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ON STRONG APPROXIMATION OF THE MULTIDIMENSIONAL EMPIRICAL PROCESS

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Let $\mathbf{X}_1, \mathbf{X}_2, \cdots$ be a sequence of i.i.d.rv's uniformly distributed over the unit square I^2. Further, let F_n be the empirical distribution function based on the sample $\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_n$. A sequence $\{B_n\}$ of Brownian bridges and a Kiefer process K is constructed such that

$$\sup_{A \in Q} |n^{\frac{1}{2}}(F_n(A) - \lambda(A)) - B_n(A)| = O(n^{-\frac{1}{19}})$$
$$\sup_{A \in Q} |n(F_n(A) - \lambda(A)) - K(A; n)| = O(n^{\frac{1}{2}\frac{2}{5}})$$

a.s. where $F_n(A)$, $B_n(A)$, $K(A; n)$ are the corresponding random measures of A, λ is the Lebesgue measure and Q is the set of Borel sets of I^2 having twice differentiable boundaries.

1. Introduction. The first strong invariance principle for empirical distribution functions (e.d.f.'s) was formulated in 1969 by Brillinger [1]:

THEOREM A. ([1]). *Let $X_1, X_1, \cdots$ be a sequence of independent $U(0, 1)$ rv's defined on a rich enough probability space. Further let $E_n(x)$ be the* e.d.f. *based on the sample $X_1, X_2, \cdots, X_n$. Then one can define a sequence $\{B_n\}$ of Brownian bridges* (B.B.'*s*) *such that*

$$\sup_{0 \leq x \leq 1} |n^{\frac{1}{2}}(E_n(x) - x) - B_n(x)| = O(n^{-\frac{1}{4}}(\log n)^{\frac{1}{2}}(\log \log n)^{\frac{1}{4}})$$

a.s. *as* $n \to \infty$.

The precise meaning of "rich enough" will not be formulated each time; it will, however, be enough to assume all the time that independent sequences of Wiener processes (W.P.'s) $\{W_n\}$ and normally distributed rv's $\{N_n\}$ which are independent of the originally given i.i.d. sequence $\{X_n\}$ can be constructed on the assumed probability space. From now on it will be assumed that the underlying probability space is rich enough in this sense.

The following stronger theorem was proved in 1975 by Komlós–Major–Tusnády [7]:

THEOREM B ([7]). *A sequence $\{B_n\}$ of* B.B.'*s can be constructed such that*

$$\sup_x |n^{\frac{1}{2}}(E_n(x) - x) - B_n(x)| = O(n^{-\frac{1}{2}} \log n)$$

a.s.

Kiefer [6] proposed to investigate the empirical process (E.P.) $\alpha_n(x) = n^{\frac{1}{2}}(E_n(x) - x)$ $(0 \leq x \leq 1;\ n = 1, 2, \cdots)$ as a process of two variables (x, n) and

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he proved:

THEOREM C ([6]). *There can be constructed a Gaussian process* (G.P.) $\{K(x; y): 0 \leqq x \leqq 1; 0 \leqq y < \infty\}$ *such that*

$$\sup_x |n^{\frac{1}{2}}\alpha_n(x) - K(x; n)| = O(n^{\frac{1}{3}}(\log n)^{\frac{2}{3}}) \qquad n \to \infty$$

a.s. *where K can be generated by a* W.P. $W(x, y)$ *of two variables as follows*:

$$K(x; y) = W(x, y) - xW(1, y) .$$

(For the definition of W see N1 in Section 2 below.)

The G.P. K will be called a Kiefer process (K.P.).

Komlós–Major–Tusnády also proved a stronger result in connection with this problem, too:

THEOREM D ([7]). *There exists a* K.P. K *such that*

$$\sup_x |n^{\frac{1}{2}}\alpha_n(x) - K(x; n)| = O(\log^2 n) \qquad n \to \infty$$

a.s.

It is natural to ask how these results can be generalized to the multidimensional case. The first result in this direction is the following:

THEOREM E ([3b]). *Let $X_1, X_2, \cdots$ be a sequence of independent* rv's *uniformly distributed over the unit cube I^d of the d-dimensional Euclidean space. Then one can define a sequence $\{B_n\}$ of* B.B.'s *and a* K.P. $\{K(\mathbf{x}; y): \mathbf{x} \in I^d; 0 \leqq y < \infty\}$ *such that*

$$\sup_{x \in I^d} |\alpha_n(\mathbf{x}) - B_n(\mathbf{x})| = O(n^{-1/(2(d+1))}(\log n)^{\frac{3}{2}}) ,$$
$$\sup_{x \in I^d} |n^{\frac{1}{2}}\alpha_n(\mathbf{x}) - K(\mathbf{x}; n)| = O(n^{(d+1)/(2(d+2))} \log^2 n)$$

a.s. *where the* E.P. $\alpha_n(\mathbf{x}) = \alpha_n(x_1, x_2, \cdots, x_d) = n^{\frac{1}{2}}(E_n(\mathbf{x}) - x_1, x_2, \cdots, x_d)$ *and $E_n(x)$ is the* e.d.f. *based on the sample* $\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_n$.

(The definition of B and K is given in Section 2 for $d = 2$; for the general case we refer to [3b].)

The main aim of this paper is to investigate the distance between the empirical measure $\alpha_n(A)$ and a Brownian measure $B_n(A)$, and between that of $\alpha_n(A)$ and the Kiefer measure $K(A; n)$, where A runs over a suitable subset of the Borel sets of the unit square. The precise meaning of these measures will be formulated in Section 2, where the main results are also summarized. All the results here will be formulated and proved in the two-dimensional case only; it appears, however, that their generalization to higher dimensions is possible via the methods of this paper.

In some aspects, the results of this exposition were suggested by a conversation of R. Pyke at Oberwolfach in November 1974 and by a paper of Dudley [4]. In fact here we do not apply Dudley's ingenious technique; our more classical definition of the Wiener measure seems to be quite suitable for the purposes of this paper. Our Theorem 1 could be obtained as a consequence of Theorem 4.2 of Dudley. Here we give a direct proof.

The proofs mostly follow those of [3b] whose minor oversights are also corrected while going along.

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2. Notations and results.

N1. The following G.P.'s will be used frequently:

(i) W.P.: $W(\mathbf{x}) = W(x_1, x_2, \cdots, x_d)(0 \leqq x_i \leqq 1; i = 1, 2, \cdots, d)$ is a separable G.P. with $EW(\mathbf{x}) = 0$, $R(\mathbf{x}_1, \mathbf{x}_2) = EW(\mathbf{x}_1)W(\mathbf{x}_2) = EW(x_{11}, x_{12}, \cdots, x_{1d}) W(x_{21}, x_{22}, \cdots, x_{2d}) = \min(x_{11}, x_{21}) \cdot \min(x_{12}, x_{22}) \cdot \cdots \cdot \min(x_{1d}, x_{2d})$,

(ii) B.B.: $B(\mathbf{x}) = B(x_1, x_2) = W(\mathbf{x}) - x_1 x_2 W(1, 1)$ $(0 \leqq x_1 \leqq 1; 0 \leqq x_2 \leqq 1)$,

(iii) K.P.: $K(\mathbf{x}, y) = K(x_1, x_2; y) = W(x_1, x_2, y) - x_1 x_2 W(1, 1, y)$ $(0 \leqq x_1 \leqq 1, 0 \leqq x_2 \leqq 1; 0 \leqq y < \infty)$.

N2. Let $\mathscr{A} = \mathscr{A}(L, M)$ $(L \geqq 1, M = 0, 1, 2, \cdots)$ be the set of those continuous vector valued functions $\mathbf{z}(t) = (x(t), y(t))$ $(0 \leqq t \leqq 1)$ for which

(i) there exists a sequence $0 = t_0 < t_1 < t_2 < \cdots < t_M < t_{M+1} = 1$ such that

$$\max\{|x'(t)|, |y'(t)|, |x''(t)|, |y''(t)|\} \leqq L$$

for every $t \neq t_i$ $(i = 0, 1, 2, \cdots, M + 1)$,

(ii) $0 \leqq x(t) \leqq 1$, $0 \leqq y(t) \leqq 1$,

(iii) $x(0) = x(1)$, $y(0) = y(1)$,

(iv) $(x(s) - x(t))^2 + (y(s) - y(t))^2 > 0$ if $0 \leqq t < s < 1$.

N3. For any $\mathbf{z}(t) \in \mathscr{A}$ define the domain $Q_{\mathbf{z}} \subset [0, 1] \times [0, 1] = I^2$ as the set of those points $(x, y) \in I^2$ which cannot be joined to any point of the boundary of I^2 without intersecting the curve $\mathbf{z}(t)$.

N4. $Q = Q(L, M) = \{Q_{\mathbf{z}} : \mathbf{z} \in \mathscr{A}(L, M)\}$.

N5. Let $R = 2^r$ $(r = 1, 2, \cdots)$ and let

$$I_{kj} = I_{kj}^{(r)} = \{(x, y) : k/R \leqq x < (k + 1)/R, j/R \leqq y < (j + 1)/R\}$$

$(k, j = 0, 1, 2, \cdots, R - 1)$.

N6. For any Borel set B of I^2 let

$$B^r = \sum_{\{(k,j):I_{kj}^{(r)} \subset B\}} I_{kj}^{(r)}, \qquad \tilde{B}^r = \sum_{\{(k,j):I_{kj}^{(r)} B \neq 0\}} I_{k,j}^{(r)} - B^r .$$

N7. Define the Wiener measure (W.M.) of I_{kj} by

$$W(I_{kj}) = W\left(\frac{k+1}{R}, \frac{j+1}{R}\right) - W\left(\frac{k+1}{R}, \frac{j}{R}\right) - W\left(\frac{k}{R}, \frac{j+1}{R}\right) + W\left(\frac{k}{R}, \frac{j}{R}\right),$$

The W.M. of a finite sum of I_{kj}'s will be defined by additivity. The W.M. of a domain $Q_{\mathbf{z}} \in Q$ will be defined as:

$$W(Q_{\mathbf{z}}) = \lim_{r \to \infty} W(Q_{\mathbf{z}}^r) .$$

(It is easy to see that the limit on the right exists with probability 1.)

N8. Let $\lambda(\cdot)$ be the Lebesgue measure on I^2; then the Brownian measure (B.M.) resp. the Kiefer measure (K.M.) of a set $Q_{\mathbf{z}} \in Q$ will be defined as:

$$B(Q_{\mathbf{z}}) = W(Q_{\mathbf{z}}) - \lambda(Q_{\mathbf{z}})W(1, 1)$$

resp.

$$K(Q_{\mathbf{z}}; y) = W(Q_{\mathbf{z}}; y) - \lambda(Q_{\mathbf{z}})W(1, 1, y) .$$

Since for any fixed $y > 0$, $y^{-\frac{1}{2}}W(\mathbf{x}; y)$ is a W.P. of two variables, $W(Q_{\mathbf{z}}; y)$ can be defined via N7.

N8. The stochastic set functions $\bar{W}(Q_{\mathbf{z}})$, $\bar{B}(Q_{\mathbf{z}})$, $\bar{K}(Q_{\mathbf{z}}; n)$ $(Q_{\mathbf{z}} \in Q)$ are stochastically equivalent versions (or versions) of the "measures" W, B, K ($\bar{W} \cong W$, $\bar{B} \cong B$, $\bar{K} \cong K$) if $P(\bar{W}(Q_{\mathbf{z}}) = W(Q_{\mathbf{z}})) = P(\bar{B}(Q_{\mathbf{z}}) = B(Q_{\mathbf{z}})) = P(\bar{K}(Q_{\mathbf{z}}; n) = K(Q_{\mathbf{z}}; n)) = 1$ for every $Q_{\mathbf{z}} \in Q$, $n = 1, 2, \cdots$. The versions $\bar{W}$ (resp. $\bar{B}$, $\bar{K}$) of W (resp. B, K) are also called W.M. (resp. B.M., K.M.).

Our first result is the following:

THEOREM 1. *For any $\varepsilon > 0$ and $t > 0$ we have*

$$P\{\sup_{\mathbf{z} \in \mathscr{A}} |\bar{W}(Q_{\mathbf{z}})| > t\} \leqq A \exp(-t^2/(2 + \varepsilon)) \tag{1}$$

and

$$P\{\sup_{\mathbf{z} \in \mathscr{A}} |\bar{B}(Q_{\mathbf{z}})| > t\} \leqq A \exp(-t^2/(2 + \varepsilon)) \tag{2}$$

where $A = (M^2L^4)^{CM^3L^4}$, $C = C(\varepsilon)$ is a positive constant depending only on ε and $\bar{W}$ resp. *$\bar{B}$. are suitable* W.M. resp. B.M.

A trivial consequence of this theorem is:

THEOREM 1*. *For any $\varepsilon > 0$ and $t > 0$ we have*

$$P\{\sup_{0 \leqq x \leqq 1; 0 \leqq y \leqq 1} |W(x, y)| > t\} \leqq C \exp(-t^2/(2 + \varepsilon))$$

where $C = C(\varepsilon)$ is a positive constant depending only on ε.

A stronger result of this latter type was obtained very recently by H. C. Chan [2].

Now we formulate our main theorem, stating that the empirical measure (E.M.) $\alpha_n(Q_{\mathbf{z}}) = \int_{Q_{\mathbf{z}}} d\alpha_n(\mathbf{x})$ $(\mathbf{z} \in \mathscr{A})$ can be uniformly approximated by a B.M. resp. K.M.

THEOREM 2. *One can define a sequence $\{\bar{B}_n(Q_{\mathbf{z}})\}$ of* B.M.*'s and a* K.M. $\bar{K}(Q_{\mathbf{z}}; n)$ *$(\mathbf{z} \in \mathscr{A})$ such that*

$$\sup_{\mathbf{z} \in \mathscr{A}} |\alpha_n(Q_{\mathbf{z}}) - \bar{B}_n(Q_{\mathbf{z}})| = O(n^{-\frac{1}{19}}) \qquad n \to \infty \tag{3}$$

$$\sup_{\mathbf{z} \in \mathscr{A}} |n^{\frac{1}{2}}\alpha_n(Q_{\mathbf{z}}) - \bar{K}(Q_{\mathbf{z}}; n)| = O(n^{\frac{12}{25}}) \qquad n \to \infty \tag{4}$$

a.s.

Up to now we investigated only the case when the sample is coming from the uniform law. In the one-dimensional case, Theorems B and D immediately imply:

THEOREM F. *Let* $Y_1, Y_2, \cdots$ *be a sequence of* i.i.d.rv'*s having common continuous distribution function* F. *Then one can define a sequence* $\{B_n\}$ *of* B.B.'*s and a* K.P. K *such that*

$$\sup_{-\infty<x<\infty} |\beta_n(x) - B_n(F(x))| = O(n^{-\frac{1}{2}} \log n) \qquad n \to \infty$$

$$\sup_{-\infty<x<\infty} |n^{\frac{1}{2}}\beta_n(x) - K(F(x); n)| = O(\log^2 n) \qquad n \to \infty$$

a.s. *where the* E.P. $\beta_n = n^{\frac{1}{2}}(F_n - F)$ *and* F_n *is the* e.d.f. *based on the sample* $Y_1, Y_2, \cdots, Y_n$.

In order to prove Theorem F one only notes that $F(Y_1), F(Y_2), \cdots$ are independent, uniformly distributed rv's.

The problem of finding the analogous generalization in the multidimensional case is not so simple. The first difficulty is to find a transformation $T: R^2 \to I^2$ mapping the sample $\mathbf{Y}_1 = (Y_{11}, Y_{12}), \mathbf{Y}_2 = (Y_{21}, Y_{22}), \cdots$ into a sample $\mathbf{X}_1 = T\mathbf{Y}_1$, $\mathbf{X}_2 = T\mathbf{Y}_2, \cdots$ coming from the uniform law. Toward this goal let F be the distribution function of $\mathbf{Y}_1$, satisfying some regularity conditions and let $G(x_2 | x_1) = P(Y_{12} < x_2 | Y_{11} = x_1)$, $H(x_1) = P(Y_{11} < x_1)$. Then define T as follows: $T(x_1, x_2) = (H(x_1), G(x_2 | x_1))$. This transformation was studied before by Rosenblatt ([8]).

Now we formulate our

THEOREM 3. *Let* $\mathbf{Y}_1 = (Y_{11}, Y_{12}), \mathbf{Y}_2 = (Y_{21}, Y_{22}), \cdots$ *be a sequence of* i.i.d.rv'*s having a common distribution function* $F(\mathbf{x}) = F(x_1, x_2)$. *Suppose that* $F(x_1, x_2)$ *is absolutely continuous and*

$$\left| \frac{\partial^2 G(x_2 | H^{-1}(x_1))}{\partial x_1^2} \right| \leqq L, \qquad \left| \frac{\partial G(x_2 | H^{-1}(x_1))}{\partial x_1} \right| \leqq L$$

(*for some* $L > 0$). *Then we can define a sequence* $\{\bar{B}_n\}$ *of* B.M.'*s and a* K.M. $\bar{K}$ *such that*

$$\sup_{\mathbf{x} \in R^2} |\beta_n(\mathbf{x}) - \bar{B}_n(TD_{\mathbf{x}})| = O(n^{-\frac{1}{19}}) \tag{5}$$

and

$$\sup_{\mathbf{x} \in R^2} |n^{\frac{1}{2}}\beta_n(\mathbf{x}) - \bar{K}(TD_{\mathbf{x}}; n)| = O(n^{\frac{12}{25}}) \tag{6}$$

a.s. *where* $\beta_n(\mathbf{x}) = n^{\frac{1}{2}}(F_n(\mathbf{x}) - F(\mathbf{x}))$, $F_n(\mathbf{x})$ *is the* e.d.f. *based on the sample* $\mathbf{Y}_1, \mathbf{Y}_2, \cdots, \mathbf{Y}_n$ *and* $D_{\mathbf{x}} = \{(a_1, a_2): 0 \leqq a_1 \leqq x_1, 0 \leqq a_2 \leqq x_2\}$ $\{\mathbf{x} = (x_1, x_2) \in R^2\}$.

Since $\sup_{-\infty<x<\infty} B(F(x)) = \sup_{0 \leqq x \leqq 1} B(x)$ (F is continuous), a simple consequence of Theorem F is the fact that the limit distribution of $\sup_{-\infty<x<\infty} \beta_n(x)$ does not depend on F. On the other hand, Theorem 3 shows that in the two-dimensional case the limit distribution of $\sup_{\mathbf{x} \in R^2} \beta_n(\mathbf{x})$ does depend on F. This is the reason that there is no appropriate analogous version of the Kolmogorov–Smirnov test in the two dimensional case.

3. Proof of Theorem 1. At first we introduce some notations:

N9. Let $\mathscr{P} = \mathscr{P}(\kappa, \tau)$ $(0 < \tau \leqq 1, \kappa = 3, 4, 5, \cdots)$ be the set of these polygons P for which

(i) $\lambda(P) \leqq \tau$,
(ii) $P \subset I^2$, and
(iii) the number of vertices of P is not more than κ.

N10. $\mathscr{P}^r = \mathscr{P}^r(\kappa, \tau) = \{P^r\colon P \in \mathscr{P}(\kappa, \tau)\}$, where P^r is defined in N6.

N11. Let $\bar{P}^r = P^{r+1} - P^r$ and

$$\bar{\mathscr{P}}^r = \bar{\mathscr{P}}^r(\kappa, \tau) = \{\bar{P}^r\colon P \in \mathscr{P}(\kappa, \tau)\},$$

N12. For any finite set Z let $C(Z)$ be the number of elements of Z.

Now we formulate some lemmas.

Lemma 1.

(i) $C(\mathscr{P}^r(\kappa, \tau)) \leqq R^{4\kappa}$,
(ii) $C(\bar{\mathscr{P}}^r(\kappa, \tau)) \leqq (2R)^{4\kappa}$,
(iii) $\lambda(\bar{P}^r) \leqq \tau$ $(P \in \mathscr{P}(\kappa, \tau);\ r = 1, 2, \cdots)$,
(iv) $\lambda(\bar{P}^r) \leqq \dfrac{3\kappa}{4R}$ $(P \in \mathscr{P}(\kappa, \tau);\ r = 1, 2, \cdots)$,

where $R = 2^r$.

Proof. Let

$$P_{mb} = \{(x, y)\colon 0 \leqq x \leqq 1,\ 0 \leqq y \leqq \min(1, mx + b)\}$$

and $P_{m_1b_1}$ is called r-equivalent to $P_{m_2b_2}$ if $P^r_{m_1b_1} = P^r_{m_2b_2}$. Each equivalence class can be represented by a line $y = mx + b$ joining two lattice points $(i_1/R, j_1/R)$ and $(i_2/R, j_2/R)$ $(0 \leqq i_1, i_2, j_1, j_2 \leqq R)$. Since the number of the pairs of lattice points is less than R^4 we get (i).

(ii) and (iii) are trivial. (iv) follows from the fact that at most 3 of the 4 cells of width 2^{-r-1} which comprise a cell of width 2^{-r} will be in $\bar{P}^r$.

Lemma 2. *For any* $\mu > 0$ *there exists a* $C = C(\mu)$ *such that*

$$P\{\sup_{P \in \mathscr{P}(\kappa,1)} |W(P) - W(P^r)| \geqq C\kappa R^{-\frac{1}{2}}(\log R)^{\frac{1}{2}}\} \leqq R^{-\mu} \tag{7}$$

and

$$P\{\sup_{P \in \mathscr{P}(\kappa,1)} |B(P) - B(P^r)| \geqq C\kappa R^{-\frac{1}{2}}(\log R)^{\frac{1}{2}}\} \leqq R^{-\mu}. \tag{8}$$

Proof. By (ii) and (iv) of Lemma 1 we have

$$P\left\{\sum_{j=0}^{\infty} \sup_{\bar{\mathscr{P}}^{r+j}} |W(\bar{P}^{r+j})| \geqq \sum_{j=0}^{\infty}\left(\frac{3\kappa}{2^{j+2}R}\right)^{\frac{1}{2}} z_j\right\}$$
$$\leqq 2\sum_{j=0}^{\infty} (2^{j+1}R)^{4\kappa}(1 - \Phi(z_j)).$$

Choosing $z_j = (8\kappa j + C_1\kappa \log R)^{\frac{1}{2}}$ with an appropriate $C_1 > 0$ we get

$$P\{\sum_{j=0}^{\infty} \sup_{\bar{\mathscr{P}}^{r+j}} |W(\bar{P}^{r+j})| \geqq CR^{-\frac{1}{2}}(\log R)^{\frac{1}{2}}\} \leqq R^{-\mu}.$$

This clearly implies

$$\lim_{r\to\infty} \sum_{j=0}^{\infty} \sup |W(\bar{P}^{r+j})| = 0 \quad \text{a.s.}$$

Hence there exists an event of probability 1 where $\lim_{r\to\infty} W(P^r)$ exists for every $P \in \mathscr{P}(\kappa, \tau)$. Since

$$\sup_{P\in\mathscr{P}(\kappa,\tau)} |W(P) - W(P^r)| \leqq \sum_{j=0}^{\infty} \sup_{\bar{P}^{r+j}\in\bar{\mathscr{P}}^{r+j}} |W(\bar{P}^{r+j})|,$$

we have (7). (8) easily follows from (7).

LEMMA 3. *For any $t > 0$ and $\varepsilon > 0$ we have*

$$(9) \qquad P\{\sup_{P\in\mathscr{P}(\kappa,\tau)} |W(P)| \geqq \tau^{\frac{1}{2}}t\} \leqq A \exp(-t^2/(2+\varepsilon))$$

and

$$(10) \qquad P\{\sup_{P\in\mathscr{P}(\kappa,\tau)} |B(P)| \geqq \tau^{\frac{1}{2}}t\} \leqq A \exp(-t^2/(2+\varepsilon))$$

where $A = (\kappa\tau^{-1})^{C\kappa}$ and $C = C(\varepsilon)$ is a positive constant depending only on ε.

PROOF. For any integer r we have

$$P = P^r + \bar{P}^r + \bar{P}^{r+1} + \cdots$$

and

$$\sup_{P\in\mathscr{P}(\kappa,\tau)} |W(P)| \leqq \sup_{P^r\in\mathscr{P}^r} |W(P^r)| + \sum_{j=0}^{\infty} \sup_{\bar{P}^{r+j}\in\bar{\mathscr{P}}^{r+j}} |W(\bar{P}^{r+j})|.$$

Further for any x; $y_0, y_1, \cdots$ we have

$$P\{\sup_{P^r\in\mathscr{P}^r} |W(P^r)| \geqq \tau^{\frac{1}{2}}x\} \leqq 2R^{4\kappa}(1-\Phi(x)),$$

$$P\left\{\sup_{\bar{P}^{r+j}\in\bar{\mathscr{P}}^{r+j}} |W(\bar{P}^{r+j})| \geqq \left(\frac{3\kappa}{2^{j+2}R}\right)^{\frac{1}{2}} y_j\right\} \leqq 2(2^{j+1}R)^{4\kappa}(1-\Phi(y_j))$$

whence

$$\begin{aligned} P&\left\{\sup_{P\in\mathscr{P}} |W(P)| \geqq \tau^{\frac{1}{2}}x + \sum_{j=0}^{\infty}\left(\frac{3\kappa}{2^{j+2}R}\right)^{\frac{1}{2}} y_j\right\} \\ &\leqq 2R^{4\kappa}(1-\Phi(x)) + 2\sum_{j=0}^{\infty}(2^{j+1}R)^{4\kappa}(1-\Phi(y_j)) \\ &\leqq 2R^{4\kappa}[x^{-1}e^{-x^2/2} + 2^{4\kappa}\sum_{j=0}^{\infty} 2^{4\kappa j} y_j^{-1} e^{-y_j^2/2}]. \end{aligned}$$

Let

$$y_j = (8\kappa j + x^2)^{\frac{1}{2}}, \qquad x = C_1 t + C_2\kappa^{\frac{1}{2}}, \qquad r = [\log_2 C_3\kappa\tau^{-1}]$$

where $C_1 = 1 - \varepsilon/6$, $C_2 = \varepsilon/18$, $C_3 = 10^4/\varepsilon$. Since $\kappa\tau^{-1} > 3$ our statements hold if $t \leqq \kappa^{\frac{1}{2}}$ provided that $C = C(\varepsilon) > 1$. Hence we can assume that $t > \kappa^{\frac{1}{2}}$. Then

$$\begin{aligned} \tau^{\frac{1}{2}}x + \sum_{j=0}^{\infty}\left(\frac{3\kappa}{2^{j+2}R}\right)^{\frac{1}{2}} y_j &\leqq \tau^{\frac{1}{2}}x + \sum_{j=0}^{\infty}\left(\frac{6\kappa}{2^{j+2}C_3}\right)^{\frac{1}{2}}[(8\kappa j)^{\frac{1}{2}} + x] \\ &\leqq \tau^{\frac{1}{2}}x(1 + 6C_3^{-\frac{1}{2}}) + (\kappa\tau)^{\frac{1}{2}}C_3^{-\frac{1}{2}}50 \\ &\leqq \tau^{\frac{1}{2}}t\{(C_1 + C_2)(1 + 6C_3^{-\frac{1}{2}}) + 50C_3^{-\frac{1}{2}}\} \leqq \tau^{\frac{1}{2}}t \end{aligned}$$

and

$$2R^{4\kappa}[x^{-1}e^{-x^2/2} + 2^{4\kappa} \sum_{j=0}^{\infty} 2^{4\kappa j} y_j{}^{-1} e^{-y_j{}^2/2}]$$
$$\leqq 2\left(\frac{C_3\kappa}{\tau}\right)^{4\kappa} [e^{-x^2/2} + 2^{4\kappa} \sum_{j=0}^{\infty} (2/e)^{4\kappa j} e^{-x^2/2}] \leqq e^{-x^2/2} 2\left(\frac{C_3\kappa}{\tau}\right)^{4\kappa} [1 + 2^{4\kappa+1}]$$
$$\leqq e^{-C_1 t^2/2} 2\left(\frac{C_3\kappa}{\tau}\right)^{4\kappa} (1 + 2^{4\kappa+1}) \leqq e^{-t^2/(2+\varepsilon)} (\kappa\tau^{-1})^A .$$

N13. For any $\mathbf{z} = \mathbf{z}(t) \in \mathscr{A}(L, M) = \mathscr{A}$ and for any $s = 1, 2, \cdots$ let $P_{\mathbf{z}}{}^s$ be the polygon having vertices

$$\mathbf{z}(t_i + jS^{-1}(t_{i+1} - t_i))$$
$$= \mathbf{z}_{ij}^s (i = 0, 1, 2, \cdots, M; j = 0, 1, 2, \cdots, S - 1; S = 2^s) .$$

Further let $T_{ij}^s(\mathbf{z}) = T_{ij}$ be the triangle with vertices $\mathbf{z}_{ij}^s$, $\mathbf{z}_{i,j+1}^s$, $\mathbf{z}_{i,2j+1}^{s+1}$.

LEMMA 4. *For any $\mathbf{z} \in \mathscr{A}(L, M)$ and for any $i=0, 1, 2, \cdots, M$; $j=0, 1, 2, \cdots, S - 1$; $s = 1, 2, \cdots$ we have*

(i) $\lambda(T_{ij}^s(\mathbf{z})) \leqq L^4 S^{-3}$,
(ii) $\lambda(Q_{\mathbf{z}} \triangle P_{\mathbf{z}}{}^s) \leqq ML^4 S^{-2}$

where $A \triangle B = (A - B) + (B - A)$, and $S = 2^s$.

LEMMA 5. *For any $\mu > 0$ one can find a $C = C(\mu) > 0$ such that*

(11) $$P\{\textstyle\sum_{l=0}^{\infty} \sum_{i=0}^{M} \sum_{j=0}^{2^{s+l}} \sup_{\mathbf{z} \in \mathscr{A}} |W(T_{ij}^{s+l}(\mathbf{z}))| \geqq CML^2 S^{-\frac{1}{2}} (\log S)^{\frac{1}{2}}\} \leqq MS^{-\mu} .$$

PROOF. Clearly we have

$$P\{\textstyle\sum_{l=0}^{\infty} \sum_{i=0}^{M} \sum_{j=0}^{2^{s+l}} \sup_{\mathbf{z} \in \mathscr{A}} |W(T_{ij}^{s+l}(\mathbf{z}))|$$
$$\geqq \textstyle\sum_{l=0}^{\infty} (M + 1)(2^{s+l} + 1) y_l L^2 2^{-\frac{3}{2}(s+l)}\}$$
$$\leqq \textstyle\sum_{l=0}^{\infty} (M + 1)(2^{s+l} + 1)(2^{3(s+l)} L^{-4})^C e^{-y_l{}^2/(2+\varepsilon)} .$$

Choosing $y_l = (C_1 l + C_2 \log S)^{\frac{1}{2}}$ with appropriate C_1 and C_2 we get (11).

Now we define the W.M. $\bar{W}(Q_{\mathbf{z}})$ ($Q_{\mathbf{z}} \in Q$) for which our statements hold. Let

$$\bar{W}(Q_{\mathbf{z}}) = \lim_{s\to\infty} W(P_{\mathbf{z}}{}^s) .$$

Lemma 5 implies the existence of an event of probability 1 where $\lim_{s\to\infty} W(P_{\mathbf{z}}{}^s)$ exists for every $Q_{\mathbf{z}} \in Q$.

The relation $P(\bar{W}(Q_{\mathbf{z}}) = W(Q_{\mathbf{z}})) = 1$ for any fixed $Q_{\mathbf{z}} \in Q$ is also immediate by Lemma 5.

The processes $\bar{B}$ and $\bar{K}$ can be defined similarly.

Using this definition and the trivial inequality

$$\sup_{\mathbf{z} \in \mathscr{A}} |\bar{W}(Q_{\mathbf{z}}) - W(P_{\mathbf{z}}{}^s)| \leqq \textstyle\sum_{l=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{2^{s+l}} \sup_{\mathbf{z} \in \mathscr{A}} |W(T_{ij}^{s+l}(\mathbf{z}))|$$

we get

LEMMA 5*. *For any $\mu > 0$ one can find a $C = C(\mu) > 0$ such that*

(11*) $$P\{\sup_{\mathbf{z} \in \mathscr{A}} |W(Q_{\mathbf{z}}) - W(P_{\mathbf{z}}{}^s)| \geqq CML^2 S^{-\frac{1}{2}} (\log S)^{\frac{1}{2}}\} \leqq MS^{-\mu}$$

and

(12) $$P\{\sup_{z\in\mathscr{A}} |B(Q_z) - B(P_z^s)| \geqq CML^2S^{-\frac{1}{2}}(\log S)^{\frac{1}{2}}\} \leqq MS^{-\mu}.$$

PROOF OF THEOREM 1. For any $\mathbf{z} \in \mathscr{A}(L, M)$ and for any $s = 1, 2, \cdots$ we have

$$Q_z = P_z^s + \sum_{l=0}^{\infty} \sum_{i=0}^{M} \sum_{j=0}^{2^{s+l}} \varepsilon_{ij} T_{ij}^{s+l}(\mathbf{z})$$

where ε_{ij} can be $+1$ or -1.
Since $W(Q_z) = \lim_{s\to\infty} W(P_z^s)$ we have

$$W(Q_z) = W(P_z^s) + \sum_{l=0}^{\infty} \sum_{i=0}^{M} \sum_{ij}^{2^s+l} \varepsilon_{ij} W(T_{ij}^{s+l}(\mathbf{z})).$$

Our last relation implies

$$\sup_{z\in\mathscr{A}} |W(Q_z)| \leqq \sup_{z\in\mathscr{A}} |W(P_z^s)| + \sum_{l=0}^{\infty} \sum_{i=0}^{M} \sum_{j=0}^{2^s+l} \sup_{z\in\mathscr{A}} |W(T_{ij}^{s+l}(\mathbf{z}))|.$$

Then by Lemma 3 we have

$$P\{\sup_{z\in\mathscr{A}} |W(P_z^s)| > x\} \leqq (SM)^{CSM} \exp(-x^2/(2+\varepsilon))$$

and

$$P\{\sup_{z\in\mathscr{A}} |W(T_{ij}^{s+l}(\mathbf{z}))| > y_l L^2 2^{-\frac{3}{2}(s+l)}\} \leqq (2^{3(s+l)}L^{-4})^C \exp(-y_l^2/(2+\varepsilon)),$$

if C is big enough whence

$$P\{\sup_{z\in\mathscr{A}}|W(Q_z)| > x + \sum_{l=0}^{\infty} (M+1)(2^{s+l}+1) y_l L^2 2^{-\frac{3}{2}(s+l)}\}$$
$$\leqq (SM)^{CSM} e^{-x^2/(2+\varepsilon)} + \sum_{l=0}^{\infty} (M+1)(2^{s+l}+1)(2^{3(s+l)}L^{-4})^C e^{-y_l^2/(2+\varepsilon)}.$$

Let

$$y_l = (C_1 l + x^2)^{\frac{1}{2}}, \qquad x = C_2 t + C_3, \qquad S = C_4(ML^2)^2$$

where $C_1, \cdots, C_4$ are appropriate real constants.
Then

$$x + \sum_{l=0}^{\infty} (M+1)(2^{s+l}+1) y_l L^2 2^{-\frac{3}{2}(s+l)} \leqq t$$
$$(SM)^{CSM} e^{-x^2/(2+\varepsilon)} + \sum_{l=0}^{\infty} (M+1)(2^{s+l}+1)(2^{3(s+l)}L^{-4})^C e^{-y_l^2/(2+\varepsilon)}$$
$$\leqq (M^3L^4)^{CM^3L^4} e^{-x^2/(2+\varepsilon)},$$

which proves (1). (2) easily follows from (1).

To complete this paragraph we formulate two further lemmas.

In the proof of the next lemma the following inequality will be used:

BERNSTEIN'S INEQUALITY (see e.g., [10], pages 387–389). *For any Borel set $Q \subset I^2$ we have*

$$P((\lambda(Q)(1-\lambda(Q)))^{-\frac{1}{2}}|\alpha_n(Q)| > x) \leqq 2e^{-x^2/5}$$

provided that $0 < x < (n\lambda(Q)(1-\lambda(Q)))^{\frac{1}{2}}$.

LEMMA 6. *Let* $0 < \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5 < 1$, $R \leqq n^{1-\varepsilon_1}$, $S \leqq n^{1-\varepsilon_2}$, $\tau \geqq n^{\varepsilon_3-1}$. *Then for any* $\mu > 0$ *one can find a* $C = C(\mu) > 0$ *such that*

(13) $$P\{\sup_{P\in\mathscr{P}(\kappa,1)} |\alpha_n(P) - \alpha_n(P^r)| \geqq CR^{-\frac{1}{2}}(\log R)^{\frac{1}{2}}\} \leqq R^{-\mu},$$

(14) $$P\{\sup_{z\in\mathscr{A}} |\alpha_n(Q_z) - \alpha_n(P_z^s)| \geqq CML^2S^{-\frac{1}{2}}(\log S)^{\frac{1}{2}}\} \leqq MS^{-\mu},$$

and

$$P\{\sup_{P\in\mathscr{P}(\kappa,\tau)}|\alpha_n(P)| \geqq \tau^{\frac{1}{2}}t\} \leqq A\exp(-t^2/(2+\varepsilon_4))\,, \tag{15}$$

where $A = (\kappa\tau^{-1})^{\bar{C}\kappa}$, $\bar{C} = \bar{C}(\varepsilon_4)$ *is a positive constant depending only on* ε_4 *and* $0 < t < n^{\varepsilon_3/2-\varepsilon_4}$.

PROOF. Clearly

$$\sup|\alpha_n(P) - \alpha_n(P^r)| \leqq \sup|\alpha_n(\bar{P}^r)| + \cdots + \sup|\alpha_n(\bar{P}^{r+j})| + n^{\frac{1}{2}}\sup\int_{\tilde{P}^{r+j+1}} dE_n(\mathrm{x}) + n^{\frac{1}{2}}\sup\lambda(\tilde{P}^{r+j+1})\,.$$

($\tilde{P}^r$ was defined in N6.)

Choose $j = [\log_2 n^{1-\varepsilon_1/2}R^{-1}]$. Then

$$n^{\frac{1}{2}}\sup\lambda(\tilde{P}^{r+j+1}) \leqq R^{-\frac{1}{2}}(\log R)^{\frac{1}{2}}\,, \tag{16}$$

$$P(n^{\frac{1}{2}}\sup\int_{\tilde{P}^{r+j+1}} dE_n(x) \geqq R^{-\frac{1}{2}}(\log R)^{\frac{1}{2}}) \leqq (R2^j)^{4\kappa}\,O(\exp(-\tfrac{1}{5}n^{\varepsilon_1/2}))\,, \tag{17}$$

$$P\left(\sup|\alpha_n(\bar{P}^{r+l})| \geqq \left(\frac{3\kappa}{2^{j+2}R}\right)^{\frac{1}{2}} z_j\right) \leqq O(e^{-z_j^{2/5}})\,, \tag{18}$$

where $z_j = (20\kappa j + C_1\kappa\log R)^{\frac{1}{2}}$ $(C_1 > 0)$.

Now (13) can be obtained from (16), (17) and (18) in the same way as we obtained (7). In order to prove (14) resp. (15) we can follow the method of proof of (11*) resp. (9).

LEMMA 7. *For any* $\mu > 0$ *there exists a* $C = C(\mu) >$ *such that*

$$P\{\sup_{z\in\mathscr{A};1\leqq k<n}|K(Q_z;k)| \geqq C(n\log n)^{\frac{1}{2}}\} \leqq An^{-\mu} \tag{19}$$

and

$$P\{\sup_{z\in\mathscr{A};n\leqq k\leqq m}|K(Q_z;k) - K(Q_z;n)| \geqq C(m-n)\log(m-n))^{\frac{1}{2}}\} \leqq A(m-n)^{-\mu} \tag{20}$$

where $A = (M^3L^4)^{CM^3L^4}$.

PROOF. Clearly we have

$$P\{\sup_{z\in\mathscr{A};1\leqq k\leqq n}|K(Q_z;k)| \geqq C(n\log n)^{\frac{1}{2}}\} \leqq nP\{\sup_{z\in\mathscr{A}}|B(Q_z)| \geqq C(\log n)^{\frac{1}{2}}\}\,.$$

Then (19) follows from Theorem 1. (20) follows in the same way.

4. Proof of Theorem 2. At first we recall a lemma proved in [3b].

LEMMA A ([3b]). *Let* π_λ *be a Poisson* rv *with mean* $\lambda > 1$. *Then for any constant* $C > 1$ *there exists a polynomial* $B(x)$ *of second order (depending only on* C*) such that for any* x $(|x| \leqq C(\log\lambda)^{\frac{1}{2}})$ *we have*

$$F_\lambda(x) = P\left(\frac{\pi_\lambda - \lambda}{\lambda^{\frac{1}{2}}} \leqq x\right) = \Phi(x) + \frac{\exp(-x^2/2)}{\lambda^{\frac{1}{2}}}f(x,\lambda)$$

where $|f(x,\lambda)| \leqq B(x)$ *and*

$$F_\lambda(x_\lambda) \approx \Phi(x_\lambda)\,, \qquad 1 - F_\lambda(x_\lambda) \approx 1 - \Phi(x_\lambda)$$

provided that $|x_\lambda| \leqq C(\log\lambda)^{\frac{1}{2}}$.

N14. Let X be a discrete rv with distribution function F where F is a step function with

$$F(x) = p_i \quad \text{if} \quad x_i < x \leqq x_{i+1} \qquad i = 0, \pm 1, \pm 2, \cdots .$$

Further let $\cdots, Z_{-2}^F, Z_{-1}^F, Z_0^F, Z_1^F, Z_2^F, \cdots$ be a sequence of independent rv's being also independent from X with distribution function

$$\begin{aligned} P(Z_i^F < x) &= 0 && \text{if} \quad x < \Phi^{-1}(F(x_i)) \\ &= \frac{\Phi(x) - F(x_i)}{F(x_{i+1}) - F(x_i)} && \text{if} \quad \Phi^{-1}(F(x_i)) \leqq x < \Phi^{-1}(F(x_{i+1})) \\ &= 1 && \text{if} \quad x \geqq \Phi^{-1}(F(x_{i+1})) . \end{aligned}$$

Define the rv $N(X)$ by

$$N(X) = Z_i^F \quad \text{if} \quad x \leqq X < x_{i+1} .$$

The following lemma is trivial again:

LEMMA 8. $P(N(X) \leqq t) = \phi(t)$.

LEMMA 9. $|N(\lambda^{-\frac{1}{2}}(\pi_\lambda - \lambda)) - \lambda^{-\frac{1}{2}}(\pi_\lambda - \lambda)| = O(\lambda^{-\frac{1}{2}} \log \lambda)$ *as* $\lambda \to \infty$, *provided that* $|\lambda^{-\frac{1}{2}}(\pi_\lambda - \lambda)| \leqq C(\log \lambda)^{\frac{1}{2}}$ *where the* O *depends only on* C.

This lemma can be proved easily using Lemma A and the method of proof of Lemma 3 of [3a].

N15. Let $\pi = \pi_n$ be an rv of Poisson distribution with parameter n and independent from the sample $\{X_i\}$. Further let $\alpha_{kj} = \alpha_{kj}(n, r)$ resp. $\beta_{kj} = \beta_{kj}(n, r)$ be the number of the elements of the sample $X_1, X_2, \cdots, X_n$ resp. $X_1, X_2, \cdots, X_\pi$ lying in the square $I_{k,j}^r$.

N16. $n^{-\frac{1}{2}}(\alpha_{kj} - nR^{-2}) = u_{kj}$, $n^{-\frac{1}{2}}(\beta_{kj} - nR^{-2}) = v_{kj}$,

$$\mathfrak{U}(P^r) = \sum_{\{(k,j): I_{jk}^r \subset P^r\}} u_{kj} , \qquad \mathfrak{V}(P^r) = \sum_{\{(k,j): I_{kj}^r \subset P^r\}} v_{kj}$$

for any $P^r \in \mathscr{P}^r(\kappa, \tau)$ (see N9).

Now we can formulate

LEMMA 10. *For any* $\mu > 0$ *there exists a* $C = C(\mu) > 0$ *such that*

(i) $P\{\sup_{P^r \in \mathscr{P}^r(\kappa,\tau)} |\mathfrak{U}(P^r) - \mathfrak{V}(P^r)| \geqq Cn^{-\frac{1}{4}}(\log n)^{\frac{3}{4}}\} \leqq n^{-\mu}$ *if* $R^2 = o(n)$,

(ii) *the* rv'*s* β_{kj} $(k, j = 0, 1, 2, \cdots, R - 1)$ *are independent with Poisson law of parameter* nR^{-2}.

This lemma can be proved in the same way as Lemma 3 of [3b] applying (i) of our Lemma 1.

We recall two lemmas of [3b].

LEMMA B ([3b]). *Let* R *be an integer and let* $\{N_{ij}\}$ $(i, j = 1, \cdots, R)$ *be a double array of independent standard normal* rv'*s defined on a probability space* $\{\Omega, \mathscr{S}, P\}$. *Then there exists a* W.P. $W(x, y)$ $(0 \leqq x \leqq 1, 0 \leqq y \leqq 1)$ *on* Ω *such that*

$$W(i/R, j/R) = R^{-2} \sum_{\alpha \leqq i; \beta \leqq j} N_{\alpha\beta}$$

$(i, j = 0, 1, 2, \cdots, R)$.

LEMMA C. *Let* $0 = t_0 < t_1 < t_2 < \cdots$ *be a sequence of real numbers and let* $\{B_i(\mathbf{x})\}$ *be a sequence of independent* B.B.'*s defined on a probability space* $\{\Omega, \mathscr{S}, P\}$. *Then there exists a* K.P. $K(\mathbf{x}; y)$ *on* Ω *such that*

$$K(\mathbf{x}; t_i) = t_1^{\frac{1}{2}} B_1(\mathbf{x}) + (t_2 - t_1)^{\frac{1}{2}} B_2(\mathbf{x}) + \cdots + (t_i - t_{i-1})^{\frac{1}{2}} B_i(\mathbf{x}) .$$

Now we prove our

LEMMA 11. *For any* $\mu > 0$ *one can construct a sequence* $\{B_i(\mathbf{x})\}$ *of* B.B.'*s such that*

$$P\{\sup_{P \in \mathscr{P}(\kappa,1)} |B_n(P) - \alpha_n(P)| \geqq C\kappa n^{-\frac{1}{6}} (\log n)^{\frac{3}{2}}\} \leqq n^{-\mu}$$

where $C = C(\mu)$ *is a positive constant depending only on* μ.

PROOF. Let $R = O(n^{\frac{1}{3}})$ and define the process $B_n(\mathbf{x})$ as follows: $N_{ij} = N(Rv_{ij})$ (see N14 an N16). Then by lemma B there exists a W.P. $W(\mathbf{x})$ such that

$$W(i/R, j/R) = R^{-1} \sum_{\alpha \leqq i; \beta \leqq j} N_{\alpha\beta}$$

and let $B(x, y) = W(x, y) - xyW(1, 1)$.

By Lemma 9,

$$|N_{ij} - Rv_{ij}| = O(Rn^{-\frac{1}{2}} \log n)$$

provided that $R|v_{ij}| \leqq C(\log n)^{\frac{1}{2}}$. Since

$$P\{\sup_{i,j} R|v_{ij}| > C(\log n)^{\frac{1}{2}}\} \leqq n^{-\mu}$$

if C is large enough, we have

$$P\{\sup_{i,j} |N_{ij} - Rv_{ij}| > CRn^{-\frac{1}{2}} \log n\} \leqq n^{-\mu} .$$

Let $\varepsilon_{ij} = N_{ij} - Rv_{ij}$ and

$$\begin{aligned} \varepsilon_{ij}^* &= \varepsilon_{ij} \qquad \text{if} \quad |\varepsilon_{ij}| \leqq CRn^{-\frac{1}{2}} \log n \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

Then

(i) the rv's ε_{ij}^* are independent, and
(ii) $|E\varepsilon_{ij}^*| = O(n^{-1})$.

Hence

$$P\{n^{\frac{1}{2}} (R \log n)^{-1} \sup_{P^r \in \mathscr{P}^r(\kappa,1)} |\sum_{\{(k,j): I_{kj}^r \subset P^r\}} (\varepsilon_{kj}^* - E\varepsilon_{kj}^*)| \geqq CR(\kappa \log n)^{\frac{1}{2}}\} \leqq n^{-\mu}$$

if C is large enough. This implies

$$P\{n^{\frac{1}{2}} (R \log n)^{-1} \sup |\sum \varepsilon_{kj}^*| \geqq CR(\kappa \log n)^{\frac{1}{2}}\} \leqq n^{-\mu}$$

and

$$P\{\sup |\sum \varepsilon_{kj}| \geqq C\kappa^{\frac{1}{2}} R^2 n^{-\frac{1}{2}} (\log n)^{\frac{3}{2}}\} \leqq n^{-\mu} .$$

That is

$$\begin{aligned} P\{\sup_{P^r \in \mathscr{P}^r(\kappa,1)} |R^{-1} \sum_{\{(k,j): I_{kj}^r \subset P^r\}} N_{kj} - \sum_{\{(k,j): I_{k,j}^r \subset P^r\}} v_{kj} | \\ \geqq C\kappa^{\frac{1}{2}} Rn^{-\frac{1}{2}} (\log n)^{\frac{3}{2}}\} \leqq n^{-\mu} . \end{aligned}$$

Especially if $P = I^2$ we have

$$P\{|W(1, 1) - n^{-\frac{1}{2}} (\pi - n)| \geqq CRn^{-\frac{1}{2}} (\log n)^{\frac{3}{2}}\} \leqq n^{-\mu} .$$

These last two relations imply

$$(21) \qquad P\{\sup_{P^r \in \mathscr{P}^r(\kappa,1)} |B(P^r) - \mathfrak{B}(P^r)| \geqq 2C\kappa^{\frac{1}{2}}Rn^{-\frac{1}{6}}(\log n)^{\frac{3}{2}}\} \leqq n^{-\mu} .$$

Now making use of Lemmas 10 and 2 one has Lemma 11.

PROOF OF (3). Let $\{B_n(\mathbf{x})\}$ be the sequence of B.B.'s constructed in Lemma 11. Then

$$\sup_{\mathbf{z}\in \mathscr{A}} |\alpha_n(Q_\mathbf{z}) - \bar{B}_n(Q_\mathbf{z})| \leqq \sup |\alpha_n(Q_\mathbf{z}) - \alpha_n(P_\mathbf{z}^s)| + \sup |\alpha_n(P_\mathbf{z}^s) - B_n(P_\mathbf{z}^s)| + \sup |B_n(P_\mathbf{z}^s) - \bar{B}_n(Q_\mathbf{z})| .$$

Let $S = O(n^{\frac{1}{9}})$; then by Lemmas 6, 5 and 11:

$$P\{\sup |\alpha_n(Q) - \alpha_n(P_\mathbf{z}^s)| \geqq CML^2S^{-\frac{1}{2}}(\log S)^{\frac{1}{2}} = Cn^{-\frac{1}{18}}(\log n)^{\frac{1}{2}}\} \leqq n^{-\mu} .$$
$$P\{\sup |B_n(Q_\mathbf{z}) - B_n(P_\mathbf{z}^s)| \geqq Cn^{-\frac{1}{18}}(\log n)^{\frac{1}{2}}\} \leqq n^{-\mu} ,$$
$$P\{\sup |\alpha_n(P_\mathbf{z}^s) - B_n(P_\mathbf{z}^s)| \geqq CMSn^{-\frac{1}{6}}(\log n)^{\frac{3}{2}} = Cn^{-\frac{1}{18}}(\log n)^{\frac{3}{2}}\} \leqq n^{-\mu} .$$

which proves (3).

LEMMA 12. *For any $\mu > 0$ one can construct a* K.P. $K(\mathbf{x}; y)$ $(\mathbf{x} \in I^2; 0 \leqq y < \infty)$ *such that*

$$(22) \qquad P\{\sup_{P \in \mathscr{P}(\kappa,1)} |n^{\frac{1}{2}}\alpha_n(P) - K(P; n)| \geqq C\kappa(\log^2 n)n^{\frac{3}{8}}\} \leqq n^{-\mu}$$

where $C = C(\mu)$ is a positive constant depending only on μ.

PROOF. Consider the sequence $0 = n_0 < n_1 < n_2 < \cdots$ of integers where $n_k = [k^\alpha]$ $(\alpha > 1)$ and let $n_k - n_{k-1} = m_k$. Denote by $\bar{\alpha}_k(\mathbf{x})$ be the E.P. based on the sample $\mathbf{X}_{n_{k-1}}, \cdots, \mathbf{X}_{n_k}$. Further let $B_k(\mathbf{x})$ be a B.B. for which

$$P\{\sup_{P^r \in \mathscr{P}^r(\kappa,1)} |B_k(P^r) - \bar{\alpha}_k(P^r)| \geqq C\kappa^{\frac{1}{2}}Rm_k^{-\frac{1}{6}}(\log m_k)^{\frac{3}{2}}\} \leqq m_k^{-\mu}$$

where $r = r_k = [\log_2 k^\beta]$ and $\beta > 1/4(\alpha - 1)$ (see (17) and Lemma 10).

By Lemma C there exists a K.P. $K(\mathbf{x}; y)$ for which

$$K(\mathbf{x}; n_k) = \textstyle\sum_{j=1}^k m_j^{\frac{1}{2}}B_j(\mathbf{x})$$

and clearly

$$n_k^{\frac{1}{2}}\alpha_{n_k}(\mathbf{x}) = \textstyle\sum_{j=1}^k m_j^{\frac{1}{2}}\bar{\alpha}_j(\mathbf{x}) .$$

Hence we have

$$P\{|n_k^{\frac{1}{2}}\alpha_{n_k}(P^r) - K(P^r; n_k)| \geqq C\kappa^{\frac{1}{2}}k^\beta(\log k)^{\frac{3}{2}} \cdot k^{\frac{1}{2}}(\log k)^{\frac{1}{2}}\} \leqq n_k^{-\mu} .$$

By Lemma 1

$$P\{\sup_{P^r \in \mathscr{P}^r(\kappa,1)} |n_k^{\frac{1}{2}}\alpha_{n_k}(P^r) - K(P^r; n_k)| \geqq C\kappa^{\frac{1}{2}}k^{\beta+\frac{1}{2}}\log^2 k\} \leqq k^{-\mu} .$$

Lemmas 3 and 6 imply

$$P\{\sup_{P \in \mathscr{P}(\kappa,1)} |n_k^{\frac{1}{2}}\alpha_{n_k}(P) - K(P; n_k)| \geqq C\kappa^{\frac{1}{2}}k^{\beta+\frac{1}{2}}\log^2 k + C\kappa k^{(\alpha-\beta)/2}\log k\} \leqq k^{-\mu} .$$

Choose $\beta = \frac{1}{3}(\alpha - 1)$. Then

$$P\{\sup_{P \in \mathscr{P}(\kappa,1)} |n_k^{\frac{1}{2}}\alpha_{n_k}(P) - K(P; n_k)| \geqq C\kappa(\log^2 k)k^{(2\alpha+1)/6}\} \leqq k^{-\mu} .$$

By Lemma 7

$$P\{\sup_{P\in\mathscr{P}(\kappa,1)} |n^{\frac{1}{2}}\alpha_n(P) - K(P;n)| \geqq C\kappa(\log^2 n)n^{(2\alpha+1)/6\alpha} + (\log n)^{\frac{1}{2}} n^{(\alpha-1)/2\alpha}\} \leqq n^{-\mu}.$$

Choosing $\alpha = 4$ we get (22).

PROOF OF (4). Clearly we have

$$\sup_{\mathbf{z}\in\mathscr{A}} |n^{\frac{1}{2}}\alpha_n(Q_{\mathbf{z}}) - \bar{K}(Q_{\mathbf{z}};n)| \leqq \sup_{\mathbf{z}\in\mathscr{A}} |n^{\frac{1}{2}}\alpha_n(P_{\mathbf{z}}^s) - K(P_{\mathbf{z}}^s;n)| + \sup_{\mathbf{z}\in\mathscr{A}} |n^{\frac{1}{2}}\alpha_n(P_{\mathbf{z}}^s) - n^{\frac{1}{2}}\alpha_n(Q_{\mathbf{z}})| + \sup_{\mathbf{z}\in\mathscr{A}} |K(P_{\mathbf{z}}^s;n) - \bar{K}(Q_{\mathbf{z}};n)|.$$

Choosing $s = [\log_2 n^\gamma]$ where $\gamma = \frac{1}{2}$ by Lemmas 5, 6 and 12 we get (4).

5. Proof of Theorem 3. The next lemma is straightforward and known.

LEMMA 13. *The transformation T has the following properties*:

(i) $F(A) = \int_A dF(\mathbf{x}) = \lambda(TA)$ *for any Borel set of* R^2,

(ii) $\lambda(B) = F(T^{-1}B)$ *for any Borel set of* I^2,

(iii) $\mathbf{X}_1 = T\mathbf{Y}_1$, $\mathbf{X}_2 = T\mathbf{Y}_2, \cdots$ *is a sequence of* i.i.d. rv'*s uniformly distributed over* I^2,

(iv) $\beta_n(\mathbf{x}) = \alpha_n(TD_{\mathbf{x}})(\mathbf{x}\in R^2)$ *where* α_n *is the* E.P. *based on the sample* $\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_n$ *and* β_n *is the* E.P. *based on the sample* $\mathbf{Y}_1, \mathbf{Y}_2, \cdots, \mathbf{Y}_n$,

(v) $\alpha_n(\mathbf{x}) = \beta_n(T^{-1}D_{\mathbf{x}})$ $(\mathbf{x}\in I^2)$.

PROOF OF (5). By Lemma 13

$$(\beta_n(\mathbf{x}) - B_n(TD_{\mathbf{x}})) = (\alpha_n(TD_{\mathbf{x}}) - B_n(TD_{\mathbf{x}})).$$

Since $TD_{\mathbf{x}} \in Q(L, 4)$ $(\mathbf{x}\in R^2)$, by Theorem 2 we have (5).

The proof of (6) is the same.

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