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ON THE CONVERGENCE OF SEQUENCES OF BRANCHING PROCESSES

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It is shown that there is a close relationship between the convergence of a sequence of normalized Galton–Watson processes and the convergence of the rowsums of a certain triangular array of independent identically distributed random variables. Using this result some limit theorems by Jirina and Lamperti are strengthened.

1. Introduction. Let $\{Z_j\}_{j=0}^{\infty}$ denote the random variables of a Galton–Watson branching process with off-spring generating function g. If $Z_0 = k$, it is a well-known fact that the generating function of Z_n is given by

$$h_n(s) = [g(g(\cdots g(s)))]^k , \tag{1.1}$$

where there are n iterations on the right-hand side of (1.1). However, for large $n \in N$, it is in general difficult to perform the iterations and obtain an explicit expression for h_n. We are therefore interested in various approximations of h_n and the corresponding probability distribution. The pioneering works in this area are due to Feller [2] and Lamperti [8] and [9].

In order to give our approximation theorems an appropriate form, we shall consider a sequence of Galton–Watson processes. The random variables of the nth process are denoted by $\{Z_j^{(n)}\}_{j=0}^{\infty}$ and the corresponding generating function by g_n. Let us also define the continuous time processes

$$Y_n(t) = \frac{Z_{[nt]}^{(n)} - b_n}{c_n} , \qquad t \in [0, \infty) , \tag{1.2}$$

where $c_n > 0$ and $b_n \in R$ are normalizing constants and $Z_0^{(n)} = a_n \to \infty$.

If we disregard translations of the whole process when we study the limiting behavior of $\{Y_n(t);\, t \in [0, \infty)\}$ as $n \to \infty$, it is enough to consider the following two cases (cf. 9):

(A) $b_n = 0$ for all $n \in N$ and $a_n/c_n \to d > 0$ as $n \to \infty$. (For simplicity we shall always assume that $a_n = b_n$ for all $n \in N$)

(B) $a_n = b_n$ for all $n \in N$, $b_n/c_n \to \infty$ as $n \to \infty$ and $\inf_{n \in N} c_n > 0$.

The plan for the present paper is as follows: In Section 2 we make some preliminary remarks on Laplace transforms and generating functions. Section 3 is devoted to Case A above. We shall prove that under general conditions the sequence $\{Y_n(t);\, t \in [0, \infty)\}_{n \in N}$ converges to some limit process $\{Y(t);\, t \in [0, \infty)\}$,

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if and only if the rowsums in a certain triangular array of independent random variables converge. We shall also derive a simple differential equation for the generating function of $Y(t)$. These two results are then used to prove the Feller–Jirina limit theorem under slightly weaker conditions than those of Jirina [7] or Lindvall [10]. In fact we can give both necessary and sufficient conditions for convergence to a certain diffusion process. Furthermore, we can prove that convergence of the finite-dimensional distributions of $\{Y_n(t);\, t \in [0, 1]\}$ to a non-degenerate limit implies convergence in the function space $D[0, 1]$ of the corresponding sequence of random elements. In Section 4 we examine Case B and give convergence criteria similar to those of Section 3.

2. Some preliminary remarks on Laplace transforms and probability generating functions. Laplace transforms and (probability) generating functions are usually defined only for probability distributions concentrated on $[0, \infty)$. We shall make the following extended definition.

DEFINITION 2.1. For a probability measure μ on $(-\infty, \infty)$ we define its Laplace transform L by

$$L(\lambda) = \int_{-\infty}^{\infty} e^{-\lambda x}\, \mu(dx)\,,$$

for all $\lambda \geqq 0$ such that the integral is finite. The generating function of μ is defined by

$$g(s) = L(-\log s)\,,$$

for all $s \in (0, 1]$ such that the right hand side is defined.

The well-known continuity theorem for Laplace transforms of measures on $[0, \infty)$ (see [3] page 431) cannot without restrictions be extended to the generalized Laplace transforms in Definition 2.1. Here we shall only consider a special case which will cover the situation we are interested in.

LEMMA 2.1. *Let $\{Z_{n,1}, Z_{n,2}, \cdots, Z_{n,n}\}_n$ be a triangular array of random variables such that*

(i) *for each n, $Z_{n,1}, Z_{n,2}, \cdots, Z_{n,n}$ are* i.i.d.
(ii) $Z_{n,j} \geqq -1,\ j = 1, 2, \cdots, n,\ n = 1, 2, \cdots$.

With $S_n = \sum_{j=1}^{n} Z_{n,j}$ we then have

$$\sup\nolimits_{n \in N, \lambda \in [0,d]} E(\exp(-\lambda S_n)) < \infty\,, \qquad \textit{for all } d > 0\,,$$

provided $\sup_{n \in N} P(|S_n| > c) \to 0$ *as* $c \to \infty$.

PROOF. Let

$$\tau_s(x) = x \qquad \text{for} \quad |x| \leqq s$$
$$\quad = \pm s \qquad \text{for} \quad |x| \geqq s$$

and put $Z'_{n,j} = \tau_s(Z_{n,j})$, $Z''_{n,j} = Z_{n,j} - Z'_{n,j}$. Then, by [3] page 308 both

$$\{nE(Z'_{n,j})\}_{n \in N} \qquad \text{and} \qquad \{nE((Z'_{n,j})^2)\}_{n \in N} \tag{2.1}$$

are bounded sequences for all sufficiently large s. Since the $|Z'_{n,j}|$ are bounded by s, there exists a constant $K = K(d)$ such that

$$|\exp(-\lambda Z'_{n,j}) - 1 + \lambda Z'_{n,j}| \leqq K(Z'_{n,j})^2 , \qquad \lambda \in [0, d]$$

and

$$\begin{aligned} E(\exp(-\lambda Z'_{n,j})) &\leqq 1 + \lambda|E(Z'_{n,j})| + KE((Z'_{n,j})^2) \\ &\leqq \exp(d|E(Z'_{n,j})|) + KE((Z'_{n,j})^2) , \qquad \lambda \in [0, d] . \end{aligned}$$

If $S_n' = \sum_{j=1}^{n} Z'_{n,j}$ it now follows from 2.1 that

$$\sup_{n \in N, \lambda \in [0,d]} E(\exp(-\lambda S_n')) < \infty .$$

Observing that $Z'_{n,j} \geqq 0$ for $s \geqq 1$ we have proved the lemma. □

REMARK. Lemma 2.1 can easily be extended to triangular arrays $\{Z_{n,1}, Z_{n,2}, \cdots, Z_{n,k_n}\}_n$, where $\{k_n\}_n$ is a sequence of integers tending to infinity.

THEOREM 2.1. *Let $\{\mu_n\}_{n \in N}$ be a sequence of probability measures and $\{k_n\}_{n \in N}$ a sequence of integers tending to infinity, such that*

(i) *for each $n \in N$, $\mu_n(-\infty, -1) = 0$*
(ii) *there exists a probability measure μ such that*

$$(\mu_n)^{*k_n} \rightarrow_w \mu .$$

Then the Laplace transforms L_n and L of $(\mu_n)^{*k_n}$ and μ, respectively, exist and, for every real number $d > 0$ and compact set $C \subseteqq (0, \infty)$, it holds that

(a) $L_n(\lambda) \rightarrow L(\lambda) > 0$ *as $n \rightarrow \infty$, uniformly in $\lambda \in [0, d]$*
(b) $\sup_{n \in N, \lambda \in C} L_n'(\lambda) < \infty$.

PROOF. Let S_n and S be random variables with probability laws $(\mu_n)^{*k_n}$ and μ, respectively. Then, for all $a \in R$,

$$\begin{aligned} |L_n(\lambda) - L(\lambda)| \leqq |&E(\exp(-\lambda \max(S_n, a))) - E(\exp(-\lambda \max(S, a)))| \\ &+ E(\exp(-\lambda S_n) I_{[S_n \leqq a]}) + E(\exp(-\lambda S) I_{[S \leqq a]}) . \end{aligned}$$

By the usual continuity theorem for Laplace transforms the first term on the right-hand side of the inequality tends to zero as $n \rightarrow \infty$, and the convergence is uniform in $\lambda \in [0, d]$ for all $a \in R$. But for every $h > 0$

$$E(\exp(-\lambda S_n) I_{[S_n \leqq a]}) < \exp(ah) E(\exp(-(\lambda + h) S_n) .$$

Hence, Lemma 2.1 implies that

$$\sup_{n \in N} E(\exp(-\lambda S_n) I_{[S_n \leqq a]}) \rightarrow 0 \qquad \text{as} \quad a \rightarrow -\infty ,$$

uniformly in $\lambda \in [0, d]$. Finally, by the Helly–Bray lemma (see [11] page 180),

$$E(\exp(-\lambda S) I_{[S \leqq a]}) \leqq \lim_{n \rightarrow \infty} E(\exp(-\lambda S_n) I_{[S_n \leqq a]}) ,$$

for every $a \in R$ such that $P(S = a) = 0$, and we can easily complete the proof of the first assertion in Theorem 2.1. The assertion on the derivatives L_n' follows similarly by straight-forward estimations of

$$\frac{1}{h}\left(\int_{-\infty}^{-a} \exp(-(\lambda + h)x)\mu_n^{*k_n}(dx) - \int_{-\infty}^{-a} \exp(-\lambda x)\mu_n^{*k_n}(dx)\right)$$

$$= \int_{-\infty}^{-a} -x \exp(-\lambda x) \frac{\exp(-hx) - 1}{-hx} \mu_n^{*k_n}(dx) . \quad \square$$

3. Convergence of sequences of normalized, non-centered Galton–Watson processes. Let $Z_j^{(n)}$, $p_k^{(n)}$ and g_n have the same meaning as in the introduction and define random elements Y_n in $D[0, 1]$ by

$$Y_n(t) = \frac{Z_{[nt]}^{(n)}}{c_n} , \qquad t \in [0, 1], \quad n \in N, \tag{3.0}$$

where $Z_0^{(n)} = c_n$ are integers tending to infinity. Let also μ_n denote the probability measure giving mass $p_k^{(n)}$ to the point $(k - 1)/c_n$.

THEOREM 3.1. *Assume that there exists a probability measure μ such that*

$$(\mu_n)^{*nc_n} \rightarrow_w \mu , \qquad \textit{as } n \rightarrow \infty .$$

Then it holds that:

(a) *The finite-dimensional distributions of $\{Y_n(t); t \in [0, 1]\}$ converge, as n tends to infinity, to those of some possibly infinite process $\{Y(t); t \in [0, 1]\}$, which is a continuous-state branching process to which we have added the absorbing state $+\infty$,*

(b) *The function $B(s, t) = -\log F(s, t)$, where $\{F(s, t); s, t \in [0, 1]\}$ for fixed t denotes the generating function of $Y(t)$, is for every $s \in (0, 1)$ the unique solution of the differential equation*

$$\frac{dB(s, t)}{dt} = -\log L(B(s, t)) ; \qquad B(s, 0) = -\log s ,$$

if $\{L(\lambda); \lambda > 0\}$ denotes the Laplace trasform of μ.

(c) *For all $t > 0$, $P(Y(t) = +\infty) > 0$ if and only if*

$$-\infty < \int_0^\delta \frac{d\lambda}{\log L(\lambda)} < 0 ,$$

for all sufficiently small $\delta > 0$. In particular $Y(t)$ is almost surely finite if the expectation of μ is finite.

In order to simplify the proof of Theorem 3.1, we start by proving three lemmas.

LEMMA 3.1. *Let $F_n(s, t)$ denote the generating function of $Y_n(t)$, $0 \leqq s, t \leqq 1$. Then $F_n(s, t)$ is a monotone function of t for fixed s.*

PROOF. Let $g_{n,k}$ denote the function g_n iterated k times. Then

$$F_n\left(s, \frac{k}{n}\right) = [g_{n,k}(s^{1/c_n})]^{c_n} .$$

In order to complete the proof of the lemma, we need only show that $g_{n,k}(u)$ is monotone as a function of k. But $g_{n,k}(u)$ is non-decreasing for $u \leqq q_n$ and non-increasing for $u \geqq q_n$, if q_n denotes the smallest nonnegative root of the equation $g_n(s) = s$. □

LEMMA 3.2. *Let* $\{H(x); x \in (0, 1)\}$ *be a function satisfying a Lipschitz condition*

$$|H(x) - H(y)| \leqq K_C|x - y| , \qquad x, y \in C ,$$

for each compact set $C \subset (0, 1)$. *Then the integral equation*

$$A(t) = c + \int_0^t A(u)H(A(u))\, du , \qquad t \in [0, T] ,$$

has at most one solution.

PROOF. The integral equation is equivalent to the differential equation

$$A'(t) = P(A(t)) , \qquad A(0) = c ,$$

where $P(x) = xH(x)$ is also locally Lipschitz. Hence, $A(t) \equiv c$ if $P(c) = 0$, while

$$\int_c^{A(t)} \frac{dy}{P(y)} = t$$

if $P(c) \neq 0$ □

LEMMA 3.3. *For every fixed* $s \in (0, 1)$,

$$\inf_{n \in N} F_n(s, 1) > 0 \qquad \textit{and} \qquad \sup_{n \in N} F_n(s, 1) < 1 .$$

PROOF. Denote the generating function of $(\mu_n)^{*(nc_n)}$ by $G_n(s)$, $s \in (0, 1]$. Then

$$G_n(s) = \left[\frac{g_n(s^{1/c_n})}{s^{1/c_n}}\right]^{nc_n} .$$

Applying Theorem 2.1 we find that μ has a finite and continuous generating function $G(s)$, $s \in (0, 1]$. Furthermore, for every $d > 0$,

$$G_n(s) = \left[\frac{g_n(s^{1/c_n})}{s^{1/c_n}}\right]^{nc_n} \to G(s) \qquad \text{as } n \to \infty , \tag{3.1}$$

uniformly in $s \in [d, 1]$. Let us now define $f_n(s)$ by

$$f_n(s) = F_n\left(s, \frac{1}{n}\right) = (g_n(s^{1/c_n}))^{c_n} = s(G_n(s))^{1/n} . \tag{3.2}$$

The branching process property then yields the identity

$$F_n\left(s, \frac{k}{n}\right) = f_n(f_n(\cdots f_n(s))) , \tag{3.3}$$

where there are k iterations on the right-hand side. By (3.2) and (3.3) we get

$$\begin{aligned} F_n\left(s, \frac{j}{n}\right) - F_n\left(s, \frac{j-1}{n}\right) \\ = F_n\left(s, \frac{j-1}{n}\right)\left[\left(G_n\left(F_n\left(s, \frac{j-1}{n}\right)\right)\right)^{1/n} - 1\right] \end{aligned} \tag{3.4}$$

and

$$(3.5)\qquad F_n\left(s, \frac{k}{n}\right) = s + \sum_{j=1}^{k} F_n\left(s, \frac{j-1}{n}\right)\left[\left(G_n\left(F_n\left(s, \frac{j-1}{n}\right)\right)\right)^{1/n} - 1\right]$$

$$= s + \int_0^{k/n} F_n(s, t)n((G_n(F_n(s, t)))^{1/n} - 1)\,dt\,.$$

First we examine the identity (3.4). By (3.1) there exists a real number $d_0 \in (0, 1)$ such that

$$\tfrac{1}{2} \leqq G_n(s) \leqq 2\,, \qquad n \in N,\ s \in [1 - d_0, 1]\,.$$

Hence there exists an integer n_0 such that

$$(3.6)\qquad \left|F_n\left(s, \frac{j}{n}\right) - F_n\left(s, \frac{j-1}{n}\right)\right| \leqq F_n\left(s, \frac{j-1}{n}\right)\frac{2\log 2}{n} \leqq \frac{2\log 2}{n}\,,$$

provided $n \geqq n_0$ and $F_n(s, (j-1)/n) \geqq 1 - d_0$. Observing that $F_n(s, 0) = s$, (3.6) yields

$$(3.7)\qquad \lim_{\delta\to 0} \sup_{0\leqq j/n\leqq\delta} \left|F_n\left(s, \frac{j}{n}\right) - s\right| = 0\,,$$

for every $s \in (1 - d_0, 1]$. In particular

$$(3.8)\qquad \lim_{\delta\to 0} \sup_{0\leqq j/n\leqq\delta} \left|\frac{F_n\left(1 - \frac{d_0}{4}, \frac{j}{n}\right) - F_n\left(1 - \frac{d_0}{2}, \frac{j}{n}\right)}{\frac{d_0}{4}} - 1\right| = 0\,.$$

Recalling that $F_n'(u, j/n)$ is a non-decreasing function of $u \in [0, 1]$, (3.8) and the mean value theorem show that

$$(3.9)\qquad \liminf_{\delta\to 0} \sup_{0\leqq j/n\leqq\delta} F_n'\left(1 - \frac{d_0}{4}, \frac{j}{n}\right) \geqq 1\,.$$

Furthermore,

$$F_n'\left(1 - \frac{d_0}{4}, \frac{j}{n}\right) - F_n'\left(1 - \frac{d_0}{2}, \frac{j}{n}\right)$$

$$\leqq \frac{4}{d_0}\left\{\int_{1-d_0/4}^{1} F_n'\left(u, \frac{j}{n}\right)du - \int_{1-3d_0/4}^{1-d_0/2} F_n'\left(u, \frac{j}{n}\right)\right\}$$

$$= \frac{4}{d_0}\left\{1 - F_n\left(1 - \frac{d_0}{4}, \frac{j}{n}\right) - F_n\left(1 - \frac{d_0}{2}, \frac{j}{n}\right) + F_n\left(1 - \frac{3d_0}{4}, \frac{j}{n}\right)\right\}\,,$$

which by 3.7 implies that

$$(3.10)\qquad \lim_{\delta\to 0} \sup_{0\leqq j/n\leqq\delta} F_n'\left(1 - \frac{d_0}{4}, \frac{j}{n}\right) - F_n'\left(1 - \frac{d_0}{2}, \frac{j}{n}\right) = 0\,.$$

But the second derivatives $F_n''(u, j/n)$ are also non-decreasing. Hence 3.10 can be strengthened to

$$\lim_{\delta\to 0} \sup_{0\leqq j/n\leqq\delta} F_n'\left(1 - \frac{d_0}{4}, \frac{j}{n}\right) - F_n'\left(u, \frac{j}{n}\right) = 0\,, \qquad u \in \left[0, 1 - \frac{d_0}{4}\right]$$

and, by (3.9), we can choose δ_0 so small that

$$F_n'\left(u, \frac{j}{n}\right) \geqq \tfrac{1}{2} \qquad \text{for all } u \in [0, 1] \text{ and } j/n \leqq \delta_0 .$$

In view of (3.3) this implies that we can write $F_n(s, 1)$ on the form

$$F_n(s, 1) = g_m(g_{m-1}(\cdots g_1(s))) ,$$

where $m \leqq 1/\delta_0 + 1$ and $g_j(s) \geqq s/2$, $j = 1, 2, \cdots, m$, and, consequently,

$$\inf_{n \in N} F_n(s, 1) > 0 , \qquad s \in (0, 1) .$$

Similar arguments show that there exists a real number $\delta_1 > 0$ such that

$$F_n\left(s, \frac{j}{n}\right) \leqq \frac{s + 1}{2} ,$$

for all $s \in [0, 1]$ and $j/n \leqq \delta_1$. But this inequality implies that

$$\sup_{n \in N} F_n(s, 1) < 1 ,$$

for each $s \in (0, 1)$. □

PROOF OF THEOREM 3.1. Recalling that $F_n(s, t)$ is a monotone function of $t \in [0, 1]$ and $F_n(s, 0) = s$, Lemma 3.3 can be strengthened to

$$\inf_{n \in N, t \in [0,1]} F_n(s, t) > 0 ,$$

for every $s \in (0, 1)$. Hence,

$$\sup_{n \in N, t \in [0,1]} |n((G_n(F_n(s, t)))^{1/n} - 1)| = K(s) < \infty ,$$

and by (3.5)

$$|F_n(s, u) - F_n(s, t)| \leqq K(s)(|u - t| + 2/n) . \tag{3.11}$$

Since $F_n(s, t)$ is a non-decreasing function of s, we can select a subsequence $\{n'\} \subset N$ such that $\lim_{n' \to \infty} F_{n'}(s, t) = F(s, t)$ exists for all $s \in (0, 1)$ and all rational $t \in [0, 1]$. But then (3.11) implies that

$$\lim_{n' \to \infty} F_{n'}(s, t) = F(s, t) \qquad \text{exists for all} \quad s \in (0, 1) , \quad t \in [0, 1] .$$

Passing to the limit in (3.5) we obtain

$$F(s, t) = s + \textstyle\int_0^t F(s, u) \log G(F(s, u))\, du , \qquad t \in [0, 1],\ s \in (0, 1) , \tag{3.12}$$

where $0 < F(s, t) < 1$ and G is the strictly positive continuous function defined in (3.1).

By the convergence theorem for Laplace transforms, $F(s, t)$ is, for each $t \in [0, 1]$, the generating function of some possibly defective probability distribution. Furthermore, Theorem 2.1 and Lemma 3.2 shows that the integral equation (3.12) determines this distribution uniquely. Applying Helly's selection theorem we can then see that $\{Y_n(t)\}_{n \in N}$, for each $t \in [0, 1]$, converges weakly to some possibly infinite random variable. The convergence of the finite-dimensional distributions of $\{Y_n(t); t \in [0, 1]\}_{n \in N}$ to those of some continuous-state branching process to which we have added the absorbing state $+\infty$ is due to Lamperti and can be proved as in [9] page 280. This completes the proof of the first part of Theorem 3.1. The second part follows by differentiation of (3.12). In order to prove the last part of Theorem 3.1 we recall (3.1). It shows that either $G(s) \geqq 1$ in the whole interval (0, 1) or $G(s) \leqq 1$ in some left-neighborhood of 1. In the first case $F(s, t) \geqq F(s, 0) = s$ and obviously.

$$\lim_{s\to 1} F(s, t) = 1\,, \qquad t \in [0, 1]\,.$$

In the second case we notice that $A(t) = \lim_{s\to 1} F(s, t)$ satisfies the integral equation

$$A(t) = 1 + \int_0^t A(u) \log G(A(u))\, du\,, \qquad t \in [0, 1]\,,$$

which can be transformed into the differential equation

$$\text{(3.13)} \qquad \frac{\partial B(t)}{\partial t} = -\log G(\exp(-B(t))) = -\log L(B(t))\,, \qquad t \in [0, 1]\,,$$

where $B(t) = -\log A(t)$ and L is the Laplace transform of μ. However, by assumption $-\log L(\lambda) \geqq 0$ in some interval $[0, \varepsilon]$ and $-\log L(0) = 0$. Hence (3.13) has a unique solution $B(t) \equiv 0$ through the origin, if

$$\int_0^\delta \frac{d\lambda}{-\log L(\lambda)} = +\infty\,, \qquad \text{for all sufficiently small } \delta > 0\,.$$

If the integral is convergent for some $\delta > 0$, there is a solution $B(t)$, $t \in [0, 1]$, which is strictly positive in the interval (0, 1] and obviously

$$\lim_{s\to 1} F(s, t) \leqq A(t) < 1\,. \qquad \square$$

Theorem 3.1 can be interpreted as an invariance principle. The weak limit of $\{Y_n(t);\, t \in [0, 1]\}_{n\in N}$ depends on the sequence of reproduction laws only through the measure μ. We shall apply this invariance principle to the Feller–Jirina limit theorem and prove it under slightly weaker conditions than those of Jirina. Actually the conditions below are the weakest possible as will be shown after Theorem 3.3. The same result can also be obtained by solving the differential equation in Theorem 3.1.

THEOREM 3.2. *Let, for each $n \in N$, $\{Z_j^{(n)}\}_{j=0}^\infty$ be a Galton–Watson process with off-spring generating function $g_n(s) = \sum_{k=0}^\infty p_k^{(n)} s^k$ and define stochastic processes $\{Y_n(t);\, t \in [0, 1]\}$ by*

$$Y_n(t) = \frac{Z_{[nt]}^{(n)}}{n}\,, \qquad t \in [0, 1],\ n \in N \ \textit{and}\ Z_0^{(n)} = n\,.$$

Assume that

(i) $m_n = \sum_{k=0}^\infty k p_k^{(n)} = 1 + \alpha_n/n$, *where* $\alpha_n \to \alpha$ *as* $n \to \infty$,
(ii) $\sigma_n^2 = \sum_{k=0}^\infty (k - m_n)^2 p_k^{(n)} \to \beta > 0$ *as* $n \to \infty$,
(iii) $\sum_{k>tn} k^2 p_k^{(n)} \to 0$ *as* $n \to \infty$, *for all* $t > 0$.

Then the finite-dimensional distributions of $\{Y_n(t);\, t \in [0, 1]\}$ converge to those of a process $\{Y(t);\, t \in [0, 1]\}$, which is a nonnegative diffusion process with drift $x \to \alpha x$

and diffusion coefficient $x \to \beta x$. *The Laplace transform of* $Y(t)$ *is given by*

$$E(\exp(-\lambda Y(t))) = \exp(-\psi_t(\lambda)) ,$$

where

$$\psi_t(\lambda) = \frac{\lambda}{1 + \frac{\beta t \lambda}{2}} \qquad \text{if} \quad \alpha = 0$$

$$= \frac{\lambda e^{\alpha t}}{1 - \frac{\beta\lambda}{2\alpha}(1 - e^{\alpha t})} \qquad \text{if} \quad \alpha \neq 0 .$$

PROOF. Let μ_n be the measure giving mass $p_k^{(n)}$ to the point $(k-1)/n$. Then

$$m_n' = \int x\mu_n(dx) = \frac{\alpha_n}{n^2}$$

$$(\sigma_n')^2 = \int (x - m_n')^2 \mu_n(dx) = \frac{\sigma_n^2}{n^2}$$

$$n^2 \cdot \int_{|x|>t} x^2 \mu_n(dx) \to 0 \qquad \text{as } n \to \infty, \text{ for all } t > 0 .$$

By the central limit theorem for triangular arrays, $(\mu_n)^{*n^2}$ tends weakly to a normal law with mean α and variance β. Applying Theorem 3.1 we find that the finite-dimensional distributions of $\{Y_n(t); t \in [0, 1]\}$ converge to those of a continuous-state branching process. It remains to prove that $\{Y_n(t)\}_n$ has the desired limit. By the invariance principle above it is even enough to find one particular sequence of distributions $\{p_k^{(n)}\}_k$ such that (i), (ii) and (iii) are satisfied and $\{Y_n(t)\}_n$ has the desired limit. For simplicity we shall only consider the case $t = 1$ and $\alpha \neq 0$. We shall choose

$$p_k^{(n)} = b_n c_n^{k-1} , \qquad k = 1, 2, 3, \cdots$$

$$p_0^{(n)} = 1 - p_1^{(n)} - p_2^{(n)} - \cdots .$$

Then $Y_n(1)$ has the generating function $(f_n(s^{1/n}))^n$, where (see [6] page 9)

$$f_n(s) = 1 - m^n\left(\frac{1 - s_0}{m^n - s_0}\right) + \frac{m^n\left(\frac{1 - s_0}{m^n - s_0}\right)^2 s}{1 - \left(\frac{m^n - 1}{m^n - s_0}\right)s} \tag{3.14}$$

$$m = f_n'(1) = \frac{b_n}{(1 - c_n)^2} \qquad \text{and} \qquad s_0 = \frac{1 - b_n - c_n}{c_n(1 - c_n)} .$$

The geometric distributions $\{p_k^{(n)}\}_k$ have variance $\sigma^2 = [b_n(1 - c_n^2) - b_n^2]/(1 - c_n)^4$. If we choose $m = \exp(\alpha/n)$ and $c_n = \beta/(2 + \beta)$, we can easily show that the conditions (i), (ii) and (iii) are satisfied. Furthermore

$$\lim_{n\to\infty} n(1 - s_0) = \frac{2\alpha}{\beta}$$

and from this the limit theorem will follow by (3.14) and some simple calculus. □

LEMMA 3.4. *Let $\{Z_j^{(n)}\}_{j=0}^{\infty}$, $\{Y_n(t);\ t \in [0, 1]\}$ and μ_n be defined as in the beginning of this section. If $Y_n(1)$ converges in distribution to some random variable Y, which is not identically zero, then $\{(\mu_n)^{*nc_n}\}_{n\in N}$ is a tight family of probability distributions.*

PROOF. Differentiating (3.3) with respect to s we obtain

$$F_n'\left(s, \frac{k}{n}\right) = \prod_{j=0}^{k-1} f_n'\left(F_n\left(s, \frac{j}{n}\right)\right).$$

Since $f_n'(s)$ is a non-decreasing function of $s \in [0, 1)$, Lemma 3.1 implies that, for $k = 0, 1, 2, \cdots, n$,

$$\{f_n'(\min(s, F_n(s, 1)))\}^k \leqq F_n'\left(s, \frac{k}{n}\right) \leqq \{f_n'(\max(s, F_n(s, 1)))\}^k . \tag{3.15}$$

However, the continuity theorem for Laplace transforms shows that

$$F_n(s, 1) \to F(s) \qquad \text{as } n \to \infty$$

and

$$F_n'(s, 1) \to F'(s) \qquad \text{as } n \to \infty ,$$

where $F(s)$ is the generating function of Y. In particular, since the convergence to $F(s)$ is uniform on $[0, 1]$.

$$\inf_{n\in N} F_n(s, 1) \to 1 \qquad \text{as } s \to 1 . \tag{3.16}$$

Obviously it is no restriction to assume that $F_n(s, 1)$ is not identically 1 for any $n \in N$. Then the convergence of the Laplace transforms also implies that

$$\sup_{n\in N} F_n(s, 1) < 1 , \qquad s \in (0, 1)$$
$$\inf_{n\in N} F_n'(s, 1) > 0 , \qquad s \in (0, 1) .$$

Recalling that $f_n'(s)$ is non-decreasing in s, (3.15) yields

$$\inf_{n\in N} (f_n'(s))^n \geqq \inf_{n\in N} F_n'(\tfrac{1}{2}, 1) > 0 ,$$

for all $s \geqq \max(\frac{1}{2}, \sup_{n\in N} F_n(\frac{1}{2}, 1))$. By (3.15) and (3.16) this implies that

$$\inf_{n\in N, 0\leqq k\leqq n} F_n'\left(s, \frac{k}{n}\right) = I(s) \geqq I(1 - \delta) > 0 \tag{3.17}$$

for some $\delta > 0$ and all $s \in [1 - \delta, 1]$. But Lemma 3.1 and the mean-value theorem show that

$$\begin{aligned}|F_n(s, 1) - s| &= \sum_{k=0}^{n-1} \left|F_n\left(s, \frac{k+1}{n}\right) - F_n\left(s, \frac{k}{n}\right)\right| \\ &= \sum_{k=0}^{n-1} \left|F_n\left(f_n(s), \frac{k}{n}\right) - F_n\left(s, \frac{k}{n}\right)\right| \\ &\geqq n|f_n(s) - s| I(\min(s, \inf_{n\in N} f_n(s))) .\end{aligned}$$

Furthermore, by (3.16) and Lemma 3,1,

$$\inf_{n\in N} f_n(s) \geqq 1 - \delta ,$$

for all s sufficiently close to 1. Hence, by (3.16) again,

$$n(f_n(s) - s) \to 0 \qquad \text{as } s \to 1, \tag{3.18}$$

uniformly in $n \in N$. If $g_n(s)$ as before denotes the generating function of the reproduction law of the nth Galton–Watson process, (3.18) can be rewritten as

$$n((g_n(s^{1/c_n}))^{c_n} - s) \to 0 \qquad \text{as } s \to 1.$$

Hence

$$n\left\{\left[\frac{g_n(s^{1/c_n})}{s^{1/c_n}}\right]^{c_n} - 1\right\} \to 0 \qquad \text{as } s \to 1,$$

uniformly in $n \in N$. Finally some simple culculus yields

$$G_n(s) = \left[\frac{g_n(s^{1/c_n})}{s^{1/c_n}}\right]^{nc_n} \to 1 \qquad \text{as } s \to 1, \tag{3.19}$$

uniformly in $n \in N$. Here $G_n(s)$ is the generating function of the measure $\nu_n = (\mu_n)^{*nc_n}$. By (3.19) it follows immediately that

$$\nu_n(-\infty, -\lambda) \to 0 \qquad \text{as } \lambda \to \infty, \tag{3.20}$$

uniformly in $n \in N$. But $G_n(s)$ can also be written on the form

$$G_n(s) = \sum_{j=1}^{nc_n} q_{-j}^{(n)} s^{-j/c_n} + \sum_{j=0}^{\infty} q_j^{(n)} s^{j/c_n} = A_n(s) + B_n(s).$$

Differentiating with respect to s we find that $A_n'(s)$ is non-decreasing in the interval (0, 1). It remains to prove that

$$\nu_n(\lambda, \infty) \to 0 \quad \text{as } \lambda \to \infty, \text{ uniformly in } n \in N. \tag{3.21}$$

Assume that the converse holds true. Then there exists an $\varepsilon > 0$ such that

$$\lim_{\delta\to0} \sup_{n\in N} \{B_n(1) - B_n(1-\delta)\} > \varepsilon.$$

By (3.19) we also obtain

$$\lim_{\delta\to0} \sup_{n\in N} \{A_n(1-\delta) - A_n(1)\} > \varepsilon. \tag{3.22}$$

But $A_n'(s)$ is increasing. Hence,

$$A_n(1-k\delta) - A_n(1) \geqq k(A_n(1-\delta) - A_n(1))$$

and by (3.22)

$$\lim_{\delta\to0} \sup_{n\in N} \{A_n(1-\delta) - A_n(1)\} > k\varepsilon$$

for every $k \in N$, which contradicts (3.19). But this means that (3.21) must hold true and $\{\nu_n\}_{n\in N}$ must be tight. □

THEOREM 3.3. *Let, for each $n \in N$, $\{Z_j^{(n)}\}_{j=0}^{\infty}$ denote a Galton–Watson process with reproduction law $\{p_k^{(n)}\}_k$. Define continuous-time processes $\{Y_n(t); t \in [0, 1]\}$ by*

$$Y_n(t) = \frac{Z_{[nt]}^{(n)}}{c_n}, \qquad Z_0^{(n)} = c_n,$$

where $c_n \to \infty$ *as* $n \to \infty$, *and let* μ_n *be the measure giving mass* $p_k^{(n)}$ *to the point* $(k-1)/c_n$. *Assume that*

(i) $\{Y_n(t)\}_{n \in N}$ *converges weakly to some random variable* $Y(t)$ *for every* $t \in [0, 1]$
(ii) $P(Y(1) > 0) > 0$.

Then the sequence $\{(\mu_n)^{*nc_n}\}_{n \in N}$ *converges weakly to some probability measure* μ, *which is uniquely determined by the probability laws of* $Y(t)$.

PROOF. Let ν_1 and ν_2 be two limit distributions of $\{(\mu_n)^{*nc_n}\}_n$ and denote their Laplace transforms by $L_1(\lambda)$ and $L_2(\lambda)$. The differential equation in Theorem 3.1 then shows that L_1 and L_2 must coincide. Hence, the tight sequence $\{(\mu_n)^{*nc_n}\}_{n \in N}$ has only one limit distribution, and it must be convergent. The uniqueness of μ is obvious. □

REMARK. Let us return to the Feller–Jirina case treated in Theorem 3.2. In the proof of that theorem we found that a sufficient condition for the sequence $\{Y_n(t); t \in [0, 1]\}_{n \in N}$ to converge to the specified diffusion is that $\{(\mu_n)^{*nc_n}\}_{n \in N}$ converges weakly to a normal law with mean α anp variance β. By Theorem 3.3 this condition is also necessary. Hence condition (iii) in Theorem 3.2 is necessary for the convergence, if (i) and (ii) hold.

Let us now turn to a discussion of weak convergence in the function space $D[0, 1]$ of a sequence of normalized Galton–Watson processes. By a famous theorem due to Prohorov a sequence $\{Y_n\}_n$ of random elements in $D[0, 1]$ is weakly convergent, if $\{Y_n\}_n$ is tight in $D[0, 1]$ and the finite-dimensional distributions of $\{Y_n(t); t \in [0, 1]\}$ converge weakly as $n \to \infty$. In view of the previous discussions in this section it is enough to consider tightness of $\{Y_n\}_n$. The main tool in proving tightness will be Theorem 2.2′ in [5], which we state as a lemma.

LEMMA 3.5. *Let for each* $n \in N$ $\{X_{n,0}, X_{n,1}, \cdots, X_{n,n}\}$ *be a real-valued Markov chain with* $X_{n,0} = 0$ *and transition probabilities* $p^{(n)}(a, \cdot)$ *satisfying*

$$P(X_{n,k+1} \in E \mid X_{n,k} = a) = p^{(n)}(a, E)$$

for all $a \in R$ *and all Borel sets* E. *Let also* $\nu_a^{(n)}$ *denote a measure defined by* $\nu_a^{(n)}(E) = p^{(n)}(a, a + E)$ *for all Borel sets* E, *and let* Y_n *denote a random element in* $D[0, 1]$ *defined by*

$$\begin{aligned} Y_n(t) &= X_{n,k}\,, && \frac{k}{n} \leqq t < \frac{k+1}{n} \\ &= X_{n,n}\,, && t = 1\,. \end{aligned}$$

Then the sequence $\{Y_n\}_n$ *is tight in* $D[0, 1]$, *if*

(i) $P(\sup_{0 \leq t \leq 1} |Y_n(t)| > \lambda) \to 0$ *as* $\lambda \to \infty$, *uniformly in* $n \in N$
(ii) $\{(\nu_a^{(n)})^{*n}\}_{a \in C, n \in N}$ *is tight for every compact subset* C *of the real line.*

If $Y_n(t)$ and μ_n are defined as in the beginning of this section, it is obvious that

$$\nu_a^{(n)} = (\mu_n)^{*ac_n}\,, \qquad a \cdot c_n = 0, 1, 2, \cdots.$$

Therefore, condition (ii) in Lemma 3.5 is equivalent to

(ii′) $\{(\mu_n)^{*nc_n}\}_{n\in N}$ is tight.

Let us now show that condition (i) is automatically fulfilled, if the assumptions of Theorem 3.3 hold true. We start by proving a simple lemma.

LEMMA 3.6. *For any given $\varepsilon > 0$ and $a > 0$ there exists an integer n_0 such that*

$$P(\textstyle\sum_{j=1}^{n_0} Y_j \geqq a) > \frac{1}{2}$$

for any sequence of nonnegative independent random variables $Y_1, Y_2, \cdots$ satisfying

$$P(Y_i > \varepsilon) > \varepsilon\,, \qquad i = 1, 2, 3, \cdots.$$

PROOF. Obviously $E(Y_i) > \varepsilon^2$. Furthermore, it is no restriction to assume that $P(Y_i \leqq 1) = 1$ so that $\mathrm{Var}\,(Y_i) \leqq 1$. A simple application of Chebyshev's inequality then yields

$$P(|\textstyle\sum_{i=1}^k Y_i - \sum_{i=1}^k E(Y_i)| > k^{\frac{2}{3}}) \leqq k^{-\frac{1}{3}}\,, \qquad k = 1, 2, 3, \cdots.$$

The remaining part of the proof is trivial. □

In Lemma 3.3 we proved that

$$\sup_{n\in N} F_n(s, 1) < 1\,, \qquad s \in (0, 1)\,,$$

which in view of Lemma 3.1 can be extended to

$$\sup_{n\in N, t\in[0,1]} F_n(s, t) < 1\,, \qquad s \in (0, 1)\,.$$

In terms of the distribution of $Y_n(t)$ this means that there exists an $\varepsilon > 0$ such that

$$P(Y_n(t) > \varepsilon) > \varepsilon \qquad n \in N,\ t \in [0, 1]\,.$$

The branching process character of $\{Y_n(t); t \in [0, 1]\}$ and Lemma 3.6 then implies that, for any $a > 0$, there exists a real number $b > 0$ such that

$$P(Y_n(1) \geqq a \mid Y_n(t) = c) > \frac{1}{2}$$

for all $n \in N$, $t \in [0, 1]$ and all $c \geqq b$. Applying the (strong) Markov property we obtain

$$P(Y_n(1) \geqq a) \geqq \frac{1}{2} P(\sup_{0\leqq t\leqq 1} Y_n(t) \geqq b)\,. \tag{3.23}$$

However, the left-hand side of (3.23) tends to zero as a tends to infinity, uniformly in $n \in N$. Hence,

$$P(\sup_{0\leqq t\leqq 1} Y_n(t) > \lambda) \to 0 \qquad \text{as } \lambda \to \infty, \text{ uniformly in } n \in N\,.$$

In other words we have now proved that convergence of the one-dimensional distributions of $\{Y_n(t); t \in [0, 1]\}$ to a non-degenerate limit implies weak convergence in $D[0, 1]$.

We shall end this discussion by summarizing all convergence results of this section.

THEOREM 3.4. *Let, for each $n \in N$, $\{Z_j^{(n)}\}_{j=0}^{\infty}$ denote a Galton–Watson process with reproduction law $\{p_k^{(n)}\}_k$. Define random elements Y_n in $D[0, 1]$ by*

$$Y_n(t) = \frac{Z_{[nt]}^{(n)}}{c_n}, \qquad t \in [0, 1],\ n \in N,$$

where $Z_0^{(n)} = c_n$ are positive integers tending to infinity. Let μ_n be the probability measure giving mass $p_k^{(n)}$ to the point $(k-1)/c_n$. Then the following statements are equivalent.

(i) *There exists a probability measure μ such that*

(a) $(\mu_n)^{*nc_n} \to_w \mu$ *as $n \to \infty$.*

(b) $\int_0^\delta d\lambda/\log L(\lambda)$ *is not negative and finite for any $\delta > 0$, if L denotes the Laplace transform of μ.*

(ii) *The one-dimensional distributions of $\{Y_n(t);\ t \in [0, 1]\}$ converge to those of an almost surely finite process $\{Y(t);\ t \in [0, 1]\}$ with $P(Y(1) > 0) > 0$*

(iii) *$\{Y_n\}_{n \in N}$ converges weakly in $D[0, 1]$ to a continuous-state branching process.*

4. Convergence of sequences of normalized and centered Galton–Watson processes. Let $\{Z_j^{(n)}\}_{j=0}^{\infty}$, $\{p_k^{(n)}\}_k$, $g_n(s)$ and μ_n have the same meaning as in Section 3. Define random elements Y_n in $D[0, 1]$ by

$$Y_n(t) = \frac{Z_{[nt]}^{(n)} - b_n}{c_n}, \qquad 0 \leqq t \leqq 1,$$

where $Z_0^{(n)} = b_n$, $b_n/c_n \to \infty$ as $n \to \infty$ and $\inf_{n \in N} c_n > 0$. We shall now, exactly as in Section 3, prove that the convergence of the sequence $\{Y_n(t);\ t \in [0, 1]\}_n$ is closely related to the convergence of the rowsums of a certain triangular array of independent random variables.

THEOREM 4.1. *Assume that there exists a probability measure μ such that*

$$(\mu_n)^{*nb_n} \to_w \mu \qquad \text{as } n \to \infty.$$

Then the finite-dimensional distributions of $\{Y_n(t);\ t \in [0, 1]\}$ converge to those of a stochastic process $\{Y(t);\ t \in [0, 1]\}$ with independent increments. Furthermore, $Y(t)$ has the generating function $(G(s))^t$, where $G(s)$ is the generating function of μ.

PROOF. Let G_n and $F_n(\cdot, t)$ denote the generating function of $(\mu_n)^{*nb_n}$ and $Y_n(t)$, respectively. Then

$$G_n(s) = \left[\frac{g_n(s^{1/c_n})}{s^{1/c_n}}\right]^{nb_n} \tag{4.1}$$

and

$$F_n\left(s, \frac{k}{n}\right) = [g_n(g_n(\cdots g_n(s^{1/c_n})))]^{b_n} s^{-b_n/c_n}, \tag{4.2}$$

where there are k iterations on the right-hand side of (4.2). In order to simplify (4.1) and (4.2) we introduce the notation

$$f_n(s) = [g_n(s^{1/c_n})]^{c_n} = sG_n(s)^{c_n/nb_n}$$

and let $f_{n,k}$ and $g_{n,k}$ denote the kth iterates of f_n and g_n, respectively. Obviously the branching process property yields

$$f_n(f_{n,k-1}(s)) = f_{n,k}(s) = [g_{n,k}(s^{1/c_n})]^{c_n}$$

which, in turn, implies that

$$F_n\left(s, \frac{k}{n}\right) = [f_{n,k}(s)]^{b_n/c_n} s^{-b_n/c_n}$$

or, equiualently,

$$f_{n,k}(s) = sF_n\left(s, \frac{k}{n}\right)^{c_n/b_n} .$$

Now we can easily see that

$$\begin{aligned} F_n\left(s, \frac{k+1}{n}\right) - F_n\left(s, \frac{k}{n}\right) &= s^{-b_n/c_n}[(f_n(f_{n,k}(s)))^{b_n/c_n} - f_{n,k}(s)^{b_n/c_n}] \\ &= \left[\frac{f_{n,k}(s)}{s}\right]^{b_n/c_n} [G_n(f_{n,k}(s))^{1/n} - 1] \qquad (4.3) \\ &= F_n\left(s, \frac{k}{n}\right)\left[G_n\left(sF_n\left(s, \frac{k}{n}\right)^{c_n/b_n}\right)^{1/n} - 1\right]. \end{aligned}$$

Summing up the equations in (4.3) with respect to k we obtain

$$F_n\left(s, \frac{k}{n}\right) = 1 + \int_0^{k/n} F_n(s, t)n[G_n(sF_n(s, t)^{c_n/b_n})^{1/n} - 1]\, dt . \tag{4.4}$$

By Theorem 2.1 G_n converges uniformly on compact subsets of (0, 1] to the continuous strictly positive function G. Hence, for all $d \in (0, 1]$,

$$\sup_{s\in[d,1], n\in N} n|G_n(s)^{1/n} - 1| = M(d) < \infty . \tag{4.5}$$

However, (4.3) shows that

$$\left|F_n\left(s, \frac{k+1}{n}\right) - F_n\left(s, \frac{k}{n}\right)\right| \leqq F_n\left(s, \frac{k}{n}\right)\frac{M(s^2)}{n} , \tag{4.6}$$

provided $F_n(s, k/n) \geqq s^{b_n/c_n}$. Furthermore, $F_n(s, t)$ is a monotone function of $t \in [0, 1]$, since Lemma 3.1 holds also in this case. If $F_n(s, t)$ is increasing,

$$F_n\left(s, \frac{k+1}{n}\right) \leqq F_n\left(s, \frac{k}{n}\right)\left(1 + \frac{M(s^2)}{n}\right) \leqq F_n\left(s, \frac{k}{n}\right)\exp\left(\frac{M(s^2)}{n}\right),$$

and, by induction,

$$F_n\left(s, \frac{k}{n}\right) \leqq \exp\left(\frac{k}{n} M(s^2)\right), \qquad k = 0, 1, 2, \cdots, n,\ n = 1, 2, \cdots . \tag{4.7}$$

If $F_n(s, t)$ is decreasing

$$F_n\left(s, \frac{k+1}{n}\right) \geqq F_n\left(s, \frac{k}{n}\right)\left(1 - \frac{M(s^2)}{n}\right) \geqq F_n\left(s, \frac{k}{n}\right)\exp\left(-\frac{2M(s^2)}{n}\right),$$

provided n is larger than some $n(s)$ and $F_n(s, k/n) \geqq s^{b_n/c_n}$. Recalling that

$b_n/c_n \to \infty$, induction gives

$$F_n\left(s, \frac{k}{n}\right) \exp\left(-\frac{2M(s^2)}{n}\right), \qquad k = 0, 1, 2, \cdots, n, \tag{4.8}$$

for all n larger than some $n_1(s)$. Hence, by (4.5), (4.7) and (4.8)

$$\sup_{t \in [0,1], n \in N} F_n(s, t) n |G_n(sF_n(s, t)^{c_n/b_n})^{1/n} - 1| = K(s) < \infty$$

for all $s \in (0, 1]$. Applying (4.4) we obtain

$$|F_n(s, t) - F_n(s, u)| \leqq (|t - u| + 2/n) K(s) .$$

Exactly as in the proof of Theorem 3.1 we can then select a subsequence $\{n'\} \subset N$ such that

$$\lim_{n' \to \infty} F_{n'}(s, t) = F(s, t) \qquad \text{exists for all} \quad s \in (0, 1], \quad t \in [0, 1] .$$

Passing to the limit in (4.4) we obtain

$$F(s, t) = 1 + \int_0^t F(s, u) \log G(s)\, du$$

i.e.

$$F(s, t) = G(s)^t , \qquad s \in (0, 1],\ t \in [0, 1] .$$

In particular the limit is independent of the sequence $\{n'\}$, which proves that

$$\lim_{n \to \infty} F_n(s, t) = G(s)^t , \qquad s \in (0, 1],\ t \in [0, 1] . \tag{4.9}$$

We shall now prove that the convergence of the generating functions $F_n(s, t)$ implies weak convergence of the random variables $Y_n(t)$ to the desired limit. Theorem 2.1 shows that $M(d) \to 1$ as $d \to 1$, which in view of (4.6) implies that

$$F_n(s, t) \to 1 \qquad \text{as} \quad s \to 1 , \quad \text{uniformly in} \quad n \in N .$$

Employing the same technique as in the proof of Lemma 3.4, we can easily see that $\{Y_n(t)\}_{n \in N}$ must be tight for each $t \in [0, 1]$. Furthermore, $Y_n(t)$ can be written on the form

$$Y_n(t) = \sum_{j=1}^{b_n} \frac{X_j^{(n)}(t) - 1}{c_n} , \tag{4.10}$$

where $X_j^{(n)}(t)$ is the number of individuals in the $[nt]$th "generation" of the nth Galton–Watson process, who are "descended" from the jth "ancestor." Hence, if $\{n'\} \subset N$ is a subsequence such that $Y_{n'}(t)$ converges in distribution to some random variable $Y(t)$, Theorem 2.1 and (4.9) imply that $F(s, t) = G(s)^t$ is the generating function of $Y(t)$. The uniqueness theorem for Laplace transforms, which holds true also for measures on $(-\infty, +\infty)$, if the transforms are finite, then shows that the tight sequence $\{Y_n(t)\}_n$ has only one limit distribution. Therefore, $\{Y_n(t)\}_n$ must converge weakly to some random variable $Y(t)$, necessarily having $G(s)^t$ as its generating function. As for the convergence of the finite-dimensional distributions and the character of the limit process we refer to Lamperti (see [9] page 283). □

Proceeding in the same spirit as in Section 3 we can also prove a converse of Theorem 4.1.

THEOREM 4.2. *Let* $\{Z_j^{(n)}\}_{j=0}^{\infty}$, $\{p_k^{(n)}\}_k$ *and* μ_n *be defined as in Section* 3. *Put*

$$Y_n(t) = \frac{Z_{[nt]}^{(n)} - b_n}{c_n}, \qquad t \in [0, 1],$$

where $Z_0^{(n)} = b_n$, $b_n/c_n \to \infty$ *as* $n \to \infty$ *and* $\inf_{n \in N} c_n > 0$. *Assume that* $Y_n(t)$ *converges weakly to some random variable* $Y(t)$ *for every* $t \in [0, 1]$. *Then, the sequence* $\{(\mu_n)^{*nb_n}\}_{n \in N}$ *converges weakly to the probability law of* $Y(1)$.

PROOF. As usual we let $F_n(s, t)$ and $F(s, t)$ denote the generating functions of $Y_n(t)$ and $Y(t)$, respectively. Writing $Y_n(t)$ as in (4.10), Theorem 2.1 yields

$$F_n(s, t) \to F(s, t) > 0 \qquad \text{as } n \to \infty,\ s \in (0, 1],\ t \in [0, 1],$$

and the convergence is uniform in $s \in [d, 1]$ for every $d \in (0, 1)$. In particular

$$\text{(4.11)} \qquad F_n(s, t) \to 1 \quad \text{as } s \to 1, \quad \text{uniformly in } n \in N.$$

With the same kind of arguments as in the proof of Lemma 3.1 we can show that $F_n(s, t)$ is a monotone function of $t \in [0, 1]$. Hence (4.11) can be strengthened to

$$\text{(4.12)} \qquad \sup_{n \in N, t \in [0,1]} |F_n(s, t) - 1| \to 0 \qquad \text{as } s \to 1.$$

However, the monotonicity of $F_n(s, t)$ also implies that the integrand in (4.4) must have the same sign for all $t \geqq 0$. By (4.12) and (4.4) again

$$\sup_{n \in N} \int_0^1 n|G_n(sF_n(s, t)^{c_n/b_n})^{1/n} - 1|\, dt \to 0 \qquad \text{as } s \to 1.$$

For any given $\varepsilon > 0$ we can then choose $s_0 < 1$ arbitrarily close to 1 and such that

$$\inf_{t \in [0,1]} n|G_n(s_0 F_n(s_0, t)^{c_n/b_n})^{1/n} - 1| < \varepsilon$$

for all $n \in N$. But at least for all n larger than some $n(s_0)$

$$s_0/(2 - s_0) \leqq s_0 F_n(s_0, t)^{c_n/b_n} \leqq (1 + s_0)/2, \qquad t \in [0, 1].$$

Hence,

$$\text{(4.13)} \qquad \inf_{s \in [s_0/(2-s_0), (1+s_0)/2]} |nG_n(s)^{1/n} - 1| < \varepsilon, \qquad n \geqq n(s_0),$$

which implies that

$$\text{(4.14)} \qquad \sup_{s \in [s_0/(2-s_0), (1+s_0)/2]} G_n(s) > e^{-2\varepsilon}$$

for all sufficiently large $n \in N$. Next we observe that $G_n(s)$ can be written on the form

$$G_n(s) = \sum_{j=1}^{nc_n} q_{-j}^{(n)} s^{-j/c_n} + \sum_{j=0}^{\infty} q_j^{(n)} s^{j/c_n} = A_n(s) + B_n(s),$$

where the A_n are non-increasing and convex, while the B_n are non-decreasing and uniformly bounded. Then by (4.13)

$$\sup_{n \in N} A_n(s_0/(2 - s_0)) < \infty,$$

which in view of the convexity of A_n implies that

(4.15) $$\sup_{n\in N} |A_n(1-d) - A_n(1)| \to 0 \qquad \text{as } d \to 0 .$$

But s_0 in (4.14) could be chosen arbitrarily close to 1. Hence, combining (4.14) and (4.15), we get

$$\sup_{s\in[s_0/(2-s_0),(1+s_0)/2]} B_n(s) = B_n((1+s_0)/2) > e^{-3\varepsilon}$$

for all sufficiently large $n \in N$, which shows that

(4.16) $$B_n(s) \to 1 \quad \text{as} \quad s \to 1 , \quad \text{uniformly in} \quad n \in N .$$

Finally, (4.15) and (4.16) implies that

$$G_n(s) \to 1 \quad \text{as} \quad s \to 1 , \quad \text{uniformly in} \quad n \in N .$$

But the methods from the proof of Lemma 3.4 will then show that the sequence

$$\{(\mu_n)^{*nb_n}\}_{n\in N} \quad \text{is tight.}$$

Furthermore, every limit distribution of this sequence must have $F(s, 1)$ as its generating function according to Theorem 4.1. Therefore, a reference to the uniqueness theorem for Laplace transforms will complete the proof of Theorem 4.2. □

Let us now prove that $\{Y_n\}_{n\in N}$ is also convergent in the function space $D[0, 1]$, if the assumptions of Theorem 4.1 hold true. Since $F_n(s, t)$ is a monotone function of $t \in [0, 1]$, the convergence of $F_n(s, t)$ to $F(s, t)$ shows that

(4.17) $$\sup_{n\in N, t\in[0,1]} F_n(s, t) < \infty , \qquad s \in (0, 1] .$$

Interpreting (4.17) and (4.12) in terms of the distribution of $Y_n(t)$ we get

(4.18) $$\sup_{n\in N, t\in[0,1]} P(|Y_n(t)| > \lambda) \to 0 \qquad \text{as } \lambda \to \infty .$$

By the branching process property there exist measures $\nu_{n,t}$ such that

$$\nu_{n,t}^{*b_n}(E) = P(Y_n(t) \in E)$$

and

$$P(Y_n(1) \in E \mid Y_n(s) = a) = \nu_{n,1-s}^{*(ac_n+b_n)}(E)$$

for all Borel sets E. Furthermore, by (4.18)

$$\{\nu_{n,t}^{*b_n}\}_{n\in N, t\in[0,1]} \quad \text{is tight.}$$

The proof is then completed in the same way as the corresponding proof in Section 3. □

We summarize all convergence results of this section in the following theorem.

THEOREM 4.3. *Let* $\{Z_j^{(n)}\}_{j=0}^\infty$, $Y_n(t)$ *and* μ_n *have the same meaning as in Theorem 4.1. Then the following three statements are equivalent.*

(i) *The one-dimensional distributions of* $\{Y_n(t); t \in [0, 1]\}$ *converge weakly to those of some stochastic process* $\{Y(t); t \in [0, 1]\}$.

(ii) *There exists a probability measure* μ *such that*

$$(\mu_n)^{*nb_n} \to_w \mu \qquad \text{as } n \to \infty$$

(iii) $\{Y_n\}_{n\in N}$ *converges weakly in the function space* $D[0, 1]$.

Furthermore, the limit process $\{Y(t);\ t \in [0, 1]\}$ *is always a process with homogeneous independent increments such that* μ *coincides with the probability law of* $Y(1)$.

We shall complete this discussion on limit theorems for sequences of Galton–Watson processes by stating an analogue of the Feller–Jirina limit theorem in the case of centered Galton–Watson processes. The proof of that theorem is an obvious consequence of Theorem 4.3 and the central limit theorem for triangular arrays.

THEOREM 4.4. *Let* $\{Z_j^{(n)}\}_{j=0}^{\infty}$, $\{p_k^{(n)}\}_k$ *and* Y_n *be defined as in the beginning of this section. Assume that*

(i) $m_n = \sum_{k=1}^{\infty} kp_k^{(n)} = 1 + \alpha_n c_n/nb_n$, *where* $\alpha_n \to \alpha$ *as* $n \to \infty$,
(ii) $\sigma_n^2 = \sum_{k=0}^{\infty} (k - m_n)^2 p_k^{(n)} = \beta_n c_n^2/nb_n$, *where* $\beta_n \to \beta > 0$ *as* $n \to \infty$,
(iii) $(nb_n/c_n^2) \sum_{k>tc_n} k^2 p_k^{(n)} \to 0$ *as* $n \to \infty$, *for every* $t > 0$.

Then, $\{Y_n\}_{n\in N}$ *converges weakly in* $D[0, 1]$ *to a Brownian motion* $\{B(t);\ t \in [0, 1]\}$ *with drift* α *and* Var $(B(1)) = \beta$. *Furthermore, condition* (iii) *is necessary for the convergence to* $\{B(t);\ t \in [0, 1]\}$, *if* (i) and (ii) *hold true.*

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