The Annals of Probability
1977, Vol. 5, No. 1, 87–99

INSENSITIVITY OF STEADY-STATE DISTRIBUTIONS OF GENERALIZED SEMI-MARKOV PROCESSES. PART I

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New proofs are given for results by Matthes concerning a certain insensitivity phenomenon displayed, for instance, by many well-known stochastic models from the areas of reliability and telephone engineering.

1. Introduction. One of the classical problems of the theory of stochastic service systems was to find a proof for the conjecture that the steady-state distribution of the number of busy servers in the $M/G/n$ loss or Erlang system depends on the service-time distribution only through its mean. The phenomenon was already known to Erlang, who gave a proof for the case of Erlang service-time distributions. The general case was settled as late as 1957 by Sewastjanow in [9]. In the meantime, the theories of telephone traffic and reliability had produced further examples of stochastic systems for which the distributions of certain life-times such as repair times, conversations lengths, or interarrival times, influence certain steady-state distributions only through their means. One such system, of equal importance as the Erlang system, is the Engset system, for which Cohen (in [1]) proved the corresponding result in 1957. Another example is provided by the semi-Markov scheme, where jumps on a countable state space S occur in Markovian fashion with the sojourn times in a given state, s, say, distributed according to a distribution function $F_s(\cdot)$ with finite mean λ_s^{-1}. If it exists, the limiting distribution, as $t \to \infty$, of the state at time t depends on the collection $\{F_s(\cdot); s \in S\}$ only through the means $\{\lambda_s^{-1}; s \in S\}$.

This background of examples motivated Matthes to study the phenomenon within a general framework rather than for just a particular system. For this purpose he introduced in [6] a rich class of models comprising many well established or potentially interesting members from such areas as reliability and telephone engineering, queuing theory, inventory theory, and others. Given a model from this class, one of his main results is a necessary and sufficient condition for the insensitivity phenomenon to prevail. This condition is of great interest not only because it usually allows in practice to actually determine whether or not the phenomenon is present but also because it allows a certain "partial balance" interpretation and thus contributes significantly to a better understanding of the phenomenon. Matthes' results along with those of later coworkers are presented in detail in [2] and [5], where also many important examples can be found.

Received January 7, 1976.
AMS 1970 *subject classifications.* Primary 60K99; Secondary 60K15, 60K20.
Key words and phrases. Generalized semi-Markov processes, insensitivity, reliability, queues.

Unfortunately, the methods of analysis in these references are heavily involved, rendering simpler approaches highly desirable.

The purpose of the present paper is to present such an approach within the framework laid down by Matthes. In Section 4 an elementary proof of the necessity of Matthes' condition is given, followed by an equally elementary proof of its sufficiency in Section 5.

In Section 3 a general concept of insensitivity is introduced. It provides a perspective from which to view insensitivity results such as Matthes' or the new result of Theorem 5.3 (i) which says that Matthes' type of insensitivity is equivalent to an apparently much weaker one. A further study of insensitivity as defined in Section 3 is contained in [7], and much remains to be done.

The notational complexity arising within this theory is considerable as illustrated in [2] and [5]. In order to keep the notation fairly simple we refrain from rigorous constructions of the various processes involved and give, in Section 2, a verbal outline only. The same concern causes us to consider the insensitivity problem under a certain restriction only (we vary only one "input" distribution). While the essence of our approach shows clearly and fully in this restricted case, some aspects of the general result get lost, and we shall therefore supplement the present paper by a sequel.

2. Generalized semi-Markov processes. Consider an organism (a "system") that is, at any time t, $t \geqq 0$, in one of the states $g \in G$, where G is a finite set. Each state g is a set itself such that $g \cap S \neq 0$ and $\bigcup_{g \in G} g \cap S = S$, where S is another finite set. There is, at any time t, $t \geqq 0$, a "living" part of the organism, consisting precisely of the elements $s \in g \cap S$, where g is the state at the given time. Each of these elements of g is living independently of the others and for a randomly distributed time. When one of these elements, say s, dies, the organism jumps to a new state, g', with probability $p(g, s, g')$ and independently of everything else present and past. Those elements living independently of each other, the jump does not affect the residual lifetime distributions of the ones in $g \cap (S/s)$ and, of course, $p(g, s, g') = 0$ unless $(g' \cap S) \supset g \cap (S/s)$. The jump may cause new elements of S to start living, i.e., $g \cap (S/s)$ may be a proper subset of $g' \cap S$. Included is the possibility that the element s—which caused the jump—is revived immediately after the time of the jump and starts a new life. All elements of $g' \cap S$ are again living independently of each other, the lifetimes of the new ones, s', s'', $\cdots$, say, being drawn from the distributions $\varphi_{s'}$, $\varphi_{s''}$, $\cdots$ independently of everything present and past. This may serve as our presentation of the class of models to be considered.

We wish to study the limiting distribution, if it exists uniquely, of the state of the organism. For this purpose, after having enumerated S as $\{s_1, \cdots, s_N\}$, we assume that a process $\{X(t), Y(t); t \geqq 0\}$ has been constructed along the following lines. At time $t = 0$ a state $g_0 \in G$ is specified and each element $s_i \in g_0 \cap S$ is assigned a positive number $y_{i,0}$, its residual lifetime. Putting $y_{i,0} = 0$ for $s_i \notin g_0 \cap S$

we let $y_0 = (y_{1,0}, \cdots, y_{N,0})$ and $(X(0), Y(0)) = (g_0, y_0)$. We now let $X(t) = g_0$, $Y_i(t) = 0$ for $s_i \notin g_0 \cap S$, and $Y_i(t) = y_{i,0} - t$, for $0 < t < \tau_1 = \min_{s_i \in g_0 \cap S} y_{i,0}$. By specifying the positive $y_{i,0}$ to be all different from each other we select uniquely the element of g_0 which dies at time τ_1. Denoting this element by s, we select at its time of death, τ_1, a new element of G according to the distribution $p(g_0, s, \cdot)$. Denoting the new state by g_1, we then obtain $y_1 = (y_{1,1}, \cdots, y_{N,1})$ by letting $y_{i,1} = 0$ if $s_i \notin g_1 \cap S$, $y_{i,1} = y_{i,0} - \tau_1$ if $s_i \in g_0 \cap (S/s)$, and by picking the remaining $y_{i,1}$, i.e., those corresponding to newly living elements, from the distributions φ_{s_i}. We then let $(X(\tau_1), Y(\tau_1)) = (g_1, y_1)$ and now continue the process beyond $t = \tau_1$ until the time of the next death just as we continued it beyond $t = 0$ until the time of the first death. It is henceforth assumed (following [5]) that the distributions φ_s, $s \in S$, are concentrated on $(0, \infty)$, have finite means λ_s^{-1} and are absolutely continuous. The assumptions ensure that simultaneous deaths occur with zero probability and that the process continues ad infinitum almost surely (see [5]). A construction similar to the one outlined here has been rigorously carried out in [5].

Some terminology is needed. The collection $\Sigma = (G, S, p)$, where p denotes the family $\{p(g, s, \cdot); g \in G, s \in g \cap S\}$ of distributions on G satisfying $p(g, s, g') = 0$ unless $g' \cap S \supset g \cap (S/s)$, is called a generalized semi-Markov scheme (GSMS); the process $\{X(t); t \geqq 0\}$ is called the generalized semi-Markov process (GSMP) based upon Σ by means of the family $\{\varphi_s; s \in S\}$; and the process $\{X(t), Y(t); t \geqq 0\}$ is called a supplemented GSMP.

The latter can be shown to be a homogeneous Markov process, following closely the arguments in [5] for a similar construction. If, for some $S' \subset S$, and for $s \in S'$ the distributions φ_s are exponential, it is sufficient for obtaining a Markov process to supplement $X(t)$ by the N-dimensional vector $Y'(t)$, where $Y_i'(t) = 0$ if $s_i \in S'$ or $s_i \notin X(t) \cap S$ and $Y_i'(t)$ equals the residual lifetime of s_i, otherwise. A process $\{X(t), Y'(t); t \geqq 0\}$ obtained in this way will be referred to as a reduced supplemented GSMP.

Our interest is focused on the unique stationary distribution of a given supplemented GSMP, where it exists, and, more specifically, on that of the GSMP. Adapting the relevant arguments in [5] to the present construction it is readily seen that the above stationary distributions exist uniquely provided that the underlying GSMS is irreducible and all distributions φ_s have (Lebesgue a.e.) positive density functions. A GSMS is called irreducible if, for every pair $g, g' \in G$, there exist finite sequences $\{g_1, \cdots, g_n\}$, $g_i \in G$, and $\{s^{(0)}, \cdots, s^{(n)}\}$, $s^{(i)} \in S$, such that $p(g, s^{(0)}, g_1)\, p(g_1, s^{(1)}, g_2) \cdots p(g_n, s^{(n)}, s') > 0$. The positivity of the density function guarantees the impossibility of a certain blocking effect (see [5]).

The examples mentioned in Section 1 can all be seen to fit the above setup. Shortage of space forces us to refer the reader to [5] for detailed presentations of many interesting examples.

3. A general insensitivity concept. Let $\Sigma = (G, S, p)$ be an irreducible GSMS and $\Phi(\Sigma)$ the collection of all families $\phi = \{\varphi_s; s \in S\}$ of distributions concentrated on $(0, \infty)$ which are absolutely continuous, have finite means, and imply the existence of a unique stationary distribution for the corresponding supplemented GSMP's based upon Σ. Let Φ be a nonempty subset of $\Phi(\Sigma)$. Then Σ is called Φ-insensitive, if every GSMP based upon Σ by means of an element of Φ has the same stationary distribution. Several special cases may serve to illustrate this concept.

If $S = \{s_1, \cdots, s_N\}$ as in the previous section, one may have

$$\Phi = \{\phi : \varphi_{s_i} \text{ exponential with fixed means } \lambda_{s_i}^{-1},\ i = 1, \cdots, k;\ \varphi_{s_{k+1}} = \cdots = \varphi_{s_N} \text{ with fixed mean } \lambda^{-1}\}\,.$$

A model from the area of telephone engineering displaying this type of intensitivity has been studied by Jacobi in [4]. The type studied by Matthes is characterised by specifying Φ as

$$\Phi = \{\phi : \varphi_{s_i} \text{ exponential with fixed mean } \lambda_{s_i}^{-1},\ i = 1, \cdots, k;\ \varphi_{s_i} \text{ arbitrary with fixed mean } \lambda_{s_i}^{-1},\ i = k+1, \cdots, N\}\,.$$

Finally, a type of insensitivity studied in [7] is described by

$$\Phi = \{\phi : \varphi_{s_i} \text{ exponential with fixed mean } \lambda_{s_i}^{-1},\ i = 1, \cdots, N-1,\ \varphi_{s_N} \text{ arbitrary with certain values of } b_{s_N}^{(\nu)}(\cdot) \text{ prescribed, where } \nu = 0, 1, \cdots, \text{ and } b_{s_N}(\cdot) \text{ the LST of } \varphi_{s_N}(\cdot)\}\,.$$

Obviously, many other types of Φ-insensitivity may crop up. Moreover, insensitivities not covered by our concept are encountered in many models, such as, for instance, insensitivity of certain characteristic values of the limiting distribution of GSMP's. The mean queue length in the $M/G/1$ model is an example.

4. Necessity of Matthes' condition. Let $\Sigma = (G, S, p)$ be an irreducible GSMS, $s_0 \in S$ fixed, and

$$\Phi_{s_0} = \{\phi : \varphi_s \text{ exponential with fixed mean } \lambda_s^{-1}, \text{ for } s \neq s_0,\ \varphi_{s_0} \text{ arbitrary with fixed mean } \lambda_{s_0}^{-1}\}\,.$$

We shall derive a necessary condition for Φ_{s_0}-insensitivity of Σ, due to Matthes. He studied the more general case of more than one nonexponential φ_s, which we shall treat in a sequel to this paper. For the family $\phi' \in \Phi_{s_0}$ corresponding to exponential φ_{s_0} it is easily seen that the GSMP $\{X(t); t \geqq 0\}$ based upon Σ by means of ϕ' is a Markov chain with stationary transition probabilities and a unique limiting distribution $\{p_g : g \in G\}$ which is the unique probability solution of the equations

$$\Lambda_g p_g = \sum_{g' \in G} p_{g'} \sum_{s \in g' \cap S} p(g', s, g)\lambda_s\,, \qquad g \in G\,, \tag{4.1}$$

where

$$\Lambda_g = \sum_{s \in g \cap S} \lambda_s\,. \tag{4.2}$$

Now consider a family $\phi'' \in \Phi_{s_0}$ corresponding to a distribution φ_{s_0} with distribution function

$$F_{s_0}(\cdot) = \pi_1 E_{\lambda_1}(\cdot) + \pi_2 E_{\lambda_2}(\cdot) * E_{\lambda_1}(\cdot)\,, \tag{4.3}$$

where $\lambda_1, \lambda_2 > 0$, $E_\lambda(t) = 1 - e^{-\lambda t}$ for $t \geqq 0$, $0 \leqq \pi_1 \leqq 1$, $\pi_1 + \pi_2 = 1$, and the symbol $*$ denotes convolution. Notice that

$$\frac{\pi_1}{\lambda_1} + \pi_2\left(\frac{1}{\lambda_2} + \frac{1}{\lambda_1}\right) = \frac{1}{\lambda_{s_0}}\,. \tag{4.4}$$

The supplemented GSMP based upon Σ by means of such a family can be constructed by first realizing, each time s_0 starts living, a random number taking on the values 1 and 2 with probabilities π_1 and π_2, respectively, and proceeding then as follows: if the number equals 1, realize for s_0 a (residual) lifetime distributed according to $E_{\lambda_1}(\cdot)$; if the number equals 2, realize for s_0 a (residual) lifetime which is the sum of two phases: a first phase distributed according to $E_{\lambda_2}(\cdot)$ followed by a second one distributed independently of the first one according to $E_{\lambda_1}(\cdot)$. Instead of the supplement $Y_{s_0}(\cdot)$ one may then introduce a supplement $K_{s_0}(\cdot)$, where $K_{s_0}(t) = 0, 1$, or 2 depending on whether, at time t, s_0 is not alive, in a phase of parameter λ_1, or in a phase of parameter λ_2, respectively. Such constructions have been more rigorously outlined in [3]. It is easily seen that the process $\{X(t), K_{s_0}(t); t \geqq 0\}$ is a Markov chain with state space $\{(g, i); g \in G, i = 0, 1, 2\}$ and stationary transition probabilities. The chain is, also obviously, irreducible and therefore has a unique limiting distribution $\{p_{g,i}; g \in G, i = 0, 1, 2\}$, where, of course, $p_{g,0} = 0$ if $s_0 \in g$ and $p_{g,1} = p_{g,2} = 0$ if $s_0 \notin g$. Letting

$$G_0 = \{g : s_0 \in g\}$$

the standard linear system for this distribution is given by

$$\begin{aligned}(\Lambda_g + \lambda_i - \lambda_{s_0})p_{g,i} = \lambda_{i+1}p_{g,i+1} &+ \textstyle\sum_{g' \in G_0} p_{g',i} \sum_{s \neq s_0; s \in g' \cap S} p(g', s, g)\lambda_s \\ &+ \pi_i \textstyle\sum_{g' \in G_0} p_{g',1} p(g', s_0, g)\lambda_1 \\ &+ \pi_i \textstyle\sum_{g' \notin G_0} p_{g',0} \sum_{s \in g' \cap S} p(g', s, g)\lambda_s\end{aligned} \tag{4.5$_1$}$$

for $g \in G_0$, $i = 1, 2$, $\lambda_3 p_{g,3} = 0$, and by

$$\Lambda_g p_{g,0} = \textstyle\sum_{g' \in G_0} p_{g',1} p(g', s_0, g)\lambda_1 + \sum_{g' \notin G_0} p_{g',0} \sum_{s \in g' \cap S} p(g', s, g)\lambda_s \tag{4.5$_2$}$$

for $g \notin G_0$.

We are now in a position to prove

THEOREM 4.1. *If Σ is Φ_{s_0}-insensitive, then the limiting distribution $\{p_g\}$ of a* GSMP *based upon Σ by means of a family $\phi \in \Phi_{s_0}$ satisfies the equations* (4.1) *and*

$$\lambda_{s_0} p_g = \textstyle\sum_{g' \notin G_0} p_{g'} \sum_{s \in g' \cap S} p(g', s, g)\lambda_s + \sum_{g' \in G_0} p_{g'} p(g', s_0, g)\lambda_{s_0}\,, \tag{4.6}$$

the latter for $g \in G_0$.

Condition (4.6) is condition "Z'" of [5]. It allows the following "flow" interpretation: if s_0 is part of g, then under equilibrium conditions the flow out of g

via inactivation of s_0 equals the flow into g via activation of s_0. This is a sharpening of the flow statement expressed by (4.1) and is known in the literature as partial balance.

PROOF. If Σ is Φ_{s_0}-insensitive, any choice of φ_{s_0} satisfying (4.3) (and (4.4)) yields a distribution $\{p_{g,i}\}$ satisfying (4.5) and the relations

$$(4.7_1) \qquad p_{g,0} = p_g , \quad g \notin G_0$$

and

$$(4.7_2) \qquad p_{g,1} + p_{g,2} = p_g , \quad g \in G_0 ,$$

where p_g are as in (4.1). By some enumeration of G let $G_0 = \{g_1, \cdots, g_n\}$ and then let

$$(4.8) \qquad \begin{aligned} -a_{ij} &= \Lambda_{g_i} - \lambda_{s_0} - \textstyle\sum_{s \neq s_0; s \in g_i \cap S} p(g_i, s, g_j)\lambda_s && \text{for } i = j = 1, \cdots, n \\ &= -\textstyle\sum_{s \neq s_0; s \in g_i \cap S} p(g_i, s, g_j)\lambda_s && \text{for } i \neq j = 1, \cdots, n , \end{aligned}$$

A the matrix (a_{ij}), P the matrix (p_{ij}) with

$$(4.9) \qquad p_{ij} = p(g_i, s_0, g_j) , \qquad i, j = 1, \cdots, n ,$$

and d, x and y the n-dimensional column vectors given by

$$(4.10) \qquad \begin{aligned} d_i &= \textstyle\sum_{g' \notin G_0} p_{g'} \sum_{s \in g' \cap S} p(g', s, g_i)\lambda_s , \\ x_i &= p_{g_i,1} , \quad \text{and} \\ y_i &= p_{g_i,2} , \quad \text{respectively.} \end{aligned}$$

Then (4.5_1) can be written in matrix form as the system of the two equations

$$(4.11) \qquad -A^T x + \lambda_1 x = \lambda_1 \pi_1 P^T x + \pi_1 d + \lambda_2 y$$

and

$$(4.12) \qquad -A^T y + \lambda_2 y = \lambda_1 \pi_2 P^T x + \pi_2 d ,$$

where the symbol T denotes the transpose of a matrix. All the quantities in (4.11) and (4.12) can be identified with those in (A4) and (A5) corresponding to the same symbols. Hence (A7) is true, which, together with (4.1), yields (4.6).

In fact we have proved more. An immediate consequence of Corollary A2 is

COROLLARY 4.1. *Let* $\phi' \in \Phi_{s_0}$ *be the family corresponding to exponential* φ_{s_0}, $\phi'' \in \Phi_{s_0}$ *a family with* φ_{s_0} *satisfying* (4.3), *where* $\lambda_1 \pi_1 \neq \lambda_0$. *Then, if* Σ *is* $\{\phi', \phi''\}$-*insensitive*, (4.1) *and* (4.6) *hold.*

The condition $\lambda_1 \pi_1 \neq \lambda_0$ cannot be dropped, as $\lambda_1 \pi_1 = \lambda_0$ implies that φ_{s_0} is exponential.

It should be noted that the family of distributions given by (4.3) and (4.4) can also be represented by letting

$$(4.3') \qquad F_{s_0}(\cdot) = \pi_1 E_{\lambda_1}(\cdot) + \pi_2 E_{\lambda_2}(\cdot)$$

and

$$\frac{\pi_1}{\lambda_1} + \frac{\pi_2}{\lambda_2} = \frac{1}{\lambda_{s_0}} \,. \tag{4.4'}$$

This representation could be used to shorten the exposition of our proof to some —rather small—extent. We prefer not to do this as (4.3) is the representation which is in line with the traditional phase method as well as the approach taken in Section 5, where an alternative representation corresponding to (4.3′) would be detrimental to the analysis.

5. Sufficiency of Matthes' condition. We let Σ and Φ_{s_0} be as in the previous section and focus on the reduced supplemented GSMP's $\{X(t),\ Y'(t);\ t \geqq 0\}$ based upon Σ by means of families from Φ_{s_0}, where $Y_i'(t) = 0$ for $s_i \neq s_0$, $Y_i'(t) = 0$ if $s_i = s_0 \notin X(t)$, and $Y'(t)$ equals the residual lifetime of s_0 if $s_i = s_0 \in X(t)$. For these processes we consider initial distributions with the property that, for all $g \in G_0$, the probability that the state is g and the residual lifetime of s_0 does not exceed x, $x > 0$, is of the form

$$p_g \lambda_{s_0} \textstyle\int_0^x (1 - F_{s_0}(t))\, dt \,, \tag{5.1}$$

where $F_{s_0}(\cdot)$ is the distribution function of φ_{s_0} and $\{p_g\}$ is a probability distribution on G. A distribution of this type is said to possess the product property. Matthes' results imply

THEOREM 5.1. *If the reduced supplemented* GSMP *based upon* Σ *by means of at least one family* $\phi \in \Phi_{s_0}$ *other than the exponential family has a stationary initial distribution possessing the product property* (5.1), *then* (4.1) *and* (4.6) *are satisfied for* $\{p_g\}$.

We have a very simple proof of this theorem for the case that $F_{s_0}(\cdot)$ is of the form

$$F_{s_0}(\cdot) = \textstyle\sum_{i=1}^k \pi_i E_\lambda^i(\cdot) \,, \tag{5.2}$$

where $1 < k < \infty$, $0 \leqq \pi_i \leqq 1$, $\sum_{i=1}^k \pi_i = 1$, and $E_\lambda^i(\cdot)$ denote the i-fold convolution of $E_\lambda^1(\cdot) = E_\lambda(\cdot)$. The class of distributions of type (5.2) is dense in the set of all distributions concentrated on $(0, \infty)$ (see [8], page 32), but we do not see how to use this fact for a proof of Theorem 5.1 in all its generality. Here is the proof of Theorem 5.1 under conditions (5.2):

In the spirit of the phase interpretation for (5.2) take the view that the life of s_0 extends with probability π_i, $i = 1, \cdots, k$, over i consecutive independent and according to $E_\lambda(\cdot)$ distributed phases, starting with phase i. Consider the analogue of the Markov chain that led to system (4.5). This chain has the state space

$$\bar{G} = \{(g, i);\ g \in G,\ i = 0, \cdots, k\} \,,$$

where $(g, 0)$ denotes a state not containing s_0 and (g, i), for $i > 0$, one that contains s_0 in its ith phase. The unique stationary initial probabilities $\bar{p}_{g,i}$ of this chain satisfy the equations

$$(5.3_1)\qquad \begin{aligned}(\Lambda_g + \lambda - \lambda_{s_0})\bar{p}_{g,i} = \lambda\bar{p}_{g,i+1} &+ \textstyle\sum_{g'\in G_0}\bar{p}_{g',i}\sum_{s\neq s_0;s\in g'\cap S} p(g', s, g)\lambda_s \\ &+ \pi_i \textstyle\sum_{g'\in G_0}\bar{p}_{g',1}p(g', s_0, g)\lambda \\ &+ \pi_i \textstyle\sum_{g'\notin G_0}\bar{p}_{g',0}\sum_{s\in g'\cap S} p(g', s, g)\lambda_s\end{aligned}$$

for $g \in G_0$, $i = 1, \cdots, k$, $\bar{p}_{g,k+1} = 0$, and the equations

$$(5.3_2)\qquad \Lambda_g\bar{p}_{g,0} = \textstyle\sum_{g'\in G_0}\bar{p}_{g',1}p(g', s_0, g)\lambda + \sum_{g'\notin G_0}\bar{p}_{g',0}\sum_{s\in g'\cap S} p(g', s, g)\lambda_s$$

for $g \notin G_0$.

On the other hand, there is a stationary initial distribution $\{p_g;\, g \in G\}$ for the GSMP under consideration such that $\{p_g;\, g \notin G_0\}$ and (5.1) for $g \in G_0$ together represent a stationary initial distribution of the reduced supplemented GSMP under consideration. The latter process has a unique stationary initial distribution as Σ is irreducible and φ_{s_0} has a positive density function. Hence we have

$$(5.4_1)\qquad \bar{p}_{g,0} = p_g\,, \quad g \notin G_0$$

and

$$(5.4_2)\qquad \textstyle\sum_{i=1}^k \bar{p}_{g,i} = p_g\,, \quad g \in G_0\,.$$

Furthermore, for $g \in G_0$, we have

$$p_g\lambda_{s_0}\textstyle\int_0^x (1 - F_{s_0}(t))\,dt = \sum_{i=1}^k \bar{p}_{g,i}E_\lambda^i(x)\,,$$

or, applying the Laplace–Stieltjes transform on both sides,

$$p_g\lambda_{s_0}\frac{1}{w}\left[1 - \textstyle\sum_{i=1}^k \pi_i\left(1 + \frac{w}{\lambda}\right)^{-i}\right] = \sum_{i=1}^k \bar{p}_{g,i}\left(1 + \frac{w}{\lambda}\right)^{-i}$$

for $w \geqq 0$. This yields the relations

$$(5.5)\qquad \bar{p}_{g,i} = (\lambda_{s_0}/\lambda)(\pi_i + \cdots + \pi_k)p_g\,, \qquad g \in G_0,\ i = 1, \cdots, k\,.$$

Inserting (5.4_1) and (5.5) in (5.3_2) results in (4.1) for $g \notin G_0$. For fixed $g \in G_0$, adding the equations (5.3_1) for all i and using (5.4) and (5.5) leads to (4.1) for $g \in G_0$. As for (4.6), consider (5.3_1) for $i = k$, say, and use (5.4) and (5.5) to obtain the equation

$$\begin{aligned}(\Lambda_g + \lambda - \lambda_{s_0})p_g\frac{\lambda_{s_0}}{\lambda}\pi_k = \frac{\lambda_{s_0}}{\lambda}\pi_k &\textstyle\sum_{g'\in G_0} p_{g'}\sum_{s\neq s_0;s\in g'\cap S} p(g', s, g)\lambda_s \\ &+ \pi_k \textstyle\sum_{g'\in G_0} p_{g'}p(g', s_0, g)\lambda_{s_0} \\ &+ \pi_k \textstyle\sum_{g'\notin G_0} p_{g'}\sum_{s\in g'\cap S} p(g', s, g)\lambda_s\,.\end{aligned}$$

Dividing by π_k ($\pi_k > 0$ is assumed) and then subtracting (λ_{s_0}/λ) times the equation (4.1) for the same fixed g as above yields (4.6).

Using the connections between product property and (4.1) and (4.6) we can now give a simple proof of the sufficiency of (4.1) and (4.6) for Φ_{s_0}-insensitivity. To start with we prove a little less, namely

THEOREM 5.2. *If there exists a distribution* $\{p_g\}$ *on* G *satisfying* (4.1) *and* (4.6), *and if* $F_{s_0}(\cdot)$ *is of type* (5.2), *then* $\{p_g\}$ *is the* (*unique*) *stationary initial distribution of the* GSMP *based upon* Σ *by means of the family* $\phi \in \Phi_{s_0}$ *for which* φ_{s_0} *is given by* $F_{s_0}(\cdot)$. *Furthermore, the* (*unique*) *stationary initial distribution of the corresponding reduced supplemented* GSMP *is given by* $\{p_g\}$ *and* (5.1).

PROOF. The function $F_{s_0}(\cdot)$ being of type (5.2) we may consider the Markov chain of the preceding proof, with stationary probabilities satisying (5.3). We are going to show that the values suggested by (5.4) and (5.5) for these probabilities yield a probability solution of (5.3). Inserting into (5.3$_2$) these values yield (4.1) for $g \notin G_0$. Inserting similarly into (5.3$_1$) for a fixed i yields the equations

$$(\Lambda_g - \lambda_{s_0}) \frac{\lambda_{s_0}}{\lambda} (\pi_i + \cdots + \pi_k) p_g$$

$$= \frac{\lambda_{s_0}}{\lambda} (\pi_i + \cdots + \pi_k) \textstyle\sum_{g' \in G_0} p_{g'} \sum_{s \neq s_0; s \in g' \cap S} p(g', s, g) \lambda_s$$

$$+ \pi_i \textstyle\sum_{g' \in G_0} p_{g'} p(g', s_0, g) \lambda_{s_0}$$

$$+ \pi_i \textstyle\sum_{g' \notin G_0} p_{g'} \sum_{s \in g' \cap S} p(g', s, g) \lambda_s - \lambda_{s_0} \pi_i p_g$$

for $g \in G_0$.

Here the last three terms on the right-hand side add up to zero as $\{p_g\}$ is assumed to satisfy (4.6). Subtracting, for fixed $g \in G_0$, (4.6) from (4.1) reveals that the above relation for $\{p_g\}$ does, in fact, hold. Thus, the values suggested by (5.4) and (5.5) for the $\bar{p}_{g,i}$ do indeed yield a probability solution of (5.3). There is, however, only one probability solution of (5.3). The properties expressed in (5.4) and (5.5) imply the statements of the theorem.

The condition (5.2) on $F_{s_0}(\cdot)$ can be removed as follows: if $F_{s_0}(\cdot)$ is continuous but not of type (5.2), there is a sequence $\{F_{s_0,n}(\cdot); n \geqq 1\}$ of distribution functions of type (5.2) with $F_{s_0,n}(t) \to F_{s_0}(t)$ for all $t \geqq 0$ as $n \to \infty$ (see [8], page 32). It has been shown in [3] that the corresponding stationary distributions considered in Theorem 5.2 converge weakly to the one corresponding to $F_{s_0}(\cdot)$, provided that

$$\lambda_{s_0,n} \textstyle\int_0^x (1 - F_{s_0,n}(t))\, dt \to \lambda_{s_0} \int_0^x (1 - F_{s_0}(t))\, dt$$

for $x \geqq 0$ as $n \to \infty$. This, however, can be arranged to hold, and hence the condition on $F_{s_0}(\cdot)$ can be dropped in Theorem 5.2. Using this fact we now sum up our findings of Sections 4 and 5 in

THEOREM 5.3. *Let* Σ *and* Φ_{s_0} *as defined in Section* 4, *and let* $\phi' \in \Phi_{s_0}$ *be the family corresponding to exponential* φ_{s_0}, $\phi'' \in \Phi_{s_0}$ *a family corresponding to a* φ_{s_0} *of type* (4.3) *with* $\lambda_1 \pi_1 \neq \lambda_0$.

(i) *If* Σ *is* $\{\phi', \phi''\}$-*insensitive, it is* Φ_{s_0}-*insensitive*;

(ii) *If* Σ *is* Φ_{s_0}-*insensitive*, (4.1) *and* (4.6) *hold for the stationary distribution of the* GSMP *based upon* Σ *by means of a* $\phi \in \Phi_{s_0}$;

(iii) *If there is a distribution on* G *satisfying* (4.1) *and* (4.6), Σ *is* Φ_{s_0}-*insensitive*;

(iv) *If Σ is Φ_{s_0}-insensitive, the stationary distribution of the reduced supplemented* GSMP *based upon Σ by means of a map $\phi \in \Phi_{s_0}$ possesses the product property.*

Theorems 5.1 and 5.2 imply moreover that Σ is Φ_{s_0}-insensitive, provided there is a family $\Phi \in \Phi_{s_0}$ other than the exponential family producing the product property. We have, however, shown this only for the case that ϕ satisfies the additional condition (5.2). Theorem 5.3(i) seems to be new. It is evident from the results in [7] or from well-known examples of GSMP's, that insensitivity with respect to two different elements of Φ_{s_0} does not in general imply Φ_{s_0}-insensitivity.

APPENDIX

THEOREM A. *Let A be an $n \times n$ conservative Q-matrix*, i.e.,

$$a_{ij} \geqq 0 \qquad \text{for} \quad j \neq i$$

and

$$\sum_j a_{ij} = 0 \quad \forall i . \tag{A1}$$

Let P be an $n \times n$ substochastic matrix. Let d be a fixed nonnegative n-dimensional vector. Let $\lambda_1, \lambda_2 > 0$ and $0 \leqq \pi_1, \pi_2 \leqq 1$ with

$$\pi_1 + \pi_2 = 1 \tag{A2}$$

and

$$\frac{1}{\lambda_1} + \pi_2 \frac{1}{\lambda_2} = \frac{1}{\lambda_0}, \tag{A3}$$

where λ_0 is fixed. If, for every choice of $\lambda_1, \lambda_2, \pi_1, \pi_2$ satisfying these conditions, the system

$$-A^T x + \lambda_1 x = \lambda_1 \pi_1 P^T x + \pi_1 d + \lambda_2 y \tag{A4}$$

$$-A^T y + \lambda_2 y = \lambda_1 \pi_2 P^T x + \pi_2 d \tag{A5}$$

has a nonnegative solution (x, y) satisfying

$$x + y = p, \tag{A6}$$

where p is a fixed vector not depending on the choice of λ_1, λ_2 and π_i, then

$$A^T p = 0 . \tag{A7}$$

Here the symbol T denotes the transpose of a matrix.

PROOF. For $\pi_2 = 0$ and hence, by (A2) and (A3), $\lambda_1 = \lambda_0$, (A5) yields

$$A^T y = \lambda_2 y .$$

The eigenvalues of A^T having nonnegative real parts, it follows that either $A^T = 0$ or $y = 0$. The former implies (A7), the latter, by (A6), $x = p$ and, by (A4)

$$-A^T p + \lambda_0 p = \lambda_0 P^T p + d . \tag{A8}$$

Adding (A4) to (A5) (for general parameters) and using (A2) and (A6) results in

$$\text{(A9)} \qquad -A^T p = \lambda_1(P^T - I)x + d$$

where I is the identity matrix. Subtracting (A8) from (A9) yields

$$\text{(A10)} \qquad (P^T - I)(\lambda_1 x - \lambda_0 p) = 0 .$$

Eliminating P^T from (A4) by means of (A10) produces

$$\text{(A11)} \qquad A^T(x - \pi_1 p) = \frac{\lambda_2}{\lambda_0}(\lambda_1 x - \lambda_0 p) ,$$

and it is also easily verified that

$$\text{(A12)} \qquad \left(-A^T + \frac{\lambda_1\lambda_2}{\lambda_0} I\right)(\lambda_1 x - \lambda_0 p) = (\lambda_1\pi_1 - \lambda_0)(d + (P^T - I)\lambda_1 x) .$$

We want to show that $\lambda_1 x = \lambda_0 p$. This follows from (A10) provided that P does not have the eigenvalue 1. Hence assume it does and that the algebraic multiplicity is k, $1 \leqq k \leqq n$. Then it is well known that P^T may be assumed to appear in the form

$$\text{(A13)} \qquad P^T = \begin{bmatrix} P_1^T & & & & R_1 \\ & P_2^T & 0 & & R_2 \\ & & \ddots & & \\ & 0 & & P_k^T & R_k \\ & & & & P_{k+1}^T \end{bmatrix} ,$$

where the P_i, $i = 1, \cdots, k$, are irreducible and stochastic and P_{k+1} does not have the eigenvalue 1. Putting

$$\lambda_1 x - \lambda_0 p = u$$

and

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_{k+1} \end{bmatrix}$$

in accordance with the partitioning underlying (A13), we conclude from (A10) that

$$\text{(A14)} \qquad u_{k+1} = 0$$

and

$$\text{(A15)} \qquad (P_i^T - I)u_i = 0 , \qquad i = 1, \cdots, k .$$

As P_i, $i = 1, \cdots, k$, has the simple eigenvalue 1, the Perron–Frobenius theorem tells that, for fixed i, either $u_i < 0$ or $u_i > 0$ or $u_i < 0$, $i = 1, \cdots, k$.

We may assume that (A13) can be written as

$$\text{(A16)} \qquad P^T = \begin{bmatrix} P^{(1)T} & 0 & R^{(1)} \\ 0 & P^{(2)T} & R^{(2)} \\ 0 & 0 & P^{(3)T} \end{bmatrix}$$

and, correspondingly,

$$u = \begin{bmatrix} u^{(1)} \\ u^{(2)} \\ u^{(3)} \end{bmatrix},$$

where $u^{(1)} \leqq 0$, $u^{(2)} > 0$, $u^{(3)} = u_{k+1} = 0$, $P^{(1)T}u^{(1)} = u^{(1)}$, $P^{(2)T}u^{(2)} = u^{(2)}$, and $P^{(3)T} = P^T_{k+1}$. Now notice that (A11) and the properties (A1) of A imply that the component sum of u equals zero. Hence either $u^{(1)} = 0$ and $u^{(2)}$ is absent which we want to show, or else $u^{(1)}$ has at least one negative component and neither $P^{(1)T}$ nor $P^{(2)T}$ are absent in (A16). It remains to deal with the latter case. Therefore let

$$A^T = \begin{bmatrix} A^T_{11} & A^T_{12} & A^T_{13} \\ A^T_{21} & A^T_{22} & A^T_{23} \\ A^T_{31} & A^T_{32} & A^T_{33} \end{bmatrix}$$

be the partitioning of A^T compatible with that of P^T in (A16). Then (A12) yields

$$\text{(A17)} \qquad \left(-A^T_{11} + \frac{\lambda_1\lambda_2}{\lambda_0} I\right) u^{(1)} = A^T_{12}u^{(2)} + (\lambda_1\pi_1 - \lambda_0)[d^{(1)} + (P^{(1)T} - I)(\lambda_1 x)^{(1)} + R^{(1)}(\lambda_1 x)^{(3)}]$$

with the obvious interpretation of the symbols I, $d^{(1)}$, $(\lambda_1 x)^{(1)}$ and $(\lambda_1 x)^{(3)}$. This relation is valid for all allowable choices of λ_1, λ_2, π_i, with x depending on the choice. Now suppose the choice is such that $\lambda_1\pi_1 - \lambda_0 > 0$, which is clearly possible. Then our assumptions about u imply that summing up the individual equations of (A17) yields a negative number on the left-hand side and a non-negative one on the right-hand side. Thus $u^{(1)} = 0$ and $u^{(2)}$ is absent, i.e., $\lambda_1 x = \lambda_0 p$, for such a choice, and (A7) follows by (A11).

Corollary A1. *The assumptions of Theorem A imply that $\lambda_1 x = \lambda_0 p$, provided that*

$$\lambda_1\pi_1 \neq \lambda_0 .$$

Proof. The assertion has been proved for the case $\lambda_1\pi_1 > \lambda_0$ by partial exploitation of (A12) in (A17). Further exploitation of (A12) yields the rest.

Corollary A2. *The assertions* (A7) *and* (A18) *are valid provided the assumptions of Theorem* A *hold for the parameter choice $\pi_2 = 0$ and one other choice such that $\lambda_1\pi_1 \neq \lambda_0$.*

The condition $\lambda_1\pi_1 \neq \lambda_0$ cannot be dropped. A probabilistic reason is given in Section 4.

Acknowledgment. We are indebted to W. Jurecka, Vienna, for asking a question which led to the present investigation.

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