The Annals of Probability
1975, Vol. 3, No. 1, 172–177

EXTREMAL PROCESSES GENERATED BY INDEPENDENT NONIDENTICALLY DISTRIBUTED RANDOM VARIABLES[1]

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Let $M_n = \max\{X_1, \cdots, X_n\}$ and $m_n(t) = (M_{[nt]} - a_n)/b_n$ $(t \geqq 1/n)$, where the $\{X_i\}$ are independent rv's and a_n and $b_n > 0$ are real constants. Suppose all the finite-dimensional laws of m_n converge to those of a stochastic process $m = \{m(t) : t > 0\}$. This paper is a study of the class of all such processes m.

0. Introduction. Let $\{X_i\}$ be a sequence of independent random variables (rv's) and let M_n denote $\max\{X_1, \cdots, X_n\}$. Suppose there exist real numbers $b_n > 0$ and a_n such that the distribution of $(M_n - a_n)/b_n$ converges to a nondegenerate distribution function (df) G as $n \to \infty$. Now define the process $m_n = \{m_n(t) : t > 0\}$ by

$$(0.1)\qquad \begin{aligned} m_n(t) &= (M_{[nt]} - a_n)/b_n && \text{if} \quad t \geqq 1/n \\ &= (X_1 - a_n)/b_n && \text{if} \quad 0 < t < 1/n\,. \end{aligned}$$

This paper treats the class of limit processes $m = \lim m_n$ which may be obtained in this manner. The limit is in the sense of convergence of all the finite-dimensional laws (fdl) of m_n to those of m.

These limit processes, the so-called *Extremal processes* have been studied by a number of authors: Dwass [1]—[2], Lamperti [4], Oliveira [9] and Resnick and Rubinovitch [10]. All of them assumed that the X_i are i.i.d. Welsch [13]—[14] generalized the results of [1] and [4] by replacing the independence of the X_i by the strong mixing property. In the present article we generalize [1] and [4] in another direction namely, we keep the independence of the X_i but allow them to be nonidentically distributed.

The joint limit processes for $(m_n^1, \cdots, m_n^k)$, where $m_n^k(t)$ is the kth largest among $\{(X_i - a_n)/b_n : i = 1, \cdots, [nt]\}$, including aspects of weak convergence, are studied in [11].

We conclude this section with some conventions. All G_t $(t > 0)$ are nondegenerate df's, except $G_0 \equiv 1$. We write G for G_1. If G_t is non-increasing in t then for $s < t$ the ratio $G_t(x)/G_s(x)$ is defined to be 0 when $G_t(x) = 0$ even if

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Received October 25, 1972; revised May 6, 1974.

[1] This paper is adapted in part from the author's Ph. D. dissertation (University of Chicago (1971)). Partly supported by the Statistics Branch, Office of Naval Research, Contract # N00014-67-A-0285-0009, and by Research Grant # NSF 2818 from the Division of the Social Sciences of the National Science Foundation.

AMS 1970 *subject classifications.* Primary 60K99, 60J25. Secondary 62E20, 62G30.

Key words and phrases. Extremal processes, convergence of finite-dimensional laws, stationary transition probabilities.

$G_s(x) = 0$. Finally, let

$$_*G = \inf\{x : G(x) > 0\}, \qquad G_* = \sup\{x : G(x) < 1\}.$$

1. The limiting process for the maximum. Suppose $\{G_t : t > 0\}$ is a family of df's on R^1 such that $G_t(x)/G_s(x)$ is a non-decreasing function of x whenever $0 < s < t$ and define a stochastic process $m = \{m(t) : t > 0\}$ as follows:

$$m(t) \leqq m(t+u) \quad \text{a.s.} \quad \forall t, u \in (0, \infty) \tag{1.1}$$

and for all $0 = t_0 < t_1 < \cdots < t_k$ and all $x_1 \leqq x_2 \leqq \cdots \leqq x_k$

$$P(\textstyle\bigcap_{i=1}^k (m(t_i) \leqq x_i)\} = \prod_{i=1}^k (G_{t_i}(x_i)/G_{t_{i-1}}(x_i)) . \tag{1.2}$$

Clearly, (1.1) and (1.2) determine a consistent set of fdl and hence a measure space exists on which such a process can be defined. *We call the class of all such processes the class M.*

THEOREM 1.1. *Let m_n be a partial maxima process as defined by* (0.1) *and suppose that for each $t > 0$ there exists a G_t such that*

$$m_n(t) \to_D G_t \qquad (n \to \infty) . \tag{1.3}$$

Then all the fdl *of m_n converge to those of $m \in M$, where the* fdl *of m are determined by the G_t as in* (1.2).

PROOF. Let $F_{ni}(x) = P\{X_i \leqq b_n x + a_n\}$. Since (1.3) holds for all $t \in (0, \infty)$, we have

$$\lim_{n\to\infty} \textstyle\prod_{i=[ns]+1}^{[nt]} F_{ni}(x) = G_t(x)/G_s(x) \qquad (0 \leqq s < t) \tag{1.4}$$

at all continuity points of G_t/G_s. Hence G_t/G_s is a non-decreasing function and a process $m \in M$ can be defined by the G_t. Now we have to show that for every $0 = t_0 < t_1 < \cdots < t_k$ and every $x_1, \cdots, x_k$

$$P\{m_n(t_1) \leqq x_1, \cdots, m_n(t_k) \leqq x_k\} \tag{1.5}$$

converges (weakly) to the same expression with n suppressed. But since m_n and m are both non-decreasing, only $x_1 \leqq x_2 \leqq \cdots \leqq x_k$ are of interest. For $0 \leqq s < t$ we define $m_n(s, t) = \max\{(X_i - a_n)/b_n : [ns] < i \leqq [nt]\}$. Then (1.5) is equal to

$$\begin{aligned} &P\{m_n(t_1) \leqq x_1, m_n(t_1, t_2) \leqq x_2, \cdots, m_n(t_{k-1}, t_k) \leqq x_k\} \\ &\qquad = \textstyle\prod_{i=1}^{[nt_1]} F_{ni}(x_1) \prod_{i=[nt_1]+1}^{[nt_2]} F_{ni}(x_2) \cdots \prod_{i=[nt_{k-1}]+1}^{[nt_k]} F_{ni}(x_k) ; \end{aligned} \tag{1.6}$$

the r.h.s. of (1.6) follows from the independence of the $\{X_i\}$. If x_i is a continuity point of $G_{t_i}/G_{t_{i-1}}$ $(i = 1, \cdots, k)$ then by (1.4) the limit of (1.6) is equal to the r.h.s. of (1.2). □

It can be seen that for each n, the process m_n has the form defined by (1.1) and (1.2). The theorem proves that this form is preserved as we pass to the limit (as $n \to \infty$).

From the multiplicative form of (1.2), one can easily see that m is a Markov process and an equivalent definition of $m \in M$ is the following: for each $t > 0$

$$P(m(t) \leqq y) = G_t(y)\,, \tag{1.7}$$

and transition probabilities $P\{m(t) \leqq y \mid m(s) = x\}$ $(s < t)$ are given by

$$\begin{aligned} p_{st}(x, y) &= G_t(y)/G_s(y) && x \leqq y \\ &= 0 && x > y\,. \end{aligned} \tag{1.8}$$

Theorem 1.1 is a generalization of Theorem 2.1 of Lamperti [4] and of Lemma 3.1 of Dwass [1]. These two papers are the first published studies of the partial maxima of i.i.d. $\{X_i\}$ in the form of functional limit theorems. Clearly, when the X_i are i.i.d., $m_n(1) \to_D G$ implies $m_n(t) \to_D G^t$ for all $t > 0$, and thus in this case, (1.2), (1.7) and (1.8) become

$$P\{\textstyle\bigcap_{i=1}^k (m(t_i) \leqq x_i)\} = G^{t_1}(x_1)G^{t_2-t_1}(x_2) \cdots G^{t_k-t_{k-1}}(x_k) \quad (0 < t_1 < \cdots < t_k;\, x_1 \leqq \cdots \leqq x_k)\,, \tag{1.9}$$

$$P(m(t) \leqq y) = G^t(y) \tag{1.10}$$

and

$$\begin{aligned} p_{st}(x, y) &= G^{t-s}(y) && x \leqq y \\ &= 0 && x > y\,, \end{aligned} \tag{1.11}$$

respectively.

2. Classification of extremal processes. Let $E \subset M$ be the class of those processes in M which are obtained as limits via (1.3).

THEOREM 2.1. *The marginals G_t of $m \in E$ satisfy one of the following relations*

$$G_t(x) = G(t^\theta(x - c) + c) \qquad \textit{for all } t > 0 \ (\theta \neq 0) \tag{2.1}$$

$$G_t(x) = G(x - c \log t) \qquad \textit{for all } t > 0 \ (\theta = 0,\ c \geqq 0)\,. \tag{2.2}$$

Moreover, if in (2.1) $\theta > 0$ *then* $G_* \leqq c$ *and if* $\theta < 0$ *then* ${}_*G \geqq c$.

PROOF. Since $m \in E$ there exists a partial maxima process m_n which satisfies (1.3). By Theorem 1 of [12] (2.1) and (2.2) follow with c arbitrary. Since G_t is non-increasing in t, we have $c \geqq 0$ in (2.2). By the same argument $t^\theta(x - c) + c \geqq x$ for $x \in ({}_*G, G_*)$ and $t \in (0, 1)$. Thus, if $\theta > 0$ then $G_* \leqq c$ and if $\theta < 0$ then ${}_*G \geqq c$. □

It follows that each limit process $m \in E$ is completely determined by a triple $\langle G, \theta, c\rangle$ where G is a df (and serves as G_1) and θ and c are real numbers. We shall identify the process m with its associated triple $\langle G, \theta, c\rangle$.

For given θ and c let $H(\theta, c)$ be the set of all limit distributions G for which $\langle G, \theta, c\rangle \in E$.

THEOREM 2.2.

(i) $H(0, 0)$ *is the set of all nondegenerate* df*'s.*
(ii) $H(0, c)$ *is empty for* $c < 0$.

(iii) $G \in H(0, c)$ *for* $c > 0$ iff $\log G(x)$ *is concave.*
(iv) $G \in H(\theta, c)$ *for* $\theta > 0$ iff $G_* \leqq c$ *and* $\log G(c - e^{-x})$ *is concave.*
(v) $G \in H(\theta, c)$ *for* $\theta < 0$ iff ${}_*G \geqq c$ *and* $\log G(c + e^{x})$ *is concave.*

PROOF. (i) Let G be an arbitrary df. We have to show that there exist a sequence of df's $\{F_n\}$ and sequences of reals $\{a_n\}$ and $\{b_n\}$ ($b_n > 0$) such that $\prod_{i=1}^{[nt]} F_i(b_n x + a_n) \to G(x)$ at all continuity points of G for all $t < 0$. The sequences $F_n(x) = G^{2^{-n}}(x)$, $a_n \equiv 0$, $b_n \equiv 1$ will do.

(ii) Obvious, since in (2.2) $c \geqq 0$.

(iii) Suppose $G \in H(0, c)$ with $c > 0$. Then the ratio $G(x)/G(x - c \log t)$ is non-decreasing in x for each $t \in (0, 1)$, hence $\log G(x)$ is concave.

Suppose now that $\log G(x)$ is concave and $c > 0$. We have to show the existence of sequences $\{F_n\}$, $\{a_n\}$ and $\{b_n\}$ such that $\prod_{i=1}^{[nt]} F_i(b_n x + a_n) \to G(x - c \log t)$ for each $t \in (0, \infty)$. Let $G_0 \equiv 1$ and for $n \geqq 1$ define $G_n(x) = G(x - c \log n)$ if $x \geqq 0$ and 0 if $x < 0$. Then $F_n(x) = G_n(x)/G_{n-1}(x)$ ($n \geqq 1$) is a df (which vanishes on $(-\infty, 0)$) because $\log G(x)$ is concave. With $a_n = c \log n$ and $b_n \equiv 1$ we have

$$\prod_{k=1}^{[nt]} F_k(b_n x + a_n) = G_{[nt]}(x + c \log n) = G(x + c \log n - c \log [nt]) \tag{2.3}$$

if $x + c \log n \geqq 0$ and 0 otherwise. Thus the l.h.s. of (2.3) converges (weakly) to $G(x - c \log t)$.

(iv) Suppose $G \in H(\theta, c)$ with $\theta > 0$. By Theorem 2.1 we have $G_* \leqq c$. Let $u = x - c$; then $G(x)/G_t(x) = G(u + c)/G(t^\theta u + c)$. It follows that $\log G(c - e^{-x})$ is concave. Conversely, suppose $\theta > 0$, $c \geqq G_*$ and $\log G(c - e^{-x})$ is concave. As in case (iii) we let $G_0 \equiv 1$ and for $n \geqq 1$ we define $G_n(x) = G(c + n^\theta x)$ if $x \geqq x_0$ and 0 if $x < x_0$, where $x_0 < 0$ is arbitrary. Then $F_n(x) = G_n(x)/G_{n-1}(x)$ is a df because $\log G(c - e^{-x})$ is concave. With $b_n = n^{-\theta}$ and $a_n = -cn^{-\theta}$ we have

$$\prod_{k=1}^{[nt]} F_k(b_n x + a_n) = G_{[nt]}(n^{-\theta}(x - c)) = G(c + [nt]^\theta n^{-\theta}(x - c)) \tag{2.4}$$

if $x \geqq c + n^\theta x_0$ and 0 otherwise. Thus the l.h.s. of (2.4) converges (weakly) to $G(c + t^\theta(x - c))$.

The proof of (v) is analog to (iv). □

REMARKS. In a sequence of papers [5]—[8] Mejzler studied the possible limit df's of $m_n(1)$, under the right negligibility condition (RNC). Namely, those df's G which are limits of $\prod_{i=1}^{n} F_i(b_n x + a_n)$ for some $\{F_n, a_n b_n\}$ under the condition that

$$\lim_{n\to\infty} \max_{1\leqq i\leqq n} (1 - F_i(b_n x + a_n)) = 0 \qquad \forall x > {}_*G .$$

He proved that the set of these G is the set of all G which satisfy one of the following conditions:

(a) $\log G(x)$ is concave,
(b) $\log G(G_* - e^{-x})$ is concave and $G_* < \infty$,
(c) $\log G({}_*G + e^{x})$ is concave and ${}_*G > -\infty$.

Notice that the third condition implies the first. Our choices of $\{F_n, a_n, b_n\}$ in the proofs of (iii)—(v) satisfy the RNC, and thus adding this condition does not reduce the classes $H(\theta, c)$ in (iii)—(v) of Theorem 2.2.

3. Extremal processes with stationary transition probabilities. Let $\{X_i\}$ be i.i.d. rv's and suppose $m_n(1) \to_D G$ where G is non-degenerate. Then there exists an extremal process $m \in E$ defined by (1.10) and (1.11) such that $m_n \to m$ (in the sense of convergence of all the fdl). As we see in (1.11) the transition probabilities of m are stationary. Moreover, up to scale and location parameters, G must belong to one of the following classes of extreme value df's: $\{\Phi_\alpha : \alpha > 0\}$, $\{\Psi_\alpha : \alpha > 0\}$ and $\{\Lambda\}$ (see [3] or [12]). For any df G the process $m = \langle G, 0, 0\rangle$ obviously possesses stationary transition probabilities, since m reduces here to a random variable ($m(t) \equiv m(1)$ a.s. for all $t > 0$). There is one other *nontrivial* class of extremal processes $m \in E$ with stationary transition probabilities. To prove this we need the following notation. For any df G we define

$$\bar{G}(x) = G(x)/G(G_*-) \qquad x < G_*$$
$$= 1 \qquad x \geqq G_* ,$$

(if $G(G_*-) = 1$ then $\bar{G} = G$).

THEOREM 3.1. *If $m = \langle G, \theta, c\rangle \in E$ (with $\theta^2 + c^2 > 0$) has stationary transition probabilities then either G or $\bar{G}$ is one of the classic extreme value* df'*s. If $\bar{G} \neq G$ then $\bar{G}$ is of $\phi_{1/\theta}$-type.*

PROOF. Since

$$H_x(t) = G_{s+t}(x)/G_s(x) \tag{3.1}$$

does not depend on s, by a routine argument we find that

$$H_x(t) = H^t(x) \tag{3.2}$$

for some $H(x)$. From (2.1) and (2.2) we get the following table

TABLE 1

	$\theta = 0$	$\theta > 0$		$\theta < 0$	
$\lim_{s\downarrow 0} G_s(x) =$	$1 \quad \forall x$	1	if $x \geqq c$	1	if $x > c$
$=$		$G(c-)$	if $x < c$	0	if $x < c$

Notice that if $\theta > 0$ then $\log G(c - 'e^{-x})$ is concave hence $G(x)$ is continuous at each $x < c$. But Theorem 2.1 implies $G_* \leqq c$ thus $G(c-) < 1$ implies $c = G_*$. Now we use the table above and take the limit in (3.1) as $s \downarrow 0$. In view of (3.2) we get $H = \bar{G}$. Hence $\bar{G}$ must satisfy either $\bar{G}^t(x) = \bar{G}(t^\theta(x - c) + c)$ or $\bar{G}^t(x) = \bar{G}(x - c \log t)$. Hence (see Theorem 2 in [13]) $\bar{G}$ is of $\phi_{-1/\theta}$-type if $\theta < 0$, of Λ-type if $\theta = 0$ and of $\phi_{1/\theta}$-type if $\theta > 0$. As follows from the table, the only case where $\bar{G} \neq G$ is $\theta > 0$ with $G(G_*-) < 1$. This completes the proof. □

Notice that in case $\theta > 0$ and $G(G_*-) < 1$ the fdl of $\bar{m} = \langle \bar{G}, \theta, G_*\rangle$ coincide with those of $m = \langle G, \theta, G_* \rangle$ conditioned by the requirement that $m(t) < G_*$ for all $t > 0$.

For every df G one can define a process $m \in M$ by putting $G_t \equiv G^t$ in (1.2), and thus get stationary transition probabilities. But by Theorem 3.1, if G is not one of the classic extreme value limit distributions then $m \notin E$.

Acknowledgment. I wish to thank Professor P. Billingsley for his guidance and encouragement during the preparation of my thesis. I also thank the referee for his comments.

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