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TRANSIENCE AND SOLVABILITY OF A NON-LINEAR DIFFUSION EQUATION[1]

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This paper is concerned with the existence of bounded solutions to an operator inequality which is a non-linear version of a discrete time diffusion equation. Here, the solvability of the inequality will be closely related to the transience of a corresponding random walk. In particular, the inequality will generally be solvable in three or more dimensions, but not in one or two dimensions if appropriate moment conditions hold.

1. The basic equation and initialization. Let Q be a probability measure on R^p, $Y \sim Q$, and consider the following operator, T, defined for real-valued bounded measurable functions, f, by

$$(Tf)(x) = Ef(x+Y) = \int f(x+y)\,dQ(y)\,. \tag{1.1}$$

This paper considers the existence of bounded nonnegative nonzero measurable solutions $\{f_n : n = 1, 2, \cdots\}$ for the operator inequality:

$$f_{n+1}(x) \geqq (Tf_n)(x) + (Tf_n)^2(x) \qquad x \in R^p,\ n = 1, 2, \cdots . \tag{1.2}$$

The existence of such solutions for (1.2) is shown to be closely related to transience of the symmetrized random walk generated by Q. In particular, Section 2 shows that if Q has a finite second moment and $p = 1$ or $p = 2$ then there is no bounded nonnegative measurable solution for (1.2) with $f_1(x) > 0$ on a set of positive Lebesgue measure. Conversely, Section 3 shows that if the random walk generated by the symmetrization of Q is transient (in particular, if $p \geqq 3$) then there is an appropriate solution for (1.2). In the remainder of this paper, all functions will be assumed to be real-valued and Borel measurable.

These results are applicable to a remaining unsolved case concerning existence of bounded solutions to a non-linear partial differential equation related to (1.2) studied by Fujita [6] (see [8]). They are also applicable to the consideration of admissibility in a certain statistical decision problem, where in fact (1.2) arose. In particular, in [7] an admissibility problem was reduced to finding real-valued nonzero solutions $\{f_n : n = 0, \pm 1, \pm 2, \cdots\}$ to the operator inequality:

$$f_{n+1}(x) - \tfrac{1}{2}f_{n+1}^2(x) \geqq T(f_n + \tfrac{1}{2}f_n^2)(x) \qquad x \in R_p,\ n = 0, \pm 1, \pm 2, \cdots . \tag{1.3}$$

The following result shows the equivalence of the two problems:

THEOREM 1.1. *There is a nonnegative bounded solution* $\{f_n : n = 1, 2, \cdots\}$ *for*

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(1.2) *with* $f_1(x) > 0$ *on* $S \subset R^p$ *if and only if there is a solution* $\{f_n^*: n = 0, \pm 1, \pm 2, \cdots\}$ *for* (1.3) *with* $f_1^*(x) \neq 0$ *for* $x \in S$.

PROOF. (i) Let $\{f_n^*\}$ be a solution for (1.3). A straightforward induction argument shows that if $f_0(x) < -a$ (with $a > 0$) for some x then $f_{-n}(y) < -(a + (n/2)a^2)$ for some y. Hence, for $\{f_n^*\}$ to be bounded, each f_n^* must be nonnegative; and, therefore, must be bounded by 2 in order to be real-valued. It trivially follows that a solution for (1.3) is also a solution for (1.2)

(ii) Let $\{f_n\}$ be a solution for (1.2) and define $f_n^*(x) = 0$ for $n = 0, -1, -2, \cdots$. Define $\{g_n: n = 1, 2, \cdots\}$ inductively by $g_1(x) = f_1(x)$ and

$$g_{n+1}(x) = (Tg_n)(x) + (Tg_n)^2(x) \qquad x \in R^p,\ n = 1, 2, \cdots . \tag{1.4}$$

Then (inductively) $0 \leqq g_n(x) \leqq f_n(x)$; and, hence, $|g_n(x)| \leqq B$ for some finite $B > 0$. So by (1.4),

$$g_{n+1}(x) \leqq (1 + B)(Tg_n)(x) . \tag{1.5}$$

Now define

$$f_n^*(x) = \frac{1}{(1+B)^2}\, g_n(x) \qquad x \in R^p,\ n = 1, 2, \cdots . \tag{1.6}$$

Then $f_n^*(x)$ is bounded, f_1^* is a multiple of f_1, and (from (1.6), (1.4), and (1.5)) for $n = 1, 2, 3, \cdots$

$$\begin{aligned} f_{n+1}^*(x) - \tfrac{1}{2} f_{n+1}^{*2}(x) &= \frac{1}{(1+B)^2}\, g_{n+1}(x) - \frac{1}{2(1+B)^4}\, g_{n+1}^2(x) \\ &\geqq \frac{1}{(1+B)^2}\,(Tg_n)(x) + \frac{1}{2(1+B)^2}\,(Tg_n)^2(x) \\ &\geqq (Tf_n^*)(x) + \tfrac{1}{2}(Tf_n^*)^2(x) . \end{aligned} \tag{1.7}$$

□

It should be noted that Berger ([1] and [2]) and Brown [4] have considered the admissibility problem in considerably more generality.

Some additional remarks about initialization of the solution should be made. The initialization condition is basically to insure that the solution is not identically zero almost everywhere. However, there is a real question of what "almost everywhere" means. If Q is discrete, we would clearly want to require $f_1(x) > 0$ for some x. For Q a lattice distribution (the case considered in [7]) it makes no difference whether $f_1(x) > 0$ at one lattice point or on an interval (since values of $f_1(x)$ in a small neighborhood of a lattice point will not affect $f_n(x)$ on the lattice). However, as the referee noted, if Q is discrete but non-lattice, we may generally reduce to a lattice case in higher dimensions (Blackwell [3] did this in a particular admissibility example). For example, the problem with Q concentrated on $\{\pm 1, \pm e, \pm \pi\}$ (in one dimension) is isomorphic to a three-dimensional problem with lattice distribution concentrated on $\{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}$. Thus, there would be bounded solutions for $f_1(x) > 0$ at one point, but there would be no bounded solution for $f_1(x) > 0$ on a set of positive Lebesgue measure. Hence, the form of initialization can be extremely important.

If Q has an absolutely continuous component, the natural meaning of "almost everywhere" is with respect to Lebesgue measure. Since such initialization also works for the discrete lattice case, it is the definition taken here.

However, this definition does seem unnatural if Q is singular but non-lattice. A more natural condition might only require that f_1 be positive on a set of positive measure under some translate Q. However, it can be shown that this latter condition always leads to a solution for (1.2) if some translate of Q has a singular component with respect to $Q * Q$ (if $f_1 = 0$ a.s. $(Q * Q)$, f_2 would be identically zero). Thus, the question of the solvability of (1.2) would be non-trivial only for distributions, Q, for which all translates are absolutely continuous with respect to $Q * Q$. It is an interesting but unresolved question as to whether any singular, non-discrete distribution can satisfy this property.

2. Insolvability when $p = 1$ or $p = 2$. This section shows that if $p = 1$ or $p = 2$, Q has finite second moment, and f_1 is positive on a set of positive Lebesgue measure, then any solution of (1.2) with initialization f_1 is unbounded. Some technical results are first required.

PROPOSITION. *Let W and T be operators defined by* (1.1) *using arbitrary distributions Q_1 and Q_2 respectively. Let S be the operator defined by* $(Sf)(x) = (Tf)(x) + (Tf)^2(x)$, *then*

$$(SW)f(x) \leqq (WS)f(x) \tag{2.1}$$

for all $x \in R^p$ and any nonnegative function f.

PROOF. By Fubini's theorem T and W commute. So

$$\begin{aligned}(SW)f(x) &= (WT)f(x) + (W(Tf)(x))^2 \\ &\leqq W(Tf)(x) + W(Tf)^2(x) = (WS)f(x) . \qquad \square\end{aligned}$$

We now show first that f_1 may be replaced by a particular initialization; and, second, that Q may be assumed to be symmetric (that is, have nonnegative real-valued characteristic function). This latter result is necessary in order to apply Lemma A.3 in the appendix. Before continuing, note that if $\{f_n\}$ is a bounded solution for (1.2), then $\{S^n f_1\}$ is also a bounded solution for (1.2) (where S is defined in the proposition).

LEMMA 2.1. *For x real, $y = (y_1, \cdots, y_p) \in R^p$, and $b > 0$ (to be determined later), define*

$$c(x) = \frac{1}{\pi b x^2}(1 - \cos bx) ; \qquad c_p(y) = \textstyle\prod_{i=1}^{p} c(y_i) . \tag{2.1}$$

(Note that $c(x)$ has triangular characteristic function.) If (1.2) *has any bounded nonnegative solution with f_1 not zero almost everywhere then there is $a > 0$ and a bounded solution for* (1.2) *with the initialization $f_1(x) = ac_p(x)$.*

PROOF. Let W_1 be the operator defined by $(W_1 f)(x) = \int f(x+y)c_p(y)\,dy$, and let W_2 be a similarly defined operator using a distribution with a continuous, bounded, strictly positive density. If f_1 is any nonnegative function which does not vanish almost everywhere (Lebesgue measure) then $(W_2 f_1)(x) > 0$ for all x (and, hence, is bounded below on the unit cube). Therefore for some $a > 0$ and $a' > 0$,

$$\begin{aligned}(W_1(W_2 f_1))(x) &= \textstyle\prod_{i=1}^p \int (W_2 f_1)(x_i + y_i)c(y_i)\,dy_i \\ &= \textstyle\prod_{i=1}^p \int (W_2 f_1)(z_i)c(z_i - x_i)\,dz_i \\ &\geqq a' \textstyle\prod_{i=1}^p \int_{-1}^1 c(z_i - x_i)\,dz_i \\ &\geqq a \textstyle\prod_{i=1}^p c(x_i) = ac_p(x)\end{aligned}$$

where the last inequality is a straightforward calculation which is not presented here. But by the proposition, if $S^n f_1$ is bounded, so is $S^n(W_2 f_1)$ and also, $S^n(W_1(W_2 f_1))$. □

LEMMA 2.2. *If there is a bounded nonnegative solution for* (1.2) *with* T *defined by* Q *then there is a bounded nonnegative solution for* (1.2) *with* T *defined by the symmetrization of* Q.

PROOF. Let W be the operator defined by $(Wf)(x) = Ef(x - y)$. Then applying the proposition inductively (and using positivity of the operators involved), $(SW)^n f_1 \leqq W^n(S^n f_1)$. Hence, if $\{S^n f_1\}$ is bounded, so is $(SW)^n f_1$. But

$$(SW)f(x) = T(Wf)(x) + (T(Wf)(x))^2 = (TW)f(x) + ((TW)f(x))^2$$

and, hence, $\{(SW)^n f_1\}$ satisfies (1.2) for the operator $(TW)f(x) = Ef(x + Y_1 - Y_2)$ which corresponds to the symmetrization of Q.

Now consider the following definitions (which will hold for both $p = 1$ and $p = 2$):

DEFINITION A. Let X be a random variable on R^p with density $c_p(x)$ (with respect to Lebesgue measure). Let $\{Y_2, Y_3, \cdots\}$ be independent and identically distributed according to Q (a distribution on R^p), let $S_1 = X$, and let $S_k = X - \sum_{i=2}^k Y_i$ for $k = 2, 3, \cdots$. Finally, let $p_k(x)$ be the density of S_k.

First note that (for $Y \sim Q$)

$$p_{k+1}(x) = \int p_k(x+y)\,dQ(y) = Ep_k(x+y) = (Tp_k)(x)\,. \tag{2.2}$$

Hence $\{p_k(x)\colon k = 1, 2, \cdots\}$ is uniformly bounded. Now, in the following proofs, we will take f bounded and continuous, let $U_m = \sum_{i=1}^m Y_i$ (with Q_m denoting the distribution of U_m) and consider

$$\begin{aligned}\frac{Ef(x + U_{n-k})p_k(x + U_{n-k})}{p_n(x)} &= \frac{\int f(x+y)p_k(x+y)\,dQ_{(n-k)}(y)}{p_n(x)} \\ &= E[f(S_k)\,|\,S_n = x]\,.\end{aligned} \tag{2.3}$$

Since $p_k(x)$ is a bounded, positive, continuous density, this conditional expectation is defined for every x; and, if f is bounded and continuous so is the conditional expectation (by dominated convergence).

Before stating the main theorems, we lastly note that if Q has finite second moment, we may assume without loss of generality that Q has zero mean and covariance matrix equal to the identity. For any Q may be transformed by an appropriate affine transformation, $g(x) = Ax + b$, to obtain a distribution Q^* with zero mean and identity covariance matrix. If $\{f_n\}$ is a solution of (1.2) with T defined by Q then $\{f_n \circ g^{-1}\}$ is a solution of (1.2) with T defined by Q^*.

THEOREM 2.1. *If $p = 1$ and Q has finite second moment, then* (1.2) *has no bounded solution with f_1 positive on a set of positive Lebesgue measure.*

PROOF. By the above remark, assume Q has zero mean and variance one; and by Lemma 2.1, let $f_1(x) = a'c(x)$. Let $a < a'$ be such that $ap_k(x) \leqq 1$ for all x and k; and define functions $\{q_k(x): k = 1, 2, \cdots\}$ as follows:

$$q_1(x) \equiv a\,, \qquad q_k(x) = (1 + ap_k(x))E[q_{k-1}(S_{k-1}) \mid S_k = x]\,. \tag{2.4}$$

By a straightforward induction argument, for $k = 2, 3, \cdots$,

$$q_k(x) = aE[\textstyle\prod_{i=2}^{k} (1 + ap_i(S_i)) \mid S_k = x]\,. \tag{2.5}$$

Note that $q_k(x) \geqq a$ for all x and k. Now, if f_n satisfies (1.2) with $f_1(x) = ac(x)$ then $f_n(x) \geqq q_n(x)p_n(x)$ for all x. This is proven by induction: if $n = 1$, the result follows since $p_1(x) = c(x)$. Assume the inequality holds for $(n - 1)$ and consider (with $Y \sim Q$)

$$\begin{aligned} f_n(x) &\geqq (Tf_{n-1})(x) + (Tf_{n-1})^2(x) \\ &= Ef_{n-1}(x + Y)\{1 + Ef_{n-1}(x + Y)\} \\ &\geqq Ep_{n-1}(x + Y)q_{n-1}(x + Y)\{1 + Ep_{n-1}(x + Y)q_{n-1}(x + Y)\} \\ &\geqq p_n(x)\,\frac{Ep_{n-1}(x + Y)q_{n-1}(x + Y)}{p_n(x)}\,\{1 + aEp_{n-1}(x + Y)\} \\ &= p_n(x)E[q_{n-1}(S_{n-1}) \mid S_n = x]\{1 + ap_n(x)\} \\ &= p_n(x)q_n(x)\,. \end{aligned}$$

Since $ap_k(x) \leqq 1$, we can apply Lemma A.1 to (2.5) to obtain

$$q_n(0) \geqq \frac{1}{a} \exp\left\{\frac{a}{4} \textstyle\sum_{k=2}^{n} E[p_k(S_k) \mid S_n = 0]\right\}\,. \tag{2.6}$$

Now $p_k(x) = 1/k^{\frac12}f_k(x/k^{\frac12})$ where f_k is the density of $S_k/k^{\frac12}$; and using the central limit theorem (see Lemma A.2 for details in case $p = 2$), one can show that $p_k(x) \approx 1/k^{\frac12}$ for $|x/k^{\frac12}| \leqq 1$ and $k \geqq K$. Therefore (for appropriate constant C_1),

$$\begin{aligned} E[p_k(S_k) \mid S_n = 0] &= \frac{1}{p_n(0)} \int p_k^2(y)\,dQ_{n-k}(y) \\ &\geqq C_1 \frac{n^{\frac12}}{k} P\left\{\left|\frac{U_{n-k}}{k^{\frac12}}\right| \leqq 1\right\} \\ &= C_1 \frac{n^{\frac12}}{k} P\left\{\left|\frac{U_{n-k}}{(n-k)^{\frac12}}\right| \leqq \frac{k^{\frac12}}{(n-k)^{\frac12}}\right\}\,. \end{aligned}$$

Again using the central limit theorem (see A.3)

$$P\left\{\left|\frac{U_{n-k}}{(n-k)^{\frac{1}{2}}}\right| \leqq \frac{k^{\frac{1}{2}}}{(n-k)^{\frac{1}{2}}}\right\} \geqq C_2 \frac{k^{\frac{1}{2}}}{(n-k)^{\frac{1}{2}}} \qquad \text{for } k \leqq n-k .$$

So for $n > 2K$,

$$\sum_{k=2}^{n} E[p(S_k) \mid S_n = 0] \geqq C_3 \sum_{k=K}^{n/2} \frac{n^{\frac{1}{2}}}{k} \frac{k^{\frac{1}{2}}}{(n-k)^{\frac{1}{2}}} \geqq C_3 \sum_{k=K}^{n/2} \frac{1}{k^{\frac{1}{2}}} \geqq C_4 n^{\frac{1}{2}} - C_5 .$$

Therefore, from (2.6),

$$p_n(0) q_n(0) \geqq \frac{C_6}{n^{\frac{1}{2}}} \exp\left\{\frac{a}{4}(C_4 n^{\frac{1}{2}} - C_5)\right\} \to +\infty \qquad \text{as } n \to \infty .$$

Hence, $\{f_n(x)\}$ cannot be bounded. □

The main additional complication when $p = 2$ is the fact that in two dimensions, $p_n(x) \approx n^{-1}$ (for $|x/n^{\frac{1}{2}}| < 1$ and n large). This leads to a lower bound for $q_n(0)$ of the form $q(0) \geqq C_1 \exp\{C_2 \log n\}$; so that $p_n(0) q_n(0) \to 0$ as $n \to \infty$. To circumvent this, the argument is iterated, defining $\tilde{q}_n(x)$ using $p_n(x) q_n(x)$ instead of $a \cdot p_n(x)$. This yields the bound $\tilde{q}_n(0) \geqq C_1 \exp[C_2 \exp\{C_3 \log n\}]$ which grows more quickly than $p_n(0)$.

THEOREM 2.2. *If $p = 2$ and Q has finite second moments, then* (1.2) *has no bounded solution with $f_1(x)$ positive on a set of positive Lebesgue measure.*

PROOF. By the remark above Theorem 2.1, assume Q has zero mean and covariance matrix equal to the identity. By Lemma 2.1, let $f_1(x) = a' c_2(x)$ and recall Definition A given earlier. Let $D > 0$ be as defined in Lemma A.2 so that $p_n(x) \leqq D/n$ for all x and $n = 1, 2, \cdots$. Let $a \leqq \min(\frac{1}{2}a', 1/D)$ so that

$$(2.7) \qquad a p_n(x) \leqq 1 \qquad \text{and} \qquad aDn^{(aD-1)} \leqq 1 \qquad \text{for } n = 1, 2, \cdots .$$

Note that it suffices to consider the initialization $f_1(x) = 2ac_2(x)$. As in (2.4), define

$$(2.8) \qquad q_1(x) \equiv 2a , \qquad q_k(x) = (1 + ap_k(x)) E[q_{k-1}(S_{k-1}) \mid S_k = x] = 2aE[\textstyle\prod_{i=2}^{k} (1 + ap_i(S_i)) \mid S_k = x] .$$

Note first that

$$(2.9) \qquad q_k(x) \geqq 2a \qquad \text{for } k = 1, 2, \cdots .$$

Also, from (2.3), we can take $Y \sim Q$ and

$$q_k(x) = (1 + ap_k(x)) \frac{Eq_{k-1}(x+Y) p_{k-1}(x+Y)}{p_k(x)} .$$

Therefore (since $ap_{k+1}(x) \leqq 1$) for $k = 1, 2, \cdots$,

$$(2.10) \qquad Eq_k(x+Y) P_k(x+Y) \geqq \tfrac{1}{2}(1 + ap_{k+1}(x)) Eq_k(x+Y) p_k(x+Y) = \tfrac{1}{2} q_{k+1}(x) p_{k+1}(x) .$$

Now define $\{q_k'(x): k = 1, 2, \cdots\}$ inductively as follows:

$$(2.11)\qquad q_1'(x) \equiv 2a\,, \qquad q_k'(x) = (1 + \tfrac{1}{2}p_k(x)q_k(x))E[q_{k-1}'(S_{k-1}) \mid S_k = x]\,.$$

Then as before, for $k = 2, 3, \cdots$,

$$(2.12)\qquad q_k'(x) = 2aE[\textstyle\prod_{i=2}^k (1 + \tfrac{1}{2}p_i(S_i)q_i(S_i)) \mid S_k = x]\,.$$

Thus, $q_k'(x) \geqq 2a$, and $(1 + \frac{1}{2}p_k(x)q_k(x)) \geqq (1 + ap_k(x))$. So using (2.11) inductively, it follows that

$$(2.13)\qquad q_k'(x) \geqq q_k(x) \qquad \text{for all } x \text{ and } k = 1, 2, \cdots\,.$$

It is now shown inductively that if $\{f_n\}$ satisfies (1.2) with $f_1(x) = 2ac_2(x)$, then

$$(2.14)\qquad f_n(x) \geqq p_n(x)q_n'(x) \qquad \text{for all } x \text{ and } n = 1, 2, \cdots\,.$$

For n = 1, (2.14) follows by definition. So assume (2.14) holds for n and consider

$$(2.15)\qquad f_{n+1}(x) \geqq (Tf_n)(x) + (Tf_n)^2(x) = Ef_n(x + Y)(1 + Ef_n(x + Y))$$

(where $Y \sim Q$). By the induction hypothesis and (2.3),

$$(2.16)\qquad Ef_n(x + Y) \geqq Ep_n(x + Y)q_n'(x + Y) = p_{n+1}(x)E[q_n'(S_n) \mid S_{n+1} = x]\,.$$

Also from (2.13) and (2.10),

$$(2.17)\qquad \begin{aligned} 1 + Ef_n(x + Y) &\geqq 1 + Ep_n(x + Y)q_n(x + Y) \\ &\geqq 1 + \tfrac{1}{2}p_{n+1}(x)q_{n+1}(x)\,. \end{aligned}$$

Therefore, combining (2.15), (2.16), and (2.17),

$$(2.18)\qquad \begin{aligned} f_{n+1}(x) &\geqq p_{n+1}(x)E[q_n'(S_n) \mid S_{n+1} = x](1 + \tfrac{1}{2}p_{n+1}(x)q_{n+1}(x)) \\ &= p_{n+1}(x)q_{n+1}'(x)\,, \end{aligned}$$

and (2.14) holds by induction.

We will now want to apply Lemma A.1 to obtain a lower bound for $q_n'(0)$. This requires the inequality, $\frac{1}{2}p_n(x)q_n(x) \leqq 1$. To obtain this, first apply Lemma A.1 to $q_n(x)$ (see (2.8)), which yields

$$\frac{1}{2a}\, q_n(x) \leqq \exp\{a \textstyle\sum_{k=2}^n E[p_k(S_k) \mid S_n = x]\}\,.$$

But by Lemma A.2, $p_k(x) \leqq D/k$ for all x; so $E[p_k(S_k) \mid S_n = x] \leqq D/k$ and

$$q_n(x) \leqq 2a \exp\left\{a \textstyle\sum_{k=2}^n \dfrac{D}{k}\right\} \leqq 2a \exp\left\{aD \textstyle\int_1^n \dfrac{dx}{x}\right\} = 2ae^{aD(\log n)}\,.$$

Therefore, using Lemma A.2 and (2.7),

$$(2.19)\qquad \tfrac{1}{2}p_n(x)q_n(x) \leqq \frac{1}{2}\,\frac{D}{n}\, 2ae^{aD(\log n)} = aDn^{(aD-1)} \leqq 1\,.$$

So applying Lemma A.1 to $q'(x)$ (see 2.12) and using Lemma A.4,

$$\log \frac{q_n'(x)}{2a} \geqq \tfrac{1}{4} \textstyle\sum_{k=2}^{n} E[\tfrac{1}{2} p_k(S_k) q_k(S_k) \mid S_n = x] \geqq \dfrac{1}{8a} q_n(x) - \tfrac{1}{4} . \tag{2.20}$$

It remains to apply Lemma A.1 once again to $q_n(0)$ to obtain a lower bound. By (2.7) $ap_k(x) \leqq 1$; so by Lemma A.1,

$$q_n(0) \geqq \frac{1}{2a} \exp \left\{ \frac{a}{4} \textstyle\sum_{k=2}^{n} E[p_k(S_k) \mid S_n = 0] \right\} . \tag{2.21}$$

By (2.3), Lemma A.2, and Lemma A.3, for $k \geqq K$ and $n > 2k$,

$$\begin{aligned} E[p_k(S_k) \mid S_n = 0] \cdot p_n(0) &= \textstyle\int p_k^2(y)\, dQ_{n-k}(y) \\ &\geqq \frac{B^2}{k^2} p\{\|U_{n-k}\| \leqq k^{\frac{1}{2}}\} \\ &= \frac{B^2}{k^2} p \left\{ \left\| \frac{U_{n-k}}{(n-k)^{\frac{1}{2}}} \right\| \leqq \frac{k^{\frac{1}{2}}}{(n-k)^{\frac{1}{2}}} \right\} \\ &\geqq \frac{B^2}{k^2} B' \left(\frac{k}{n-k} \right) = \frac{B^2 B'}{k(n-k)} . \end{aligned}$$

Hence (since $p_n(0) \leqq D/n$), for $n > 2K$,

$$\begin{aligned} \frac{a}{4} \textstyle\sum_{k=2}^{n} E[p_k(S_k) \mid S_n = 0] &\geqq \frac{a}{4} \textstyle\sum_{k=K}^{n/2} \left(\frac{B^2 B'}{Dk} \frac{n}{n-k} \right) \\ &\geqq C \textstyle\sum_{k=K}^{n/2} \frac{1}{k} \geqq C_1 \log n - C_2 \end{aligned} \tag{2.22}$$

for appropriate $C_1 > 0$ and $C_2 > 0$. Therefore (from (2.21) and (2.22)),

$$\frac{1}{8a} q_n(0) \geqq \frac{1}{16a^2} \exp\{C_1 \log n - C_2\} = C_3 n^{C_1}$$

for $n > 2K$ and appropriate $C_3 > 0$. Therefore, from (2.20),

$$q_n'(0) \geqq 2a \exp\{C_3 n^{C_1} - \tfrac{1}{4}\} \qquad \text{for } n > 2K .$$

But $p_n(0) \geqq B/n$; and, hence, $p_n(0) q_n'(0) \to +\infty$ as $n \to \infty$. Theorem 2.2 therefore follows from (2.14). □

3. Solvability and relation to recurrence.

THEOREM 3.1. *Let $\phi(t)$ denote the characteristic function of Q (defined on R^p), and let $B = \{(t_1, \cdots, t_p) : |t_i| \leqq 1, i = 1, \cdots, p\}$. Suppose*

$$\int_B \frac{dt}{1 - |\phi(t)|} < +\infty . \tag{3.1}$$

Then (1.3) *has a bounded solution.*

PROOF. As in Definition A of Section 2, let Y_1 have density $p_1(x) = c_p(x)$ (with

$b = 1$), so that its characteristic function is

$$\begin{aligned} \phi_1(t) &= \prod_{i=1}^{p} (1 - |t_i|) \qquad & t \in B\,; \\ &= 0 \qquad & t \notin B\,. \end{aligned}$$

Let $\{Y_2\, Y_3 \cdots\}$ be independent and identically distributed according to Q, and let $p_n(x)$ be the density of $S_n = Y_1 - \sum_{i=2}^{n} Y_i$ for $n = 2, 3, \cdots$. Then (as in Section 2), $p_{n+1}(x) = (Tp_n)(x)$.

By the Parseval relation, for $n = 1, 2, \cdots$,

$$\begin{aligned} p_n(x) &= \frac{1}{2\pi} \int \phi_1(t)\phi^{n-1}(t)e^{-itx}\, dt \\ &\leqq \frac{1}{2\pi} \int_B |\phi(t)|^{n-1}\, dt = c_n \end{aligned} \tag{3.2}$$

where the last equality defines c_n. Now, by monotone convergence, (3.1) implies that $\sum_{n=1}^{\infty} c_n < +\infty$. Hence, letting $b_n = \prod_{i=1}^{n} (1 + c_i)$, there is $a > 0$ such that $ab_n \leqq 1$ for $n = 1, 2, \cdots$. Define

$$f_n(x) = ab_n p_n(x) \qquad \text{for } x \in R^p,\ n = 1, 2, 3, \cdots. \tag{3.3}$$

Then $0 \leqq f_n(x) \leqq ab_n c_n \leqq c_n \to 0$ (so $\{f_n\}$ is bounded). Also

$$\begin{aligned} (Tf_n)(x) + (Tf_n)^2(x) &= ab_n p_{n+1}(x)[1 + ab_n p_{n+1}(x)] \\ &\leqq ab_n p_{n+1}(x)(1 + c_{n+1}) \\ &= ab_{n+1} p_{n+1}(x) = f_{n+1}(x)\,. \end{aligned} \tag{3.4}$$

□

COROLLARY 3.1. *Let Q' denote the symmetrized version of Q. If the random walk generated by Q' is transient, then* (3.1) *holds* (*for* Q).

PROOF. If the random walk is transient, then (see Feller [3] page 578). $(1 - |\phi(t)|^2)^{-1}$ is integrable in a neighborhood of the origin. But $(1 - |\phi(t)|)^{-1} \leqq 2(1 - |\phi(t)|^2)^{-1}$ for all t; and, hence, (3.1) holds. □

The corollary immediately implies that (1.2) has a bounded solution if $p \geqq 3$ (since all three-dimensional random walks are transient). It also shows that solutions will exist for appropriate stable laws if $p = 1$ or $p = 2$. In particular, if $p = 1$ and Q stable with index $\alpha < 1$ (or, in fact, in the domain of attraction of such a law), then (1.2) has a nonnegative bounded solution. This also holds if Q is stable with index $\alpha < 2$ in two dimensions.

In light of the results here, it is plausible to conjecture that (1.2) has a bounded nonnegative solution if and only if the random walk generated by the symmetrization of Q is transient. Since the results of Section 2 require second moments, it is easy to find classes of distribution in one or two dimensions which are not covered by the present results. Some of these, however, can be obtained by more or less direct extensions of the argument in Theorem 2.2. The argument requires essentially that the characteristic function $(\phi(t/a_n))^n$ remain bounded above zero (in a neighborhood of the origin) where $a_n \to +\infty$ as slow as or more slowly than n (so that $\sum_{i=1}^{n}(1/a_i) \geqq c \log n$). Thus, if $p = 1$ and if Q is in the domain of attraction of a stable law with index $a \geqq 1$, the argument of Section 2 should imply that (1.2) has no bounded nonnegative solution. The argument may even be extended to cover some distributions for which $a_n \to +\infty$ more quickly than n. For example, if $a_n \approx n \log n$, the induction argument in Theorem 2.2 could be iterated yet a third time to show that (1.2) has no bounded nonnegative solutions. Nonetheless, it seems clear that the argument of Section 2 requires too much regularity to cover the general recurrent situation, and the general conjecture above would probably require an entirely different argument.

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APPENDIX

LEMMA A.1. *Let $\{f_i : i = 1, \cdots, n\}$ be nonnegative, measurable functions on R^p, and let E denote expectation with respect to an arbitrary probability measure on R^p. Then*

(a) $E[\prod_{i=1}^{n}(1 + f_i(X))] \leqq \exp\{E[\sum_{i=1}^{n} f_i(X)]\}$.

(b) *If $f_1(x) \leqq 1$ almost surely,*

$$E[\prod_{i=1}^{n}(1 + f_i(X))] \geqq \exp\{\tfrac{1}{4}E[\sum_{i=1}^{n} f_1(X)]\}\ .$$

PROOF. Straightforward.

LEMMA A.2. *Consider random variables on R^2 as follows: let Y_1 have density $c_2(x)$—that is, characteristic function $\phi^*(t) = (1 - |t_1|/b)(1 - |t_2|/b)$ for $|t_1| < b$ and $|t_2| < b$ (and zero otherwise) for an appropriate constant $b > 0$ chosen in the proof—and let $Y_2, Y_3, \cdots$ be independent and identically distributed with mean zero, covariance matrix the identity, and characteristic function $\phi(t)$. Let p_k be the density of $S_k = Y_1 - \sum_{i=2}^{k} Y_i$. Then there are absolute constants $B > 0$, $D > 0$, and an integer $K > 1$ (depending only on the distribution of Y_2) such that*

(a) *$p_k(x) \geqq B/k$ for $||x|| \leqq k$ and $k \geqq K$,*

(b) *$p_k(x) \leqq D/k$ for all x and $k = 1, 2, \cdots$.*

PROOF. First $p_k(x) = k^{-1}f_k(x/k^{\frac{1}{2}})$ where f_k is the density of $(k^{\frac{1}{2}})^{-1}S_k$. Now, using the Taylor expansion for ϕ about zero (see Feller [3] page 489), we may choose $b < 1$ so that $|\phi(t)| \leqq \exp\{-\frac{1}{4}||t||^2\}$ for $|t_1| < b$ and $|t_2| < b$. Let $d_k(z) = kc_2(zk^{\frac{1}{2}})$. Then

$$f_k(x) - \frac{1}{2\pi}\int d_k(x-y)e^{-\frac{1}{2}||y||^2}\,dy$$
$$= \frac{1}{2\pi}\int \phi^*\left(\frac{t}{k^{\frac{1}{2}}}\right)\left(\phi^{k-1}\left(\frac{t}{k^{\frac{1}{2}}}\right) - e^{-\frac{1}{2}||t||^2}\right)e^{-i(t,x)}\,dt$$

(by the Parsevel relation). Therefore (letting Φ denote the normal density),

$$
\begin{aligned}
|f_k(x) - d_k{}^*\Phi| &\leqq \frac{1}{2\pi} \int \left|\phi^*\left(\frac{t}{k^{\frac{1}{2}}}\right)\right|\left|\phi^{k-1}\left(\frac{t}{k^{\frac{1}{2}}}\right) - e^{-\frac{1}{2}\|t\|^2}\right| dt \\
&\leqq \frac{1}{2\pi} \int_{|t_i|<bk^{\frac{1}{2}}} \left|\phi^{k-1}\left(\frac{t}{k^{\frac{1}{2}}}\right) - e^{-\frac{1}{2}\|t\|^2}\right| dt \qquad \text{(A.1)} \\
&\leqq \frac{1}{2\pi} \int_{|t_i|<A} \left|\phi^{k-1}\left(\frac{t}{k^{\frac{1}{2}}}\right) - e^{-\frac{1}{2}\|t\|^2}\right| dt + \frac{1}{\pi} \int_F e^{-\frac{1}{4}\|t\|^2} dt
\end{aligned}
$$

where $F = \{(t_1, t_2) : |t_1| \geqq A \text{ or } |t_2| \geqq A\}$.

By the central limit theorem, the first term converges to zero as $k \to \infty$. The second term can be made arbitrarily small by choosing A large. Choose $B = \inf_{\|x\|\leqq 1} \Phi(x) - \varepsilon$. Now, clearly, $(d_k{}^*\Phi)(x) \to \Phi(x)$ uniformly in x, so we can choose K_1 for which $(d_k{}^*\Phi)(x) \geqq B + \frac{2}{3}\varepsilon$ for $\|x\| \leqq 1$ and $k > K_1$. Now choose A so that the second integral in (A.1) is less than $\varepsilon/3$, and choose $K > K_1$ so that if $k > K$, the first integral in (A.1) is less than $\varepsilon/3$. Then $f_k(x) \geqq B$ for $\|x\| \leqq 1$ and $k > K_1$; from which (a) follows.

Part (b) can be obtained from (A.1) as follows:

$$|f_k(x) - (d_k{}^*\Phi)(x)| \leqq \frac{1}{\pi} \int e^{-\frac{1}{4}\|t\|^2} dt = 4 .$$

Thus, since $(d_k{}^*\Phi)(x) \leqq 1/2\pi$ for all x, $f_k(x) \leqq 1/2\pi + 4 = D$; from which part (b) follows.

LEMMA A.3. *Let Q be an arbitrary symmetric probability distribution in R^2 with zero mean and covariance matrix the identity. Let $\{X_1, X_2, \cdots, X_m\}$ be independent and identically distributed according to Q, and define$U_m = \sum_{i=1}^m X_i$. Then, there is a constant $B' > 0$ (depending only on Q) such that for any value a, $0 \leqq a \leqq 1$,*

$$p\left\{\frac{\|U_m\|}{m^{\frac{1}{2}}} \leqq a\right\} \geqq B'a^2 .$$

PROOF. Let G_m be the distribution of $U_m/m^{\frac{1}{2}}$ letting $b = (2^{\frac{1}{2}})^{-1} a$ and using the Parseval relation,

$$
\begin{aligned}
p\left\{\left\|\frac{U_m}{m^{\frac{1}{2}}}\right\| \leqq a\right\} &= \int_{\|y\|\leqq a} dG_m(y) \\
&\geqq \int_{-b}^{b} \int_{-b}^{b} \left(1 - \frac{|y_1|}{b}\right)\left(1 - \frac{|y_2|}{b}\right) dG_m(y) \\
&= 2b^2 \iint \left(\frac{1 - \cos t_1 b}{b^2 t_1{}^2}\right)\left(\frac{1 - \cos t_2 b}{b^2 t_2{}^2}\right) \phi^m\left(\frac{t}{m^{\frac{1}{2}}}\right) dt_1\, dt_2 .
\end{aligned}
$$

Now, by the central limit theorem, $\phi^m(t/m^{\frac{1}{2}}) \to e^{-\frac{1}{2}\|t\|^2}$ uniformly on compact sets; and, hence, there is $c > 0$ such that for m large, $\phi^m(t/m^{\frac{1}{2}}) \geqq \frac{1}{2}$ for $|t_i| \leqq c$. Thus, we may choose $c' < 2\pi$ so that for all m $\phi^m(t/m^{\frac{1}{2}}) \geqq \frac{1}{2}$ for $|t_i| \leqq c'$. Also, for $|t_i| \leqq c'$, $|t_i b| \leqq c'$, and there is $B > 0$ such that $(1 - \cos bt_i/b^2t_i{}^2) \geqq B$ for

$|t_i| \leqq c'$. Therefore (since $b^2 = \frac{1}{2}a^2$ and $\phi(t)$ is positive by symmetry of Q)

$$P\left\{\left\|\frac{U_m}{m^{\frac{1}{2}}}\right\| \leqq a\right\} \geqq a^2 \int_{-c'}^{c'} \int_{-c'}^{c'} B^2 \cdot \tfrac{1}{2}\, dt_1\, dt_2$$
$$= \tfrac{1}{2}B^2(2c')^2 a^2 = B'a^2$$

where $B' = \frac{1}{2}B^2(2c')^2$.

LEMMA A.4. *Let* $\{S_k : k = 1, 2, \cdots\}$ *and* $\{p_k : k = 1, 2, \cdots\}$ *be as in Definition* A *of Section* 2, *and let* $\{q_k : k = 1, 2, \cdots\}$ *be defined by* (2.8). *Then*

$$\sum_{k=2}^{n} E[\tfrac{1}{2}p_k(S_k)q_k(S_k) \mid S_n = x] \geqq \frac{1}{2a} q_n(x) - 1 .$$

PROOF (by induction). For $n = 2$, the left-hand side is

$$\tfrac{1}{2}p_2(x)q_2(x) \geqq ap_2(x) = E[1 + ap_2(S_2) \mid S_2 = x] - 1$$
$$= \frac{1}{2a} q_2(x) - 1 .$$

So assume the inequality holds for $(n - 1)$ and consider the following:

$$\begin{aligned}
&\sum_{k=2}^{n} E[\tfrac{1}{2}p_k(S_k)q_k(S_k) \mid S_n = x] \\
&\quad = \sum_{k=2}^{n-1} E[\tfrac{1}{2}p_k(S_k)q_k(S_k) \mid S_n = x] + \tfrac{1}{2}p(x)q_n(x) \\
&\quad = \sum_{k=2}^{n-1} E\{E[\tfrac{1}{2}p_k(S_k)q_k(S_k) \mid S_n, S_{n-1}] \mid S_n = x\} + \tfrac{1}{2}p_n(x)q_n(x) \\
&\quad = \sum_{k=2}^{n-1} E\{E[\tfrac{1}{2}p_k(S_k)q_k(S_k) \mid S_{n-1}, Y_n] \mid S_n = x\} + \tfrac{1}{2}p_n(x)q_n(x) \\
&\quad = E\{\textstyle\sum_{k=2}^{n-1} E[\tfrac{1}{2}p_k(S_k)q_k(S_k) \mid S_{n-1}] \mid S_n = x\} + \tfrac{1}{2}p_n(x)q_n(x) \\
&\quad \geqq E\left\{\frac{1}{2a} q_{n-1}(S_{n-1}) - 1 \mid S_n = x\right\} + \tfrac{1}{2}p_n(x)q_n(x) \\
&\quad = \frac{1}{2a} E[q_{n-1}(S_{n-1}) \mid S_n = x] - 1 \\
&\qquad\quad + \frac{1}{2a} \cdot ap_n(x)(1 + ap_n(x))E[q_{n-1}(S_{n-1}) \mid S_n = x] \\
&\quad \geqq \frac{1}{2a} (1 + ap_n(x))E[q_{n-1}(S_{n-1}) \mid S_n = x] - 1 \\
&\quad = \frac{1}{2a} q_n(x) - 1
\end{aligned}$$

where the third equality uses independence of S_{n-1} and Y_n, the next inequality uses the induction hypothesis, and the last inequality uses the fact that $1 + ap_n(x) \geqq 1$.

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