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A NOTE ON THE CENTRAL LIMIT THEOREM IN BANACH SPACES[1]

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We show how a recent theorem of J. Hoffmann-Jørgensen and G. Pisier can be formulated in such a way as to include a theorem of N. Jain and M. Marcus. We also obtain some central limit theorems on $L^p[0, 1]$, for $1 \leqq p < 2$.

In this note we show how the recent results of J. Hoffmann-Jørgensen and G. Pisier (see [6], [7]) on central limit theorems can be formulated in such a way as to provide an alternate proof of a result of N. Jain and M. Marcus [9]. (For previous results on central limit theorems on $C(S)$ see Dudley–Strassen [3], Giné [4], and Dudley [2].) In addition the same technique yields a central limit theorem on $L^p[0, 1]$ for $1 \leqq p \leqq 2$. Finally, some relations to the laws of iterated logarithm of J. Kuelbs [10], [11] and G. Pisier [14] will be given.

Let G be a Banach space and μ a Radon probability measure on G satisfying:

$$\int \|x\|_G^2 \mu(dx) < \infty \quad \text{and} \tag{1}$$

$$\int x\mu(dx) = 0 \,. \tag{2}$$

Now let $X_1, X_2, \cdots$ be a sequence of independent G-valued random variables with distribution μ. μ is said to *satisfy the central limit theorem* on G if there exists a Gaussian Radon probability γ on G such that the distributions, μ_n, of $(X_1 + \cdots + X_n)/n^{\frac{1}{2}}$ converge $\|\cdot\|$-weakly to γ, i.e., for every bounded norm-continuous real-valued function f on G,

$$\int f\, d\mu_n \to \int f\, d\gamma \,.$$

In the following $\{\varepsilon_i\}_{i=1}^\infty$ will denote a sequence of independent random variables such that $\Pr(\varepsilon_i = 1) = \Pr(\varepsilon_i = -1) = \frac{1}{2}$. E and F will denote Banach spaces. As usual $\mathbb{E}$ will denote expectation.

DEFINITION. A linear map $v : E \to F$ is of *type* 2 if $\sum_{i=1}^n \varepsilon_i v(x_i)$ converges in F a.s. for all sequences $\{x_i\} \subseteq E$ such that $\sum_{i=1}^\infty \|x_i\|_E^2 < \infty$. A Banach space G is said to be *type* 2 if the identity map on G is type 2.

The following theorem is essentially contained in [6], Theorems 4.1 and 4.2 (see also [7]). The proof requires only obvious modifications.

THEOREM 1. *$v : E \to F$ is of type 2 if and only if for every Radon probability μ on E satisfying* (1) *and* (2) *above, $\mu \circ v^{-1}$ satisfies the central limit theorem on F.*

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Now let (S, d) be a compact metric space and $C(S)$ the space of continuous functions on S with the supremum norm. For a d-continuous metric ρ on S let $N_\rho(S, \varepsilon)$ denote the minimal number of ρ-balls of diameter $\leqq 2\varepsilon$ which cover S. Also let $H_\rho(S, \varepsilon) = \log_e N_\rho(S, \varepsilon)$. We put

$$\mathrm{Lip}\,(\rho) = \left\{x \in C(S) : q(x) \equiv \sup_{s \neq t} \frac{|x(s) - x(t)|}{\rho(s, t)} < \infty\right\}.$$

Also, for $x \in \mathrm{Lip}\,(\rho)$ put

$$||x||_\rho = q(x) + |x(a)|,$$

where a is some fixed element in S.

DEFINITION. A d-continuous metric ρ on S is said to *imply Gaussian continuity* (or ρ is GCI) if whenever $\{X(t)\}_{t \in S}$ is a separable Gaussian process such that $E|X(t) - X(s)|^2 \leqq \rho^2(t, s)$, then X has continuous sample paths a.s.

EXAMPLES. (i) If ρ satisfies

$$\int_{0+} H_\rho^{\frac{1}{2}}(S, u)\, du < \infty, \tag{3}$$

then Theorem 3.1 [1] yields that ρ is *GCI*.

(ii) If ρ satisfies

$$\rho^2(t, s) = \mathbb{E}|Z(t) - Z(s)|^2, \tag{4}$$

where $\{Z(t)\}_{t \in S}$ is a continuous Gaussian process, then Lemma 2.1 [13] implies that ρ is GCI.

COROLLARY 1. *Let ρ be* GCI *and let μ be a Radon probability on* Lip (ρ) *satisfying* (1) *and* (2). *Then μ satisfies the central limit theorem on* $C(S)$.

PROOF. By Theorem 1, we need only show that the natural inclusion i: $\mathrm{Lip}\,(\rho) \to C(S)$ is of type 2. Therefore let $\{x_i\} \subseteqq \mathrm{Lip}\,(\rho)$ satisfy $\sum ||x_i||_\rho^2 = C < \infty$. To show $\{\sum_{i=1}^n \varepsilon_i x_i\}$ converges in $C(S)$ a.s., it is enough to show that $\{\sum_{i=1}^n \eta_i x_i\}$ converges in $C(S)$ a.s., where $\{\eta_i\}$ is a sequence of independent standard normal random variables (see Corollaries 3.3 and 4.4 in [5]). We also have

$$\begin{aligned}\mathbb{E}|\textstyle\sum_{j=1}^\infty \eta_j[x_j(t) - x_j(s)]|^2 &= \textstyle\sum_{j=1}^\infty |x_j(t) - x_j(s)|^2 \\ &\leqq \textstyle\sum_{j=1}^\infty ||x_j||_\rho^2 \rho^2(t, s) = C\rho^2(t, s).\end{aligned}$$

Hence, since ρ is GCI, a separable version of $X(t, \omega) = \sum_{j=1}^\infty \eta_j(\omega) x_j(t)$ has continuous sample paths a.s. But now Theorem 4.1 [8] implies that the series $\sum_{j=1}^\infty \eta_j x_j$ converges in $C(S)$ a.s. □

In the first draft of this paper it was erroneously claimed that Corollary 1, applied to the case where $\int_{0+} H_\rho^{\frac{1}{2}}(S, u)\, du < \infty$, *is* the Jain–Marcus central limit theorem. However Lip (ρ) is rarely separable. Hence if μ is a probability measure on $C(S)$ such that $\mu(\mathrm{Lip}\,(\rho)) = 1$, it may not induce a Radon measure on Lip (ρ). The error was noticed by N. Jain, who mentioned it at a conference in Oberwolfach (July, 1975).

The gap is filled below. The proof is due to J. Hoffmann-Jørgensen (Oberwolfach, July, 1975).

THEOREM 2 (Jain–Marcus [9]). *Let ρ be a d-continuous metric on S satisfying* (3). *Also, let μ be a Borel probability measure on $C(S)$ satisfying*:

(a) $$\int \|x\|_\rho^2 \mu(dx) < \infty$$

(b) $$\int x\mu(dx) = 0$$

and

(c) $$\mu(\text{Lip}\,(\rho)) = 1 \,.$$

Then μ satisfies the central limit theorem on $C(S)$.

PROOF. Since $\int_{0+} H_\rho^{\frac{1}{2}}(S, u)\, du < \infty$, there exists a continuous function g on $(0, \alpha]$, where $\alpha = \sup_{s,t \in S} \rho(s, t)$, such that $0 < g(u) \uparrow \infty$ as $u \downarrow 0$ and $\int_{0+} H_\rho^{\frac{1}{2}}(S, u) g(u)\, du < \infty$. Now put $h(\lambda) = \int_0^\lambda g(u/2)\, du$ and $\rho'(s, t) = h(\rho(s, t))$. Then ρ' is a d-continuous metric satisfying:

(5) For $\varepsilon > 0$ there exists $\delta > 0$ such that $\rho(s, t) \leqq \delta$ implies $\rho(s, t) \leqq \varepsilon\rho'(s, t)$.

Hence for $a \geqq 0$, $B_\rho(a) = \{x \in \text{Lip}\,(\rho); \|x\|_\rho \leqq a\}$ is compact in Lip (ρ'). But then since $\mu(B_\rho(n)) \uparrow 1$, μ is supported on a separable subspace of Lip (ρ'), and hence is Radon on Lip (ρ').

Note it is easily seen that $N_{\rho'}(\varepsilon) = N_\rho(h^{-1}(2\varepsilon)/2)$, and hence $\int_{0+} H_{\rho'}^{\frac{1}{2}}(S, u)\, du < \infty$. Therefore Corollary 1 applies to Lip (ρ'). □

Let us return to the general case. Now for $\{x_i\} \subseteq E$ satisfying $\sum \|x_i\|^2 < \infty$, $\sum \varepsilon_i x_i$ defines a cylinder set measure of type 2 (for the definitions of type, order and Radonifying see [15]). Hence if v is a 2-Radonifying, v is of type 2-Rademacher. Now for Banach spaces E and F, $v: E \to F$ is 2-Radonifying if and only if v is 2-absolutely summing (see Theorems 3.4 and 3.9 [15]). By Theorem 4.3 [12] any continuous v from an $\mathscr{L}_\infty$-space to a $\mathscr{L}_p$-space ($1 \leqq p \leqq 2$) is 2-absolutely summing. But any $v: \mathscr{L}_\infty \to L_p$, $2 \leqq p < \infty$ is type 2-Rademacher, since L_p is a G_1-space (see discussion preceding Theorem 3.1 [6]). Hence we have

COROLLARY 2. *If μ is a Borel probability on* $C[0, 1]$ *satisfying* (1) *and* (2) *then μ satisfies the central limit theorem on* $L_p[0, 1]$ *for any* $1 \leqq p < \infty$.

REMARK. In [14], G. Pisier proves that if G is a type 2 space, then the law of the iterated logarithm holds for all mean zero Radon probability measures (on G) with finite variance. He also points out that the techniques of this paper then allow you to obtain the log log law of J. Kuelbs [10], [11] on $C[0, 1]$. It is of course clear that Theorem 2 and Corollary 2 remain valid if one replaces "the central limit theorem" by the "law of the iterated logarithm."

REFERENCES

[1] DUDLEY, R. M. (1967). The sizes of compact subsets of Hilbert space and continuity of Gaussian processes. *J. Functional Analysis* **1** 290–330.

[2] DUDLEY, R. M. (1974). Metric entropy and the central limit theorem in $C(S)$. *Ann. Inst. Fourier* **24** 49–60.

[3] DUDLEY, R. M. and STRASSEN, V. (1969). The central theorem and ε-entropy. *Lecture Notes in Mathematics* **89** 224–231.

[4] GINÉ, E. (1974). On the central limit theorem for sample continuous processes. *Ann. Probability* **2** 629–641.

[5] HOFFMANN-JØRGENSEN, J. (1974). Sums of independent Banach space valued random variables. *Studia Math.* **52** 159–186.

[6] HOFFMANN-JØRGENSEN, J. (1975). The strong law of large numbers and the central limit theorem in Banach spaces. Proceedings of the seminar on random series, convex sets, and geometry of Banach spaces. Various Publications Series No. 24, 74–99.

[7] HOFFMANN-JØRGENSEN, J. and PISIER, G. (1976). The law of large numbers and the central limit theorem in Banach spaces. *Ann. Probability* **4** 587–599.

[8] ITÔ, K. and NISIO, M. (1968). On the convergence of sums of independent Banach space valued random variables. *Osaka J. Math.* **5** 25–48.

[9] JAIN, N. C. and MARCUS, M. B. (1975). Central limit theorems for $C(S)$-valued random variables. *J. Functional Analysis* **19** 216–231.

[10] KUELBS, J. (1976). The law of the iterated logarithm in $C[0, 1]$. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **33** 221–235.

[11] KUELBS, J. The law of the iterated logarithm in $C(S)$. Preprint.

[12] LINDENSTRAUSS, J. and PELCZYNSKI, A. (1968). Absolutely summing operators in L^p-spaces and their applications. *Studia Math.* **29** 275–326.

[13] MARCUS, M. B. and SHEPP, L. A. (1972). Sample behavior of Gaussian processes. *Proc. Sixth Berkeley Symp. Math. Statist. Prob.* **2** 423–441, Univ. of California Press.

[14] PISIER, G. (1976). Sur la loi du logarithme itéré dans les espaces de Banach. *Lecture Notes in Mathematics* **526** 203–210.

[15] SCHWARTZ, L. (1971). Probabilités cylindriques et applications radonifiantes. *J. Fac. Sci. Univ. Tokyo Sect. IA* **18** 139–286.

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