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A GAUSSIAN CORRELATION INEQUALITY FOR SYMMETRIC CONVEX SETS[1]

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If $\mathsf{n}(x)$ is the standard normal density on R^2 and if $A = -A$ and $B = -B$ are convex subsets of R^2 then

$$\int_{A\cap B} \mathsf{n}(x)\, d^2x \geqq \left(\int_A \mathsf{n}(x)\, d^2x\right)\left(\int_B \mathsf{n}(x)\, d^2x\right).$$

1. Summary and introduction. A function $h(x)$ defined for $x \in R^n$ is called quasi-concave if for any $x_1, x_2 \in R^n$ and $0 < \lambda < 1$, $h(\lambda x_1 + (1-\lambda)x_2) \geqq \min\{h(x_1), h(x_2)\}$. Our main result is

THEOREM 1. *Let $f(x)$ and $g(x)$ be even smooth quasi-concave functions of $x \in R^2$. Suppose also that the gradients $\nabla f(x)$ and $\nabla g(x)$ never vanish for $x \neq 0$. Then for any nonnegative $\phi(x) = \phi(|x|)$ that is a decreasing function of $|x|$,*

$$\int_{R^2} \nabla f(x) \cdot \nabla g(x)\phi(x)\, d^2x \geqq 0\,, \tag{1.1}$$

provided only that the integral converges. Here $\nabla f(x) \cdot \nabla g(x)$ denotes the scalar product $\sum_1^2 f_{x_i}(x) g_{x_i}(x)$.

From Theorem 1 we deduce

THEOREM 2. *Let $\mathsf{n}(x) = (2\pi)^{-1}\exp(-|x|^2/2)$, $x \in R^2$, denote the standard normal probability density on R^2. If A and B are balanced* (i.e., $A = -A$ *and* $B = -B$) *convex subsets of R^2 then*

$$\int_{A\cap B} \mathsf{n}(x)\, d^2x \geqq \left(\int_A \mathsf{n}(x) d^2x\right)\left(\int_B \mathsf{n}(x)\, d^2x\right). \tag{1.2}$$

Theorem 2 and the more detailed Theorem 3 represent improvements of the earlier results of Khatri [5], Šidák [10, 11] and others. See especially [4], Section 3, and the references given there.

Our proof of Theorem 1 does not seem to generalize to R^n with $n > 2$. However, if (1.1) is true in R^n then the deduction of Theorem 2 is also valid in R^n with $n > 2$.

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2. Proof of Theorem 1. Because $f(x)$ is even and quasi-concave, the set $F(\lambda) \equiv \{x : f(x) \geqq \lambda\}$ is for each λ a closed convex balanced subset of R^2. Moreover from the assumption that $\nabla f(x) \neq 0$ for $x \neq 0$ it follows whenever $\{0\} \neq F(\lambda)$ that the boundary $F'(\lambda)$ of $F(\lambda)$ is either empty or $F'(\lambda) = \{x : f(x) = \lambda\}$ is a

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smooth balanced convex curve in R^2. Similar comments hold for the sets $G(\mu) = \{x : g(x) \geqq \mu\}$ and $G'(\mu) = \{x : g(x) = \mu\}$.

For each $x \neq 0$ we denote by $\theta(x)$, $0 \leqq \theta(x) < \pi$, the angle between the vectors $\nabla f(x)$ and $\nabla g(x)$. Thus $\cos\theta(x) = \eta_1(x) \cdot \eta_2(x)$ where $\eta_1(x) = -\nabla f(x)\|\nabla f(x)\|^{-1}$ is the outward pointing normal to $F'(f(x))$ at x and $\eta_2(x) = -\nabla g(x)\|\cdot\nabla g(x)\|^{-1}$ is the normal to $G'(g(x))$ at x. Set $A = \{x \in R^2 : \frac{1}{2}\pi < \theta(x) < \pi\}$ and for $x \in A$ define $y = \alpha(x)$ to be the first point of $G'(g(x))$ which can be reached from x by traversing the curve $F'(f(x))$ in the counterclockwise direction (see Figure 1).

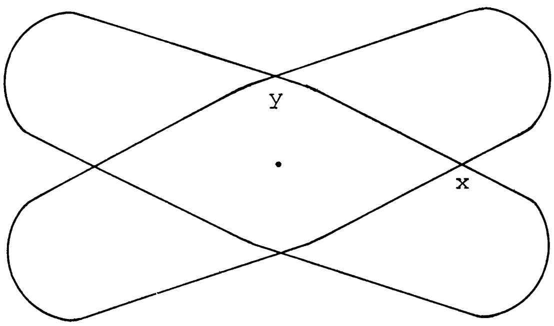

FIG. 1.

Elementary geometric considerations now show for each $x \in A$ that:

(2.1) $$|y| \leqq |x| \quad \text{and hence} \quad \phi(x) \leqq \phi(|y|)\,.$$

(2.2) The angle $\theta(y)$ is acute with $\cot\theta(y) \geqq -\cot\theta(x) > 0$.

Define $A_0 = \{x \in A : \theta(\alpha(x)) \neq 0\}$ and consider the map $\Phi : R^2 \to R^2$ with $\Phi(x) = (f(x), g(x))$. The Jacobi determinant of Φ is

(2.3) $$\begin{aligned} |J_\Phi(x)| &= |f_{x_1}(x)g_{x_2}(x) - f_{x_2}(x)g_{x_1}(x)| \\ &= |\nabla f(x)|\,|\nabla g(x)| \sin(\theta(x))\,. \end{aligned}$$

For $x \in A$, $\Phi(\alpha(x)) = \Phi(x)$ and thus $\Phi(A - A_0) = \Phi(\alpha(A - A_0))$ contains only critical values of the function Φ. By the theorem of Sard and Brown (e.g., [6]), $\Phi(A - A_0)$ has zero Lebesgue measure. But (2.3) shows $|J_\Phi(x)| \neq 0$ for $x \in A$ and hence $A - A_0$ has zero Lebesgue measure.

When restricted to A_0 the function $\alpha(x)$ is easily seen to be continuous. Applying the chain rule to the identity $\Phi(\alpha(x)) = \Phi(x)$ we find $\alpha(x)$ is differentiable on A_0 and its Jacobi determinate $|J_\alpha(x)|$ satisfies

(2.4) $$|J_\alpha(x)| = |J_\Phi(x)|\,|J_\Phi(\alpha(x))|^{-1}\,.$$

We can now prove the inequality (1.1). The sets A_0 and $\alpha(A_0)$ are disjoint. Because $\nabla f(x) \cdot \nabla g(x) \geqq 0$ unless $x \in A$, and because $A - A_0$ has measure zero, (1.1) will follow if we show

(2.5) $$\int_{A_0} \nabla f(x) \cdot \nabla g(x)\phi(x)\,d^2x + \int_{\alpha(A_0)} \nabla f(x) \cdot \nabla g(x)\phi(x)\,d^2x \geqq 0\,.$$

Using in order (2.4), (2.3), (2.2) and (2.1) we have

$$\begin{aligned}\int_{\alpha(A_0)} \nabla f(x)\cdot\nabla g(x)\phi(x)\,d^2x &= \int_{A_0} \nabla f(\alpha(x))\cdot\nabla g(\alpha(x))\phi(\alpha(x))|J_\alpha(x)|d^2x\\ &= \int_{A_0} \nabla f(\alpha(x))\cdot\nabla g(\alpha(x))\phi(\alpha(x))|J_\Phi(x)|\,|J_\Phi(\alpha(x))|^{-1}\,d^2x\\ &= \int_{A_0} \cot(\theta(\alpha(x))\phi(\alpha(x))|J_\Phi(x)|\,d^2x\\ &\geqq -\int_{A_0} \cot(\theta(x))\phi(x)|J_\Phi(x)|\,d^2x\\ &= -\int_{A_0} \nabla f(x)\cdot\nabla g(x)\phi(x)\,d^2x\,,\end{aligned}$$

thus proving Theorem 1.

COMMENT. The inequality (1.1) may be extended to a wider class of functions $f(x)$ and $g(x)$ than quasi-concave. Following Sherman [9], we bring in the norm $||u||_* = \max\{||u||_1, ||u||_\infty\}$ on bounded integrable functions and the $||\ ||_*$ closed convex cone $\mathscr{C}$ generated by the indicator (= characteristic) functions $1_A(x)$ of balanced convex sets $A \subset R^n$. Sherman showed $\mathscr{C}$ is closed under convolution. More recently Davidovic, Karenbljum and Hacet [3], Prékopa [7], [8] and others have shown that the convolution $u * v$ of two logarithmically concave functions is also log concave. This shows that indicator functions of balanced convex sets may be approximated by smooth log concave functions $f(x)$ with $\nabla f(x) \neq 0$ for $x \neq 0$. Since even log concave functions are also quasi-concave we may state

COROLLARY 1. *The inequality* (1.1) *holds for smooth Sherman functions f and g in* $\mathscr{C}$.

A more detailed exposition of the proof of Theorem 1 may be found in Antell [2].

3. Proof of Theorem 2. We begin by stating a more detailed and general theorem. Let the random vector $(X_1, \cdots, X_n, Y_1, \cdots, Y_n)$ be normally distributed with mean zero and covariance matrix Σ where

$$\Sigma = \begin{pmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{12}^* & \Sigma_{22}\end{pmatrix}$$

and $\Sigma_{11} = (EX_iX_j)$, $\Sigma_{22} = (EY_iY_j)$ and $\Sigma_{12} = (EX_iY_j)$. Set $X = (X_1, \cdots, X_n)$ and $Y = (Y_1, \cdots, Y_n)$ and for balanced convex subsets A and B of R^n set

$$p(\Sigma) = P\{X \in A;\ Y \in B\}\,.$$

For each λ, $0 \leqq \lambda \leqq 1$, the matrix

$$\Sigma_\lambda = \begin{pmatrix}\Sigma_{11} & \lambda\Sigma_{12}\\ \lambda\Sigma_{12}^* & \Sigma_{22}\end{pmatrix}$$

is also a normal covariance matrix and we may consider the probability $p(\Sigma_\lambda)$ as a function of λ, $0 \leqq \lambda \leqq 1$.

THEOREM 3. *Under the above conditions, if rank* $(\Sigma_{12}) \leqq 2$ *then* $p(\Sigma_\lambda)$ *is an increasing function of* λ, $0 \leqq \lambda \leqq 1$.

REMARKS. The probabilities $P\{X \in A\}$ and $P\{Y \in B\}$ do not depend on λ. Moreover, $\lim_{\lambda \to 0} p(\Sigma_\lambda) = P\{X \in A\}P\{Y \in B\}$. Thus Theorem 3 implies

$$P\{X \in A;\ Y \in B\} \geqq P\{X \in A\}P\{Y \in B\}\ . \tag{3.1}$$

If one takes $n = 2$ and $\Sigma_{11} = \Sigma_{12} = \Sigma_{22} = I$, then (3.1) becomes (1.2) and we see that Theorem 2 is a special case of Theorem 3.

The proof of Theorem 3 is conceptually simpler in the special case when $n = 2$ and $\Sigma_{11} = \Sigma_{12} = \Sigma_{22} = I$. We advise considering only this special case on the first reading.

PROOF. Without loss of generality we may assume rank $(\Sigma_{11}) =$ rank $(\Sigma_{22}) = n$. Then by introducing canonical variates (see [1], Chapter 12), we may further assume that $\Sigma_{11} = \Sigma_{22} = I$ and that Σ_{12} is diagonal with nonnegative entries. Let $\Sigma_{12} = \operatorname{diag}(\lambda_1, \cdots, \lambda_n)$ with $1 \geqq \lambda_1 \geqq \lambda_2 \geqq \cdots \geqq 0$. Since rank $(\Sigma_{12}) \leqq 2$ we have $\lambda_3 = \lambda_4 = \cdots = \lambda_n = 0$.

The functions $1_A(x)$ and $1_B(y)$ are both even and log concave. By the results of Davidovic et al., we may approximate 1_A and 1_B by smooth even log concave functions. Thus it suffices to show for arbitrary smooth even rapidly decreasing log concave functions $F(x)$ and $G(y)$ that the expectation

$$\mathscr{E}(\lambda) = EF(X)G(Y)$$

corresponding to the covariance matrix Σ_λ is an increasing function of λ, $0 \leqq \lambda \leqq 1$.

When conditioned on X_1, X_2, Y_1, Y_2 the variables $F(X)$ and $G(Y)$ are independent. Moreover,

$$E\{F(X) \mid X_1, X_2, Y_1, Y_2\} = E\{F(X) \mid X_1, X_2\} = f(X_1, X_2)\ ,$$

where

$$f(x_1, x_2) = (2\pi)^{-(n-2)/2} \int F(x) \exp(-\tfrac{1}{2} \textstyle\sum_3^n x_j^2)\, dx_3 \cdots dx_n\ ,$$

and

$$E\{G(Y) \mid X_1, X_2, Y_1, Y_2\} = E\{G(Y) \mid Y_1, Y_2\} = g(Y_1, Y_2)\ ,$$

where

$$g(y_1, y_2) = (2\pi)^{-(n-2)/2} \int G(y) \exp(-\tfrac{1}{2} \textstyle\sum_3^n y_j^2)\, dy_3 \cdots dy_n\ .$$

Thus

$$\mathscr{E}(\lambda) = Ef(X_1, X_2)g(Y_1, Y_2)\ .$$

Now Prékopa [8] has shown that if $H(x_1, \cdots, x_k, y_1, \cdots, y_l)$ is log concave then $h(x_1, \cdots, x_k) = \int H(x_1, \cdots, x_k, y_1, \cdots, y_l)\, dy_1 \cdots dy_n$ is log concave. Thus, both of the above functions $f(x_1, x_2)$ and $g(y_1, y_2)$ are log concave.

The conditional density of (Y_1, Y_2) given $(X_1, X_2) = (x_1, x_2)$ is

$$p(\lambda, x_1, x_2; y_1, y_2) = \prod_{j=1}^2 [2\pi(1 - (\lambda\lambda_j)^2)]^{-1} \exp\left\{-\frac{1}{2} \frac{(\lambda\lambda_j x_j - y_j)^2}{1 - (\lambda\lambda_j)^2}\right\}\ .$$

Setting $\mathsf{n}(x_1, x_2) = (2\pi)^{-1} \exp\{-(x_1^2 + x_2^2)/2\}$ and

$$g(\lambda, x_1, x_2) = \int p(\lambda, x_1, x_2; y_1, y_2) g(y_1, y_2)\, dy_1\, dy_2$$

we find

$$\mathscr{E}(\lambda) = \int f(x_1, x_2) g(\lambda, x_1, x_2) \mathsf{n}(x_1, x_2)\, dx_1\, dx_2\, .$$

Let $\lambda = e^t$, $-\infty < t \leqq 0$. Then direct computation gives

$$\frac{\partial}{\partial t} g(e^t, x_1, x_2) = \left(\sum_{j=1}^{2} \frac{\partial^2}{\partial x_j^2} - x_j \frac{\partial}{\partial xj}\right) g(e^t, x_1, x_2)$$

and an easily justified integration by parts yields

$$\frac{d}{dt} \mathscr{E}(e^t) = \int \nabla f(x_1, x_2) \cdot \nabla g(e^t, x_1, x_2) \mathsf{n}(x_1, x_2)\, dx_1\, dx_2\, . \tag{3.2}$$

Since $p(\lambda; x_1, x_2; y_1, y_2)$ is log concave in (x_1, x_2, y_1, y_2) the results of Prékopa show that $g(e^t, x_1, x_2)$ is log concave in (x_1, x_2). Since n is a radially decreasing function, Corollary 1 of Theorem 1 implies $(d/dt)\mathscr{E}(e^t) \geqq 0$ and completes the proof of Theorem 3.

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