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ASYMPTOTIC RESULTS FOR ESTIMATORS IN A SUBCRITICAL BRANCHING PROCESS WITH IMMIGRATION[1]

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It is shown that certain estimators of the offspring and immigration means in a subcritical simple branching process with immigration are strongly consistent and obey the central limit theorem and law of the iterated logarithm under natural conditions.

1. Introduction and main results. Let $\{X_n, n = 0, 1, 2, \cdots\}$ be a simple branching process with immigration, that is, a temporally homogeneous Markov chain with state space $I = \{0, 1, 2, \cdots\}$ and transition probabilities

$$P(X_n = j \mid X_{n-1} = i) = \{f^{*i} * b\}_j , \qquad n \geqq 1 ;$$

the probability distributions $\{f_i\}$ and $\{b_i\}$ are called the offspring and immigration distributions respectively.

The principal aim of this paper is to provide asymptotic results for estimators of the two means

$$\lambda_1 = \Sigma i f_i , \qquad \lambda_2 = \Sigma i b_i ,$$

in the case when the condition

$$\text{(A)} \qquad 0 < \lambda_1 < 1 ; \qquad b_0 < 1$$

is satisfied. Some of the results obtained constitute improvements of the work of Heyde and Seneta (1972, 1974).

The estimators investigated are

$$\hat{\lambda}_{1,n} = \frac{\sum_{i=1}^n X_i(X_{i+1} - n^{-1}S_n)}{\sum_{i=1}^n (X_i - n^{-1}S_n)^2}$$

and

$$\hat{\lambda}_{2,n} = \frac{S_n}{2n} \cdot \frac{\sum_{i=1}^n (X_{i+1} - X_i)^2}{\sum_{i=1}^n (X_i - n^{-1}S_n)^2}$$

of λ_1 and λ_2 respectively; S_n throughout denotes $\Sigma_1{}^n X_i$.

We will need the following notation: let $\sigma_1{}^2$, $\sigma_2{}^2$ denote the variances of the offspring and immigration distributions respectively, and put

$$\mu = \lambda_2/(1 - \lambda_1) ,$$

$$c^2 = \mu\sigma_1{}^2 + \sigma_2{}^2 ,$$

$$\gamma = \frac{1}{1 - \lambda_1{}^3}\left\{\Sigma(j - \lambda_2)^3 b_j + \mu\Sigma(j - \lambda_1)^3 f_j + \frac{3\lambda_1\sigma_1{}^2 c^2}{1 - \lambda_1{}^2}\right\} .$$

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We also write c^2 as k_2^2, and put

$$k_1 = \left(\gamma\sigma_1^2 + \frac{c^4}{1-\lambda_1^2}\right)^{\frac{1}{2}} (1-\lambda_1^2)/c^2 .$$

Let N_i denote unit normal random variables.

The main results are as follows:

THEOREM. *Assume condition* A. *If* $c^2 < \infty$,

$$\hat{\lambda}_{i,n} \to \lambda_i \quad \text{a.s.} \qquad i = 1, 2 . \tag{1.1}$$

For $i = 1$ *or* 2,

$$n^{\frac{1}{2}}(\hat{\lambda}_{i,n} - \lambda_i)/k_i \to_{\mathscr{D}} N_i , \tag{1.2}$$

and

$$\limsup \frac{n^{\frac{1}{2}}(\hat{\lambda}_{i,n} - \lambda_i)}{k_i(2\log\log n)^{\frac{1}{2}}} = 1 \quad \text{a.s.} \tag{1.3}$$

$$\liminf \frac{n^{\frac{1}{2}}(\hat{\lambda}_{i,n} - \lambda_i)}{k_i(2\log\log n)^{\frac{1}{2}}} = -1 \quad \text{a.s.},$$

as long as $k_i < \infty$.

REMARK. These results for $i = 2$ improve those implied by Heyde and Seneta (1972, 1974) (who actually deal with estimators of λ_1 and μ). The present estimator of λ_1 is somewhat different from their D_n or D_n^*. However, the present techniques can be used to improve their results concerning these two estimators.

2. Proofs. The theorem follows from known results about Markov chains once a few facts are established. Firstly, note that (as is not difficult to verify) the state space I contains a countable irreducible set I^* on which $\{X_1, X_2, \cdots\}$ have their support. In view of A, the state

$$\kappa \equiv \inf\{i : b_i > 0\} = \inf\{i : i \in I^*\}$$

is accessible at all times $n \geqq 1$, so that I^* is aperiodic. Let $\mathscr{F}_{I^*}$ be the σ-field of all subsets of I^*.

Heathcote (1966) showed that under certain conditions (which Quine (1970b) weakened to condition A),

$$\sum_{j=1}^{\infty} b_j \log j < \infty \Rightarrow \{X_n\} \quad \text{positive recurrent}$$

$$\sum_{j=1}^{\infty} b_j \log j = \infty \Rightarrow \forall\, \theta > 0 : P(X_n < \theta) \to 0 \qquad \text{as } n \to \infty .$$

Thus if $\sum b_j \log j < \infty$, I^* forms a positive class. If ϕ is any function from $\mathscr{F}_{I^*} \times \mathscr{F}_{I^*}$ to the real line, it therefore follows from Theorems 1.1 and 1.3 of Billingsley (1961) that in this case, if $\{\pi_i\}$ is the limiting distribution of $\{X_n\}$,

$$n^{-1}\sum_1^n \phi(X_j, X_{j+1}) \to \sum_{i,j} \pi_i P(X_1 = j \mid X_0 = i)\phi(i, j) \quad \text{a.s.} \tag{2.1}$$

as long as the limiting series converges absolutely. The reader should bear in mind that the limiting distribution is independent of the distribution of X_0.

It can be shown, for example by differentiation of the functional equation

$$\phi(t) = \rho(t) + \phi(\chi(t)) ,$$

where $\exp\phi(t) = \sum \pi_j \exp(tj)$, $\exp\rho(t) = \sum b_j \exp(tj)$, and $\exp\chi(t) = \sum f_j \exp(tj)$, that the stationary distribution $\{\pi_i\}$ has mean μ, variance $c^2/(1 - \lambda_1^2)$ and central third moment γ (the finiteness of these stationary moments is assured by Fatou's lemma and (2.4) below; see also Quine (1970b), Section 3).

By suitable choices of ϕ we may now show that when the limits are finite,

$$\begin{aligned} &n^{-1}S_n \to \mu \quad \text{a.s.,} \\ (2.2) \qquad &n^{-1}\Sigma_1^n(X_{i+1} - X_i)^2 \to \frac{2c^2}{1 + \lambda_1} \quad \text{a.s.,} \\ &n^{-1}\Sigma_1^n X_i^2 \to \frac{c^2}{1 - \lambda_1^2} + \mu^2 \quad \text{a.s.,} \\ &n^{-1}\Sigma_1^n(X_i - \mu)^3 \to \gamma \quad \text{a.s.} \end{aligned}$$

For example, to prove the second part one puts $\phi(i, j) = (j - i)^2$ in (2.1) so that

$$\begin{aligned} n^{-1}\Sigma_1^n(X_{i+1} - X_i)^2 &\to \textstyle\sum_{i,j} \pi_i(j - i)^2 P(X_1 = j \mid X_0 = i) \quad \text{a.s.} \\ &= 2 \textstyle\sum_j j^2\pi_j - 2 \sum_i i\pi_i \sum_j jP(X_1 = j \mid X_0 = i) \end{aligned}$$

since $\{\pi_j\}$ is a stationary distribution, and

$$= \frac{2c^2}{1 + \lambda_1}$$

after a little algebra. Equation (1.1) follows quickly from (2.2).

Before proceeding to prove the remainder of the theorem, some moment results must be established. These results serve to identify various constants in the theorem. It should be borne in mind that the moment restrictions we will put on X_0 at this stage are not necessary for the validity of the theorem. Writing $EX_n = \mu_n$, we have

$$\begin{aligned} \operatorname{Var} S_n &= \operatorname{Var} S_{n-1} + \operatorname{Var} X_n + 2 \textstyle\sum_{r=1}^{n-1} E(X_r - \mu_r)(X_n - \mu_n) \\ (2.3) \qquad &= \operatorname{Var} S_{n-1} + \operatorname{Var} X_n \\ &\quad + 2 \textstyle\sum_{r=1}^{n-1} [\lambda_1^{n-r} \operatorname{Var} X_r + \mu\mu_r + \lambda_1^{n-r}\mu_r(\mu_r - \mu) - \mu_n\mu_r] \\ &= \operatorname{Var} S_{n-1} + \operatorname{Var} X_n + R_n , \end{aligned}$$

say. The sort of arguments used in Quine (1970a; 1970b, Section 3) show that

$$(2.4) \qquad \begin{bmatrix} \mu_r \\ \operatorname{Var}(X_r) \\ E(X_r - \mu_r)^3 \end{bmatrix} = Q^r \begin{bmatrix} \mu_0 \\ \operatorname{Var} X_0 \\ E(X_0 - \mu_0)^3 \end{bmatrix} + (I - Q)^{-1}(I - Q^r) \begin{bmatrix} \lambda_2 \\ \sigma_2^2 \\ \sum (j - \lambda_2)^3 b_j \end{bmatrix}$$

where

$$(I - Q)^{-1} = \begin{bmatrix} 1-\lambda_1 & 0 & 0 \\ -\sigma_1^2 & 1-\lambda_1^2 & 0 \\ -F_3 & -3\lambda_1\sigma_1^2 & 1-\lambda_1^3 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} \dfrac{1}{1-\lambda_1} & 0 & 0 \\ \dfrac{\sigma_1^2}{(1-\lambda_1)(1-\lambda_1^2)} & \dfrac{1}{1-\lambda_1^2} & 0 \\ \dfrac{F_3}{(1-\lambda_1)(1-\lambda_1^3)} + \dfrac{3\lambda_1\sigma_1^4}{(1-\lambda_1)(1-\lambda_1^2)(1-\lambda_1^3)} & \dfrac{3\lambda_1\sigma_1^2}{(1-\lambda_1^2)(1-\lambda_1^3)} & \dfrac{1}{1-\lambda_1^3} \end{bmatrix}$$

F_3 denoting $\Sigma(j - \lambda_1)^3 f_j$. It follows without trouble that if c^2, $E(X_0^2) < \infty$,

$$\text{(2.5)} \qquad \operatorname{Var} X_n \to \frac{c^2}{1-\lambda_1^2},$$

and that if γ, $E(X_0^3) < \infty$,

$$\text{(2.6)} \qquad E(X_n - \mu_n)^3 \to \gamma .$$

It also follows after some computation that

$$\Sigma_1{}^j R_n \sim \frac{2j\lambda_1 c^2}{(1-\lambda_1)(1-\lambda^2{}_1)}$$

which, together with (2.3) and (2.5), shows that if c^2, $EX_0^2 < \infty$,

$$\text{(2.7)} \qquad n^{-1} \operatorname{Var} S_n \to \frac{c^2}{1-\lambda_1^2} + \frac{2\lambda_1 c^2}{(1-\lambda_1)(1-\lambda_1^2)} = \frac{c^2}{(1-\lambda_1)^2}.$$

We now turn to the results (1.2) and (1.3) when $i = 2$. It is clear from the definition of $\hat{\lambda}_{2,n}$ and from (2.2) that both results will be proved if we can prove analogous results about $n^{-1}S_n$. But these results are merely applications of the theory in Section 1.16 of Chung (1967). Assume $c^2 < \infty$, so that in particular $\{X_i, i \geqq 1\}$ has support on a positive class. Then using (2.7), and Theorems 1.16.1 and 1.16.3 of Chung (ibid.) (we take his X_i to be our X_{i+1}),

$$\text{(2.8)} \qquad (1-\lambda_1)(S_n - n\mu)/cn^{\frac{1}{2}} \to_{\mathscr{D}} N_3 .$$

This result was given previously by Pakes (1971) under the additional conditions $f_0 + f_1 < 1$, $b_0 > 0$ and $X_0 = 0$.

Define $T = \inf\{j : j \geqq 1, X_j = \kappa\} I(X_0 = \kappa)$. Chung (ibid.) shows that if in addition to A, $ET^2 < \infty$ and $ES_T{}^2 < \infty$, then the law of the iterated logarithm holds for S_n, specifically, writing $\nu_n = cn^{\frac{1}{2}}/(1-\lambda_1)$,

$$\limsup \frac{S_n - n\mu}{\nu_n(2 \log\log n)^{\frac{1}{2}}} = 1 \quad \text{a.s.},$$

with the corresponding result for the lim inf. We remark that T relates to state

κ for convenience only; as noted by Chung (ibid., page 102) the second moment conditions hold for all $i \in I^*$ or for none. In passing we note that in the same reference it is shown that in this case the distributions of

$$\max_{1 \leq j \leq n} \nu_n^{-1}[S_j - j\mu], \qquad \max_{1 \leq j \leq n} \nu_n^{-1}|S_j - j\mu|$$

converge to known laws. More details may be found in Chung's monograph. The condition $ET^2 < \infty$ holds if λ_2, $\sum jf_j \log^+ j < \infty$, since in this case the chain is geometrically ergodic, as noted by Pakes (1971). Pakes actually requires a stronger condition than A, but inspection of his proof, bearing in mind the strengthening in Quine (1970b, pages 414, 418) of the theorem of Heathcote (1966), shows that A suffices. ES_T^2 can be shown to be finite when $c^2 < \infty$: we have

$$ES_T^2 = E\Sigma_1^T X_j^2 + 2E \sum_{j=2}^T \sum_{i=1}^{j-1} X_i X_j \,. \tag{2.9}$$

Now the finiteness of $E\Sigma_1^T X_j^2$ when $c^2 < \infty$ can be shown using arguments along the lines of Heyde and Seneta (1972), pages 241–242; we note in passing that in our case,

$$ES_T = \mu ET \,,$$

which can be proved directly or using Theorem 1.14.5 of Chung (1967). As for the cross product term,

$$E \sum_{j=2}^T \sum_{i=1}^{j-1} X_i X_j = E \sum_{j=2}^\infty \sum_{i=1}^{j-1} X_i X_j I(T \geqq j) = \sum_{j \geq 2} e_j \,, \quad \text{say,}$$

and

$$\begin{aligned} e_j &= E \sum_{i=1}^{j-1} I(T \geqq j) X_i \{\lambda_1 X_{j-1} + \lambda_2\} \\ &\leqq \lambda_1 e_{j-1} + \lambda_1 d_{j-1} + \lambda_2 E \sum_{i=1}^{j-1} I(T \geqq j) X_i \,, \end{aligned}$$

where $\Sigma d_j \equiv E\Sigma X_j^2 I(T \geqq j) = E\Sigma_1^T X_j^2 < \infty$ as noted above. Thus the cross product term in (2.9) will be finite if $E \sum_{j=2}^\infty S_{j-1} I(T \geqq j)$ is. But this equals

$$\begin{aligned} E\Sigma_1^{T-1} S_j &\leqq ETS_T \leqq ET^2 \max_{1 \leq j \leq T} X_j \\ &\leqq E^{\frac{1}{2}} T^4 E^{\frac{1}{2}} \max_{1 \leq j \leq T} X_j^2 \leqq E^{\frac{1}{2}} T^4 E^{\frac{1}{2}} \Sigma_1^T X_j^2 \end{aligned}$$

so that the finiteness of (2.9) when $c^2 < \infty$ follows from geometric ergodicity (which implies *all* moments of T are finite). It is now clear that the LIL result for S_n implies (1.3) in case $i = 2$.

The easiest way to prove the remainder of the theorem appears to be to introduce a functional of the process:

$$M_n = (X_n - \mu)(X_{n+1} - \lambda_1 X_n - \lambda_2) \,.$$

We note in passing that $\Sigma_1^n M_j$ is a martingale relative to the σ-field $\mathscr{F}_{n+1}$ generated by $X_0, \cdots, X_{n+1}$; see also Heyde and Seneta (1972).

It is easy to check that $\mathbf{X}_n = (X_{n+1}, X_n)$ is Markovian. If

$$T' = \inf(i \geqq 1 : X_i = X_{i+1} = \kappa) \,;$$

then

$$P(T' = 1 \mid X_0 = X_1 = \kappa) = b_\kappa f_0^\kappa = \alpha \,,$$

say, and if

$$T = \inf(i > 1 : X_i = \kappa) - 1 ,$$
$$T'' = \inf(i \geqq 1 : X_{T+i} = X_{T+i+1} = \kappa) ,$$

then for $j > 2$,

$$\begin{aligned} P(T' = j \mid X_0 = X_1 = \kappa) &= \textstyle\sum_{i=2}^{j-1} P(T = i, T'' = j - i \mid X_0 = X_1 = \kappa) \\ &= \textstyle\sum_{i=2}^{j-1} P(T'' = j - i \mid T = i, X_0 = X_1 = \kappa)P(T = i \mid X_0 = X_1 = \kappa) \\ &= \textstyle\sum_{i=2}^{j-1} P(T' = j - i \mid X_0 = X_1 = \kappa)P(T = i \mid X_0 = X_1 = \kappa) , \end{aligned}$$

so that if H and G are the p.g.f.'s of T' and T, conditional on $X_0 = X_1 = \kappa$, then

$$H(s) = \alpha s + H(s)\{G(s) - \alpha s\} .$$

Rearranging and differentiating, one finds that, conditional on $X_0 = X_1 = \kappa$,

$$ET' = \alpha^{-1}ET , \qquad ET'^2 = \alpha^{-1}(ET^2 - 2ET + 2\alpha^{-1}(ET)^2)$$

and that the rth conditional moments of T' and T converge or diverge together. However, as noted earlier, $\sum b_j \log j < \infty$ implies positive recurrence of $\{X_n\}$ so that $E(T \mid X_0 = X_1 = \kappa) < \infty$ and hence $E(T' \mid X_0 = X_1 = \kappa) < \infty$. Thus in this case, $\{\mathbf{X}_n\}$ has support on a positive class. We now apply the results of Chung (1967, Section 16) to the functional M_n of $\mathbf{X}_n$, taking his X_n to be our $\mathbf{X}_{n+1}$. Firstly, we see that M_n will obey the central limit theorem as long as

$$B^2 = \lim n^{-1}\Sigma_1^n EM_j^2 < \infty .$$

However, using (2.5) and (2.6) one finds

$$B^2 = \sigma_1^2\gamma + \frac{c^4}{1 - \lambda_1^2} .$$

Thus when $\gamma < \infty$,

$$n^{-\frac{1}{2}}\Sigma_1^n M_i/B \to_{\mathscr{D}} N_4 .$$

Furthermore,

$$n^{\frac{1}{2}}(\hat{\lambda}_{1,n} - \lambda_1) = \frac{n^{-\frac{1}{2}}\Sigma_1^n M_i - (1 - \lambda_1)n^{\frac{1}{2}}(\mu - n^{-1}S_n)^2 + \mu n^{-\frac{1}{2}}\{X_{n+1} - X_1\}}{n^{-1}\Sigma_1^n X_i^2 - n^{-2}S_n^2} . \tag{2.10}$$

Using (2.2) we see the denominator converges a.s. to $c^2/(1 - \lambda_1^2)$. Using (2.2) and (2.8), we see that the second term in the numerator converges in probability to zero, and the last term does likewise since X_n converges in distribution. Equation (1.2), for $i = 1$, follows.

It also follows from (2.10) and the previous a.s. results concerning S_n that in order to prove equation (1.3) for $i = 1$ it suffices to prove the LIL result for $\Sigma_1^n M_i$. In view of earlier arguments it should be clear that to do this we need only establish the finiteness of $E(\Sigma_1^{T'} M_j)^2$. However, the cross product terms vanish since M_j is a martingale difference sequence. The finiteness of the diagonal terms can be established using the earlier methods. Details are omitted.

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