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A PROBABILISTIC PROOF OF BLACKWELL'S RENEWAL THEOREM

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A coupling method is used to give another proof of Blackwell's renewal theorem. The nature of the proof is probabilistic, using in an essential way the nonlattice property of the lifetime distribution and the Hewitt-Savage zero-one law.

1. Introduction. As a complement to the interesting analytical methods to prove Blackwell's renewal theorem (cf. Feller [4], XI. 1–2, pages 346 ff. and Breiman [2], 10.1–3, pages 216 ff.), it is the purpose of this note to show how a certain probabilistic method applies, namely a coupling with an independent, stationary renewal process. The use of couplings goes back to Doeblin [3]. They have attracted much interest in recent years: for an introduction and further references, see Griffeath [6].

Of great interest in the present context is a paper by Pitman [8], which pays considerable attention to discrete renewal theory. In his investigations of random walks, covering the renewal theorem, Ornstein [7] makes use of a coupling argument in the proof of his Theorem 0.7. He benefits from the Chung–Fuchs theorem on recurrence.

In this note, a new type of coupling is introduced. Our aim is to present a proof that is as short as possible and leans on no theorem that demands a technically complicated proof.

2. The proof.

2.1. *Preliminaries.* Let $X_1, X_2, \cdots$, be i.i.d. nonnegative random variables with $0 < E[X_i] = \mu < \infty$ and distribution function F of nonlattice type, i.e., its support is not a subset of the multiples of any single number. With $S_0 = 0$ and $S_n = \sum_1^n X_i$, $\langle S_n \rangle_0^\infty$ is the *renewal process* with *renewals* at each S_n, $n \geqq 0$. For $B \subset [0, \infty)$, let

$$N(B) = \#\{i;\, S_i \in B\}\,, \qquad U(B) = \mathbf{E}[N(B)]\,.$$

Blackwell's renewal theorem states that for every finite $A > 0$,

$$U[x, x + A] \to A/\mu \qquad \text{as} \quad x \to \infty\,.$$

Let the random variables $X_0', X_1', \cdots$, be independent, also independent of $\langle X_i \rangle_1^\infty$. Here, $X_1', X_2', \cdots$, all have distribution F, X_0' has a density

$$f_0(y) = (1 - F(y))/\mu\,, \qquad y \geqq 0\,.$$

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With $S_n' = \sum_0^n X_j'$, $\langle S_n' \rangle_0^\infty$ is the "coupling process," a delayed renewal process which is strictly stationary in the sense that $\langle S'_{\nu(t)+n} - t\rangle_{n=0}^\infty$ has the same distribution as $\langle S_n' \rangle_0^\infty$ for every $t \geqq 0$; here,

$$\nu(t) = \min\{j \geqq 0; S_j' \geqq t\}.$$

To understand that stationarity, see Feller [4], pages 353–355. Obviously, the t above may be replaced by any nonnegative random variable independent of $\langle X_j' \rangle_0^\infty$: the strict stationarity is retained. The notations N' and U' have the obvious meaning.

The idea in the proof is to show that, sooner or later, a renewal from $\langle S_n \rangle_0^\infty$ comes "close" to one from $\langle S_n' \rangle_0^\infty$. After that has occurred, both processes behave much like one another, because the $\langle S_n \rangle_0^\infty$ process is unchanged probabilistically on replacing the X_i's in its definition by the proper X_j''s onwards from where $\langle S_n \rangle_0^\infty$ and $\langle S_n' \rangle_0^\infty$ first come close (cf., e.g., Example 1 of Griffeath [6]). But $\langle S_n' \rangle_0^\infty$ is stationary, hence $\langle S_n \rangle_0^\infty$ is eventually so too, approximately, and we have $U[x, x+A] \approx U'[x, x+A] = A/\mu$ for large x.

2.2 *The details.* For $i \geqq 0$, let

$$Z_i = \min\{S_j' - S_i; S_j' - S_i \geqq 0, j \geqq 0\} = S'_{\nu(S_i)} - S_i,$$

and for any fixed $\delta > 0$, let

$$A_i = \{Z_j < \delta \text{ for some } j \geqq i\}.$$

We have

$$A_0 \supset A_1 \supset \cdots \supset \textstyle\bigcap_{i=0}^\infty A_i = A_\infty = \{Z_i < \delta \text{ i.o.}\}.$$

But by the stationarity of the process $\langle S_n' \rangle_0^\infty$ and the fact that $\langle S_{i+n} - S_i \rangle_{n=0}^\infty$ is a sequence with distribution independent of i, $\langle Z_{i+n} \rangle_{n=0}^\infty$ is also a sequence with distribution independent of i. Hence all the A_i's have the same probability, and in particular, $\mathbf{P}(A_0) = \mathbf{P}(A_\infty)$. Now $\mathbf{P}(A_\infty \mid X_0' = t)$ equals 0 or 1 for every t by virtue of the Hewitt–Savage zero-one law (cf. [4], page 122) applied to the i.i.d. sequence $(X_1, X_1'), (X_2, X_2'), \cdots$. That F is nonlattice renders $\mathbf{P}(A_0 \mid X_0' = t) > 0$ for every t, cf. [4], page 144, Lemma 2. These observations and the equality

$$\int \mathbf{P}(A_0 \mid X_0' = t) f_0(t)\, dt = \mathbf{P}(A_0) = \mathbf{P}(A_\infty) = \int \mathbf{P}(A_\infty \mid X_0' = t) f_0(t)\, dt,$$

force $\mathbf{P}(A_\infty \mid X_0' = t)$ to be 1 a.e. with respect to the distribution of X_0'. Hence $\mathbf{P}(A_\infty) = 1$, a fortiori $\mathbf{P}(A_0) = \mathbf{P}(Z_i < \delta \text{ for some } i) = 1$. Let

$$T = \min\{i; Z_i < \delta\}, \qquad T' = \min\{j; S_j' \geqq S_T\},$$

so that $T' = \nu(S_T)$. With

$$N''[x, x+A] = N([x, x+A] \cap [0, S_T]) + N'([x + Z_T, x + A + Z_T] \cap (S'_{T'}, \infty))$$

we certainly have $N''[x, x + A] =_{\mathscr{D}} N[x, x + A]$. Hence

$$\begin{aligned} U[x, x + A] &= \mathbf{E}[N''[x, x + A]] \\ &= \mathbf{E}[N([x, x + A] \cap [0, S_T])] + \mathbf{E}[N'[x + Z_T, x + A + Z_T]] \\ &\quad - \mathbf{E}[N'([x + Z_T, x + A + Z_T] \cap [0, S'_{T'}])] \\ &= V_1(x) + V_2(x) - V_3(x)\,, \quad \text{say.} \end{aligned}$$

Since we can choose arbitrarily small $\delta > 0$, we can make $V_2(x)$ arbitrarily close to A/μ uniformly in x by virtue of the fact that $Z_T < \sigma$ implies

$$(A - \delta)/\mu = U'[x + \delta, x + A] \leqq V_2(x) \leqq U'[x, x + A + \delta] = (A + \delta)/\mu\,.$$

Next, since $N[x, x + A]$ is stochastically no larger than $N[0, A]$, we have that

$$\begin{aligned} V_1(x) &\leqq \mathbf{E}[N[x, x + A] \cdot I(x \leqq S_T)] \\ &\leqq a \cdot \mathbf{P}(x \leqq S_T) + \mathbf{E}[N[0, A]; N[0, A] \geqq a]\,, \end{aligned}$$

an inequality valid for any $a > 0$. Now $\mathbf{E}(N[0, A]) = U(A) < \infty$ so the last term tends to zero for $a \to \infty$, and then because $S_T < \infty$ a.s. the term $a\mathbf{P}(x \leqq S_T) \to 0$ for $x \to \infty$. We show similarly that $V_3(x) \to 0$ as $x \to \infty$, and the proof is complete.

3. Concluding remarks. The same arguments provide another simple proof of the discrete renewal theorem, with the simplification that by taking $\delta <$ span of the lattice of the distribution F, we automatically have $Z_T = 0$.

To prove $U[x, x + A] \to 0$ in the case $\mu = \infty$, it is tempting to use a truncation argument. However, no such proof has been settled, as far as I know. Nevertheless, there is a short analytical one, along the following lines (cf. Freedman [5], pages 23 ff.). Let $\beta = \limsup_{t\to\infty} U(t, t + 1] = \lim_{k\to\infty} U(t_k, t_k + 1]$. Conditioning with respect to S_i yields $\beta = \lim_{k\to\infty} \int U(t_k - y, t_k + 1 - y]F^{*i}(dy)$ for every i; this fact and the asymptotic density of the support of U render $\liminf_{k\to\infty} U(t_k - j, t_k + 2 - j] \geqq \beta$ for integers $j \geqq$ some j_0. The relations $1 = \int_0^{t_k} (1 - F(t_k - y))U(dy) \geqq (1 - F(2)) \cdot U(t_k - 2, t_k] + (1 - F(4)) \cdot U(t_k - 4, t_k - 2] + \cdots$ and a limiting argument force β to be 0, because $\sum_{i=1}^{\infty} (1 - F(2i)) = \infty$.

We have paid attention to nonnegative X_i's only. The extension to a result about random walks with drift follows from the result about such X_i's by an analysis of the imbedded ladder height process, as was proved in detail by Blackwell himself [1].

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