq \theta' + \varepsilon$, then*

$$E(X - X')^+ \leqq \varepsilon v(\theta) , \tag{3.2.3}$$

where E refers to expectation under $\theta \wedge \theta'$, and X and X' are the coordinate variables.

3.3. *Relationship between strategies, stochastic processes, and incremental processes.* A strategy σ gives the distribution of f_1 and the conditional distribution of f_{n+1} given p, for every partial history $p = (f_1, \cdots, f_n)$. Thus, under σ, f_1, f_2, $\cdots$ is a stochastic process. And, of course, a stochastic process $f_1, f_2, \cdots$ with a *fixed assignment* of conditional probabilities determines the strategy σ. Since, in the remainder of this note, it will be convenient to employ the customary language of stochastic processes, a *stochastic process* here is a strategy, or what is the same thing, a *coordinate* stochastic process $f_1, f_2, \cdots$ with a fixed assignment of conditional probabilities. There should be no confusion if a symbol such as "f_2" is used to designate a random variable, the projection onto the second coordinate, as well as the element of F that happens to occur in the second position of the history $h = (f_1, f_2, \cdots)$.

When each f_i is real, and the initial state $f = f_0$ is fixed, each history h determines, and is determined by, the *increments* $x_n = f_n - f_{n-1}$.

For fixed f, the strategy σ determines, and is determined by, the distribution of x_1 and the conditional distribution of x_{n+1} given the past. The distribution of x_1 is the *initial lottery* of σ, and the conditional distribution of x_{n+1} given p is the *lottery* employed by σ after the partial history $p = (f_1, \cdots, f_n)$.

Henceforth, the initial lottery employed by σ will be designated by σ_0, and the lottery employed after p will be designated by $\sigma(p)$.

Similarly, each two-dimensional strategy σ^* determines a two-dimensional stochastic process $f_1^*, f_2^*, \cdots$ with fixed conditional distributions. When the initial state $f^* = f_0^*$ is fixed, this process determines, and is determined, by the incremental process $x_1^*, x_2^*, \cdots$, where

$$x_i^* = f_i^* - f_{i-1}^* \\ = (x_i, x_i') \\ = (f_i - f_{i-1}, f_i' - f_{i-1}') \,. \tag{3.3.1}$$

Thus the lottery $\sigma^*(p^*)$ employed by σ^* after p^* is the conditional joint distribution of (x_{n+1}, x_{n+1}') given $p^* = (f_1^*, \cdots, f_n^*)$.

3.4. *Infimum of two strategies.* An *infimum* of a pair of strategies σ and σ' is a two-dimensional strategy σ^* such that, for each p^*, with component partial histories p and p',

$$\sigma^*(p^*) = \sigma(p) \wedge \sigma'(p) \,, \tag{3.4.1}$$

where $\sigma^*(p^*)$, $\sigma(p)$ and $\sigma'(p)$ are lotteries.

Notice two aspects of this definition. The first is the asymmetric treatment of σ and σ', for it is the lottery that σ' employs after p rather than after p' that is relevant. The second is that the definition is framed in terms of lotteries $\sigma(p)$, $\sigma'(p)$ and $\sigma^*(p^*)$, rather than gambles, in accordance with the convention adopted in the preceding sections. In detail, $\sigma^*(p^*)$, the conditional joint distribution of (x_{n+1}, x_{n+1}'), given that the past of the process under σ^* has traced out the partial history p^*, is the infimum of $\sigma(p)$ with $\sigma'(p)$, where $\sigma(p)$ is the conditional distribution of x_{n+1} given that the past of the process under σ has traced out p, and $\sigma'(p)$ is the conditional distribution of x_{n+1}' given that the past of the process under σ' has traced out the same partial history p.

Here is another definition. Write $\sigma \leqq \sigma' + \varepsilon$ if

$$\sigma(p) \leqq \sigma'(p) + \varepsilon \qquad \text{for all } p \,. \tag{3.4.2}$$

In contrast to the preceding definition, this one does not threaten to be misunderstood. For here, were both $\sigma(p)$ and $\sigma'(p)$ to refer to gambles, rather than lotteries, used after p, the meaning would not be altered.

Let $\sigma \wedge \sigma'$ designate the infimum of σ and σ'.

LEMMA 3.4.1. *Suppose that σ and σ' are strategies with initial fortune f_0 and that $\sigma \leqq \sigma' + \varepsilon$. Then, under $\sigma \wedge \sigma'$, the process $f_0^*, f_1^*, \cdots$ with $f_0^* = (f_0, f_0)$ satisfies:*

$$E[(x_{n+1} - x_{n+1}')^+ \mid f_i^* : i \leqq n] \leqq \varepsilon v(\sigma(p)) \\ = \varepsilon v(\sigma(f_1, \cdots, f_n)) \,. \tag{3.4.3}$$

PROOF. Apply Lemma 3.2.2.

We use the notation $(j \leqq t)$ to designate the indicator function of the event $(j \leqq t)$, which accords with convenient notational devices suggested by de Finetti [1].

LEMMA 3.4.2. *Let $f, f_1, f_2, \cdots$ be a nonnegative stochastic process with increments*

$x_1, x_2, \cdots$, and let t be a stop rule for $f, f_1, f_2, \cdots$. Then

$$E(\textstyle\sum_j (j \leqq t)E[-x_j \mid f_1, \cdots, f_{j-1}]) \leqq f \,. \tag{3.4.4}$$

If, in addition, it is supposed that $0 < f, f_1, f_2, \cdots$ is subfair with values in the unit interval, then

$$E(\textstyle\sum_j (j \leqq t) \operatorname{Var}(x_j \mid f_1, \cdots, f_{j-1})) < 2f \,. \tag{3.4.5}$$

PROOF OF (3.4.4). Verify that, for L, the left-hand side of (3.4.4),

$$L = E(-\textstyle\sum_{j \leqq t} x_j) = E(f - f_t) = f - Ef_t \leqq f \,. \tag{3.4.6}$$

That (3.4.5) holds, is shown in [3].

Of course, when $v(\theta)$ is the negative of the mean of θ, (3.4.4) states the same as

$$E(\textstyle\sum_j (j \leqq t)E[v(\sigma(f_1, \cdots, f_{j-1}))]) \leqq f \,. \tag{3.4.7}$$

Moreover, if $v(\theta)$ is interpreted as one-half of the variance of θ, then also (3.4.5) states that (3.4.7) holds, where now the inequality is even strict.

Henceforth, the underlying v is either the negative of the mean, or one-half of the variance. With this understanding, Lemma 3.4.3 is, in effect, two propositions.

LEMMA 3.4.3. *Let σ and σ' be strategies based on the nonnegative reals with the same initial fortunes, and suppose that $\sigma \leqq \sigma' + \varepsilon\delta$. Then, under $\sigma \wedge \sigma'$,*

$$\operatorname{Prob}(\exists j : f_j - f_j' \geqq \delta f) \leqq \varepsilon \,. \tag{3.4.8}$$

PROOF. According to Sub-section 3.1, the left-hand side of (3.4.8) is majorized by $(\delta f)^{-1}E(\sum_{j \leqq t} (x_j - x_j')^+)$, where t is the first j for which $f_j - f_j' \geqq \delta f$.

For the rest of the proof, compute thus:

$$\begin{aligned} E \textstyle\sum_{j \leqq t} (x_j - x_j')^+ &= E(\textstyle\sum_j (j \leqq t)(x_j - x_j')^+) \\ &= E(\textstyle\sum_j E[(j \leqq t)(x_j - x_j')^+ \mid f_1^*, \cdots, f_{j-1}^*]) \\ &= E(\textstyle\sum_j (j \leqq t)E[(x_j - x_j')^+ \mid f_1^*, \cdots, f_{j-1}^*]) \\ &\leqq \varepsilon\delta E(\textstyle\sum_j (j \leqq t)E[v(\sigma(f_1, \cdots, f_{j-1}))]) \\ &\leqq \varepsilon\delta f \,, \end{aligned} \tag{3.4.9}$$

where the first and second inequalities hold by Lemma 1 and by (3.4.7), respectively.

3.5. *The stability Theorem 2, slightly generalized.*

THEOREM 2*. *Suppose that for some σ available in Γ at f,*

$$\sigma \leqq \sigma' + \varepsilon\delta \,. \tag{3.5.1}$$

Then there is a σ^* available in $\Gamma \times \Gamma^c$ at (f, f) which is an (ε, δ)-imitator of σ.

Plainly, Theorem 2 is immediate from Theorem 2*.

Also, since σ^* is an (ε, δ)-imitator of σ, it is equally plain that, under σ^*,

$$\operatorname{Prob}(\exists i : f_i' \geqq 1 - 2\delta) \geqq u(\sigma) - \varepsilon \,, \tag{3.5.2}$$

where $u(\sigma)$ is the probability under σ that for some i, $f_i \geqq 1$.

PROOF OF THEOREM 2*. Let β be a real number. If θ is the distribution of f, let $\beta\theta$ denote the distribution of βf. If θ is the joint distribution of f and g, let $\beta\theta$ denote the joint distribution of f and βg.

Let $\alpha = (1 + \delta)^{-1}$, and, in terms of σ, σ' and α, define the two-dimensional strategy σ^*, thus. The initial lottery of σ^*, say, σ_0^*, is the infimum of σ_0, the initial lottery of σ, with $\alpha\sigma_0'$, where σ_0' is the initial lottery of σ'. In symbols,

$$\sigma_0^* = \sigma_0 \wedge \alpha\sigma_0' . \tag{3.5.3}$$

Similarly, for each partial history $p^* = (f_1^*, \cdots, f_n^*)$,

$$\sigma^*(p^*) = \sigma(p) \wedge \alpha\sigma'(p) , \tag{3.5.4}$$

except, if, for some $j \leqq n$, $\alpha f_j > f_j'$, let

$$\sigma^*(p^*) = \sigma(p) \wedge \delta(0) , \tag{3.5.5}$$

where $\delta(0)$ is the trivial lottery that assigns probability one to the singleton $\{0\}$.

This σ^* satisfies the conclusion of Theorem 2*. Why? That σ^* extends σ is obvious from its definition. To see that σ^* is available in $\Gamma \times \Gamma'$ at (f, f), it plainly suffices to verify that $\alpha\sigma'(p)$ is available in Γ' at f_n' if $\alpha f_j \leqq f_j'$ for all $j \leqq n$. For this, recall that since $\sigma'(p)$ is available in Γ' at f_n, and since Γ' is a casino, $\alpha\sigma'(p)$ is available in Γ' at αf_n, and hence also at f_n', since $\alpha f_n \leqq f_n'$.

Therefore, to complete the proof, it is only necessary to verify that under σ^*,

$$\operatorname{Prob}(\exists i : f_i \geqq f_i' + 2\delta) \leqq \varepsilon . \tag{*}$$

For this purpose, consider the two-dimensional process $(f, f), (f_1, f_1''), (f_2, f_2''), \cdots$ governed by the strategy $\sigma \wedge \sigma'$. Of course, $\sigma'(p)$ is the conditional distribution of $f_{n+1}'' - f_n''$ given the past. Since $\sigma \wedge (\alpha\sigma') = \alpha(\sigma \wedge \sigma')$, as far as verification of (*) is concerned, it may be supposed that

$$f_0' = f ; \qquad f_1' = f + \alpha(f_1'' - f) ; \tag{3.5.6}$$

and, inductively,

$$f_{n+1}' = f_n' + \alpha(f_{n+1}'' - f_n'') , \tag{3.5.7}$$

unless, for some $j \leqq n$, $\alpha f_j > f_j'$, in which event,

$$f_{n+1}' = f_n' . \tag{3.5.8}$$

To complete the proof of Theorem 2*, three preliminary lemmas are needed.

LEMMA 3.5.1. *If $\alpha f_j \leqq f_j'$ for all $j \leqq n$, then*

$$\begin{aligned} f_j' &= \alpha f_j'' + (1 - \alpha) f \\ &= f + \alpha \textstyle\sum_{i=1}^{j} (f_i'' - f_{i-1}'') \end{aligned} \tag{3.5.9}$$

for all $j \leqq n + 1$.

PROOF. By induction on n.

LEMMA 3.5.2. *If $f_j - f_j'' < \delta f$ for all $j \leqq n$, then $\alpha f_j \leqq f_j'$ for all $j \leqq n$.*

PROOF. For $n = 0$, it is clear, since $f_0 = f_0' = f_0'' = f$. Suppose now that for some n, $f_j - f_j'' < \delta f$, for all $j \leqq n$. Then, by induction, $\alpha f_j \leqq f_j'$ for all $j \leqq n - 1$, and hence, by Lemma 3.5.1, $f_j' = \alpha f_j'' + (1 - \alpha)f$ for $j \leqq n$. Also, since $f_n - f_n'' < \delta f$,

$$\begin{aligned} \alpha f_n &< \alpha f_n'' + \alpha\delta f \\ &= \alpha f_n'' + (1 - \alpha)f \\ &= f_n' . \end{aligned} \tag{3.5.10}$$

This completes the proof.

LEMMA 3.5.3. *If, for every* $j \leqq n$, $f_j - f_j'' < \delta f$, *then for every* $j \leqq n + 1$, $f_j - f_j' < 2\delta$.

PROOF. According to Lemma 3.5.2, $\alpha f_j \leqq f_j'$ for $j \leqq n$. Hence, by Lemma 3.5.1, $f_j' = \alpha f_j'' + (1 - \alpha)f$ for $j \leqq n + 1$.
Therefore,

$$\begin{aligned} f_j - f_j' &= f_j - \alpha f_j'' - (1 - \alpha)f \\ &< f_j'' + \delta f - \alpha f_j'' - (1 - \alpha)f \\ &= (1 - \alpha)f_j'' + \delta f - (1 - \alpha)f \\ &< (1 - \alpha) + \delta + 0 \\ &< \delta + \delta + 0 , \end{aligned} \tag{3.5.11}$$

where the fact that $f_j'' \leqq 1$ is used.

The proof of Theorem 2* is now completed thus.

$$\operatorname{Prob}(\exists j : f_j - f_j' \geqq 2\delta) \leqq \operatorname{Prob}(\exists j : f_j - f_j'' \geqq \delta f) \leqq \varepsilon , \tag{3.5.12}$$

where the first inequality holds by Lemma 3.5.3, and the second by Lemma 3.4.3.

4. Remainder of the proof of continuity (Theorem 1).

4.1. *The uselessness of randomization.* Let v be a bounded utility defined on F. Define v^* on $F^* = F \times F'$ thus

$$v^*(f, f') = v(f) . \tag{4.1.1}$$

Let V be the U of the gambling problem (Γ, v), that is, in the notation of [6], $V = \Gamma v$. Similarly, $V^* = \Gamma^* v^*$ is the U of (Γ^*, v^*).

PROPOSITION 4.1.1. *For all* $f \in F$ *and* $f' \in F'$,

$$V^*(f, f') = V(f) . \tag{4.1.2}$$

PROOF. Plainly,

$$V(f) \leqq V^*(f, f') , \tag{4.1.3}$$

for, in determining $V(f)$, a sup of $v(\sigma)$ or, equivalently of $v^*(\sigma^*)$, is taken only over those σ^* which depend only on the first coordinates of the fortunes f_i^*. For the reverse inequality, apply Theorem 2.12.1 in [6]. That is, first verify that

$$V(f) = v^*(f, f') . \tag{4.1.4}$$

Second, if $Q(f, f')$ is defined as $V(f)$, then Q is excessive for Γ^*. That is, for $\gamma^* \in \Gamma^*(f^*)$, with $f^* = (f, f')$,

$$\begin{aligned} \gamma^* Q &= \int Q(s, t)\, d\gamma^*(s, t) \\ &= \int V(s)\, d\gamma(s) \\ &\leqq V(f) \\ &= Q(f, f')\,, \end{aligned} \tag{4.1.5}$$

where γ is the marginal of γ^* on F. That the inequality holds is the content of Theorem 2.14.1 in [6]. From Theorem 2.12.1 in [6] conclude that $Q \geqq V^*$. The proof is complete.

A *randomized strategy* available in Γ at f is any strategy σ^* available in Γ^* at (f, f') for any f'.

Interchanging the roles of Γ and Γ' one obtains:

COROLLARY 4.1.1. For any f, v, ε^* and randomized σ^* available in Γ' at f, there is an (ordinary) strategy σ'' available in Γ' at f such that

$$v(\sigma'') > v^*(\sigma^*) - \varepsilon^*\,. \tag{4.1.6}$$

Though not important for the main purposes of this paper, we note that Proposition 4.1.1 and Corollary 4.1.1 apply to all abstract gambler's problems. Indeed, F and F' can be arbitrary sets, and Γ^* can be the product of any Γ based on F with any Γ' based on F'. In particular, for all f', $\Gamma'(f')$ could be the set of all gambles on F'.

4.2. *The end of the proof of continuity* (*Theorem* 1). Theorem 1 follows easily from the following more general formulation, which is sometimes applicable when Theorem 1 is not.

THEOREM 1*. *Suppose that for some ε_1-optimal strategy σ in Γ at f, there is a σ' in Γ' at f, such that $\sigma \leqq \sigma' + \varepsilon\delta$, and that the oscillation of U' on every interval of length 2δ is at most ε_2. Then*

$$U'(f) \geqq U(f) - (\varepsilon + \varepsilon_1 + \varepsilon_2)\,. \tag{4.2.1}$$

Of course, in Theorem 1*, ε_2 may be the smallest number such that $U'(g) - U'(h) \leqq \varepsilon_2$ for all g and h for which $|g - h| \leqq 2\delta$. Incidentally, as shown in [2], when U' is a casino function, $1 - U'(1 - 2\delta)$ is the minimal permissible ε_2.

PROOF OF THEOREM 1*. By Theorem 2*, there is a σ^* available in $\Gamma \times \Gamma'$ at (f, f) such that, the σ^*-probability that for some i, $f_i' \geqq 1 - 2\delta$ is at least $U(f) - \varepsilon_1 - \varepsilon$. Since σ^* is a randomized strategy available in Γ' at f, Corollary 4.1.1 applies to v, the indicator function of $[1 - 2\delta, 1]$, to yield a σ'' in Γ' at f under which

$$\operatorname{Prob}(\exists i: f_i \geqq 1 - 2\delta) \geqq U(f) - \varepsilon_1 - \varepsilon - \varepsilon^*\,. \tag{4.2.2}$$

Moreover, by hypothesis,

$$U'(1 - 2\delta) \geqq 1 - \varepsilon_2\,. \tag{4.2.3}$$

As is easily seen, $U'(f)$ is no less than the product of the left-hand sides of (4.2.2) and (4.2.3). Hence,

$$U'(f) \geqq U(f) - (\varepsilon + \varepsilon_1 + \varepsilon_2 + \varepsilon^*) . \tag{4.2.4}$$

Since ε^* is arbitrary, the proof is complete.

PROOF OF THEOREM 1. Given $\Gamma' = \Gamma^c$ and $\varepsilon > 0$, choose δ so that (4.2.3) holds with $\varepsilon_2 = \frac{1}{2}\varepsilon$, and let $\varepsilon' = \delta\varepsilon_2$. Now let Γ be subfair and satisfy $\Gamma \leqq \Gamma' + \varepsilon'$; let σ be an ε_1-optimal strategy in Γ at f. There is a σ' in Γ' at f such that $\sigma \leqq \sigma' + \varepsilon'$. According to the conclusion of Theorem 1*,

$$\begin{aligned} U'(f) &\geqq U(f) - (\tfrac{1}{2}\varepsilon + \varepsilon_1 + \varepsilon_2) \\ &= U(f) - (\varepsilon + \varepsilon_1) . \end{aligned} \tag{4.2.5}$$

Since ε_1 is arbitrary, the proof is complete.

5. Some remarks on the applicability and inapplicability of the continuity and stability theorems.

For an illustration of the applicability of the theorems offered here, consider red-and-black casinos Γ and $\Gamma' = \Gamma^c$ with parameters w and w',

$$0 < w' < w \leqq \tfrac{1}{2} . \tag{5.1}$$

Case 1. $v = v_1$, where $v_1(\theta)$ is the negative of the mean of θ. As is easily checked for v_1, $\Gamma \leqq \Gamma^c + \varepsilon$ for

$$\varepsilon = 2(w - w')/1 - 2w . \tag{5.2}$$

Hence, if the inequality in (5.1) is strict, the hypotheses of the theorems are satisfiable, and the conclusions therefore apply. In particular, if w and w' are much closer to each other than they are to $\frac{1}{2}$, Theorem 1 implies that U and U' are close to each other, and Theorem 1* yields an upper bound to their difference.

If, however, $w = \frac{1}{2}$, the hypothesis that $\sigma \leqq \sigma' + \varepsilon$ is not satisfiable for $v = v_1$. Moreover, no hypothesis can then guarantee the conclusion of Theorem 2. For no matter how close Γ' may be to Γ, there will certainly be a strategy σ, available in Γ at $\frac{1}{2}$, that increases the gambler's fortune, sometime in the future, to 1, with probability $\frac{1}{2}$, and that employs such tiny bets, that no strategy σ' available in Γ' at $\frac{1}{2}$ can imitate σ. For, if σ' employs bets significantly larger than does σ, it plainly does not imitate σ; whereas if σ' employs bets of size comparable to those employed by σ, then the probability that, under σ', the gambler's fortune increases to 1 is microscopic, as can easily be seen, for example with the help of [5].

Case 2. $v = v_2$, where $v_2(\theta)$ is one-half of the variance of θ.

As is easily verified, even if $w < \frac{1}{2}$, there is no finite ε for which $\Gamma \leqq \Gamma' + \varepsilon$, so Theorems 1 and 2 are inapplicable. By good fortune, even for $w \leqq \frac{1}{2}$, Theorem 1* is satisfied for suitable σ, σ', ε's and δ, thus yielding quantitative information, in particular, about the difference between $U'(f)$ and f.

Returning to more general houses Γ and Γ', it would be desirable to find a less restrictive notion of closeness for Γ and Γ' which would guarantee that U is within a prescribed ε of U'. Even the kind of continuity established here by Theorems 1 and 1* does not seem to be improvable to an assertion about uniform continuity, unless, possibly, a neighborhood of the fair casinos were deleted from the domain of the mapping $\Gamma \to U$. This can be contrasted with the fact, which we do not prove here, that once there no longer is insistence on a prescribed ε, there is definable even a very weak notion of convergence such that, if a sequence of subfair casinos Γ_n converges to a fair casino Γ, then $U_n(f)$ converges to $U(f) = f$.

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