The Annals of Probability
1975, Vol. 3, No. 4, 737–738

ON DERIVATIVES OF CHARACTERISTIC FUNCTIONS[1]

BY STEPHEN J. WOLFE
University of Delaware

If k is a positive odd integer, it is shown that it is possible to construct a characteristic function $f(t)$ such that $f^{(k)}(0)$ exists but $f^{(k)}(t_m)$ does not exist for a sequence of numbers $\{t_m\}$ where $t_m \to 0$ as $m \to \infty$.

1. Introduction. Various authors have studied the relationships between the asymptotic behavior of a distribution function and the behavior of its characteristic function near the origin. A discussion of this work can be found in [2].

Let $f(t)$ be the characteristic function of a distribution function $F(x)$. It is well known that if k is a positive even integer, the existence of $f^{(k)}(0)$ implies the existence of the kth absolute moment of $F(x)$ and thus the existence of $f^{(k)}(t)$ for all real t. If k is a positive odd integer, the existence of $f^{(k)}(0)$ does not imply the existence of the kth absolute moment of $F(x)$. Thus it is of interest to ask the following question: If k is a positive odd integer and $f^{(k)}(0)$ exists, does $f^{(k)}(t)$ exist for all t or at least for all t in some neighborhood of the origin? In this note, it will be shown that if k is a positive odd integer then it is possible to construct a characteristic function $f(t)$ such that $f^{(k)}(0)$ exists but $f^{(k)}(t_m)$ does not exist for a sequence of numbers $\{t_m\}$ where $t_m \to 0$ as $m \to \infty$. This construction depends on a result of Boas [1] that if $1 - F(x) + F(-x) = o(x^{-1})$ as $x \to +\infty$ then $f'(t)$ exists if and only if

$$\lim_{T\to+\infty} \int_{-T}^{T} xe^{ixt}\, dF(x)$$

exists.

2. Construction. For each positive integer n, let $F_n(x)$ be the distribution function with masses $c_n/j^2 \ln j$ concentrated at the points $\pm j$ for $j = 2^n, (2^n)5, (2^n)9, \cdots$, where c_n is chosen so that the sum of the masses is 1. Let $F(x) = \sum_{n=1}^{\infty} 2^{-n} F_n(x)$ and let $f(t)$ be the characteristic function of $F(x)$. Let

$$h_n(t, T) = \int_{-T}^{T} x \sin xt\, dF_n(x)$$

and let m be an integer that is greater than 1. If $n < m - 1$ the first 2^{m-n-2} terms of the sequence $\{\sin [2^{n-m}(1 + 4k)\pi]\}_{k=0}^{\infty}$ are positive, the next 2^{m-n-2} terms are negative, and so on. Thus $\lim_{T\to+\infty} h_n(\pi/2^m, T)$ exists and is positive. If $n = m - 1$ then $\lim_{T\to+\infty} h_n(\pi/2^m, T) = +\infty$. If $n > m - 1$ then $h_n(\pi/2^m, T) = 0$ for $T \geqq 0$. If follows that

$$\lim_{T\to+\infty} \int_{-T}^{T} x \sin (\pi x/2^m)\, dF(x) = +\infty$$

and thus $f'(\pi/2^m)$ does not exist.

Received October 21, 1974.
[1] This research was supported by The National Science Foundation Grant GP-43986.
AMS 1970 *subject classifications.* Primary 60E05, Secondary 42A72.
Key words and phrases. Characteristic function, derivative.

For each n there is a positive constant D_n such that

$$1 - F_n(x) + F_n(-x) \leqq D_n \int_{x-1}^{\infty} (y^2 \ln y)^{-1} \, dy$$

if $x > 2$. Thus $1 - F_n(x) + F_n(-x) = o(x^{-1})$ as $x \to +\infty$ for each n and it follows that $1 - F(x) + F(-x) = o(x^{-1})$ as $x \to +\infty$. Since $F(x)$ is symmetric it follows that $f'(0)$ exists.

If k is a positive odd integer that is greater than 1 and n is a nonnegative integer let

$$G_n(x) = b_n \int_{-\infty}^{x} y^{1-k} \, dF_n(y)$$

where b_n is chosen so that $G_n(x)$ is a distribution function. Let $G(x) = \sum_{n=1}^{\infty} 2^{-n} G_n(x)$ and let $g(t)$ be the characteristic function of $G(x)$. It is easy to see that $g^{(k)}(0)$ exists but $g^{(k)}(t_m)$ does not exist for a sequence of numbers $\{t_m\}$ where $t_m \to 0$ as $m \to \infty$.

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Department of Mathematics
University of Delaware
Newark, Delaware 19711