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DISTRIBUTION INEQUALITIES FOR THE BINOMIAL LAW

BY ERIC V. SLUD[1]

New Mexico State University

We prove that the probability of at least k successes, in n Bernoulli trials with success-probability p, is larger than its normal approximant if $p \leqq \frac{1}{4}$ and $k \geqq np$ or if $p \leqq \frac{1}{2}$ and $np \leqq k \leqq n(1-p)$. A local refinement is given for $np \leqq k \leqq n(1-p)$, $k \geqq 2$, and for $p \leqq \frac{1}{4}$, $k \geqq n(1-p)$. Bounds below for individual binomial probabilities $b(k, n, p)$ are also given under various conditions. Finally, we discuss applications to significance tests in one-way layouts.

1. Introduction. The classical Poisson and de Moivre–Laplace limit theorems (see, for example, Feller (1957), chapters 6, 7) deal with approximations in law or in density to the binomial Bin (n, p) asymptotically as $np \to \lambda$ or $n \to \infty$, respectively by the Poisson law Poi (λ) with mean λ and the normal law $\mathcal{N}(np, np(1-p))$. A long line of contributions paralleling the history of probability itself, and possibly culminating in a paper of Prohorov (1953), sharpens the rates of convergence and extends the conditions. But only within the last fifteen years have there been some scattered results giving inequalities among these distribution functions, more in the spirit of Laplace's continued fraction for $\int_a^\infty \exp(-t^2/2)\, dt/(2\pi)^{\frac{1}{2}} \equiv 1 - \Phi(a)$ than of his limit theorem.

The primary emphasis of the present paper is in this newer direction, on inequalities rather than approximations for binomial tail probabilities. To chart progress in the area, we list five known domination relations between binomial tails and their classical approximants. Here

$$p(k, \lambda) = \exp(-\lambda)\lambda^k/k!\,, \qquad P(k, \lambda) = \textstyle\sum_{j=0}^{k} p(j, \lambda)\,, \qquad \bar{P}(k, \lambda) = \textstyle\sum_{j=k}^{\infty} p(j, \lambda)\,,$$
$$b(k, n, p) = \tbinom{n}{k} p^k (1-p)^{n-k}\,, \qquad q = 1-p\,, \qquad B(k, n, p) = \textstyle\sum_{j=0}^{k} b(j, n, p)\,,$$
$$\bar{B}(k, n, p) = \textstyle\sum_{j=k}^{n} b(j, n, p)\,, \qquad \text{and} \qquad 0 \leqq p \leqq 1\,, \quad \lambda \geqq 0\,,$$

with $k \leqq n$ nonnegative integers.

(i) (Bohman, 1963). $\bar{P}(k, \lambda) \geqq 1 - \Phi((k-\lambda)/\lambda^{\frac{1}{2}})$.

(ii) (Anderson, Samuels, 1965). If $k \leqq n^2p/(n+1)$, then $\bar{P}(k+1, np) \leqq \bar{B}(k+1, n, p)$.

(iii) (Anderson, Samuels, 1965). If $k \geqq np + 1$, then $\bar{P}(k, np) \geqq \bar{B}(k, n, p)$.

(iv) If $k \geqq np + 1$, $\bar{P}(k, np) \geqq \max(\bar{B}(k, n, p), 1 - \Phi((k-np)/(npq)^{\frac{1}{2}}))$.

(v) If $k \leqq np$, $\bar{B}(k, n, p) \geqq \bar{P}(k, np) \geqq 1 - \Phi((k-np)/(np)^{\frac{1}{2}})$.

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Bohman's inequality (i) says that for all $\lambda > 0$ the $\mathcal{N}(\lambda, \lambda)$ random variable is stochastically smaller than the Poi (λ) variable. (ii) and (iii) stem from a line of investigation initiated by Samuels (1965) in his thesis and continued in Jogdeo, Samuels (1968). In (iv), $k > np = \lambda$ allows us to replace $1 - \Phi((k - \lambda)/\lambda^{\frac{1}{2}})$ a fortiori by $1 - \Phi((k - \lambda)/(\lambda q)^{\frac{1}{2}})$ in (i), but we note that $k - np \leqq 0$ in (v). Both (iv) and (v), which follow easily from (i)—(iii), suggest that it is worth looking for inequalities of the form $\bar{B}(k, n, p) \geqq 1 - \Phi((k - np)/(npq)^{\frac{1}{2}})$. Dudley conjectured in a seminar that this tail-inequality holds for $k \geqq np$, $p \geqq \frac{1}{4}$, and showed (in a private communication) that it does when $\bar{B}(k, n, p)$ is replaced by its Peizer–Pratt (1968) approximant and $k \geqq np + \frac{1}{2}$, $p \leqq \frac{1}{4}$. In Section 2 we establish the tail-inequality for $k \geqq np$, $p \leqq \frac{1}{4}$ and for $np \leqq k \leqq n(1 - p)$, and we strengthen it in Section 3 to various local refinements for individual binomial probabilities including a bound below for error in Theorem 3.2.

There is also a systematic method for checking which inequalities among binominal, normal and Poisson tail probabilities can hold for arbitrarily large n. It is based on the following large-deviations result of Chernoff (1952):

THEOREM 1.1. *If $\{X_j\}_{j=1}^{\infty}$ is a sequence of independent identically distributed random variables, with some $t_0 > 0$ for which $E(\exp(t_0 X_1)) < \infty$, then for $a > 0$,*

$$(^\circ) \qquad n^{-1} \log (\Pr\{X_1 + \cdots + X_n > an\}) \to \log(\rho^+(X_1, a))$$
$$\textit{where} \quad \rho^+(X_1, a) \equiv \min_t E(\exp(t \cdot (X_1 - a))) .$$

We remark also that if X_1 is respectively normal, Poisson or binomial, then so is $X_1 + \cdots + X_n$, and moment-generating functions $E(\exp(tX_1))$ exist. Hence if a is taken $> (<)\, p$, $(^\circ)$ gives asymptotic information about upper (lower, using $-X_i$ instead of X_i) tail probabilities for all three distributions.

(α) If X_1 is Bin$(1, p)$, $\log \rho^+(X_1, a) \equiv F(a, p) = a \log(p/a) + (1 - a) \log((1 - p)/(1 - a))$, and $(^\circ)$ says $n^{-1} \log \bar{B}(an, n, p) \to F(a, p)$.

(β) If X_1 is $\mathcal{N}(p, p(1 - p))$, then $\log \rho^+(X_1, a) \equiv G(a, p) = -(a - p)^2/(2p(1 - p))$.

(γ) If X_1 is Poi (p) distributed, then $\log \rho^+(X_1, a) \equiv K(a, p) = a \log(p/a) + a - p = \lim_{n\to\infty} (n^{-1} \log \bar{P}(an, np))$.

We readily check that for $a > p$, $F(a, p) = a \log(p/a) + (1 - a) \log((1 - p)/(1 - a)) = a \log(p/a) + (1 - a) \log(1 + (a - p)/(1 - a)) < a \log(p/a) + a - p = K(a, p)$, which is the asymptotic version of (iii). Similar calculations work, as they must, for (i), (ii).

To find the proper conditions for the inequalities of Sections 2 and 3, we try to assure $F(a, p) \geqq G(a, p)$ for $a \geqq p$. At $a = 1$, this says $\log p + q/(2p) \geqq 0$, which holds for all $p \leqq \eta \equiv \exp(-(1 - \eta)/(2\eta))$ because the function $p \exp(q/(2p))$ decreases in p. (The number η is slightly larger than $\frac{1}{4}$.). $F(p, p) = G(p, p) = 0$, and $F(q, p) \geqq G(q, p)$ whenever $(q - p)/(2pq) + \log(p/q) \geqq 0$, which is true for $0 \leqq p \leqq \frac{1}{2}$ because the left-hand side is decreasing in $p < \frac{1}{2}$ and 0 at $p = \frac{1}{2}$.

The function $F(a, p) - G(a, p)$ is convex and increasing in a for $p \leqq a \leqq q$, and concave for $q \leqq a \leqq 1$. From the above information when $a = p$, q, or 1, we conclude that if $p \leqq a \leqq q$ then both $F(a, p) \geqq G(a, p)$ and $\partial F/\partial a \geqq \partial G/\partial a$, while if $q \leqq a \leqq 1$ and $p \leqq \frac{1}{4}$, then $F(a, p) \geqq G(a, p)$.

It is worth noting that $F(a, p) \geqq G(a, p)$ does not hold for $a < p$ or if $G(a, p)$ is replaced by $-(a - p)^2/(2p)$, so that (v) cannot be extended to $k > np$, and $\bar{B}(k, n, p) \geqq 1 - \Phi((k - np)/(npq)^{\frac{1}{2}})$ will not hold for $k < np$.

We discuss applications of our inequalities in Section 4, especially to the construction of statistical tests conservative with respect to type II errors.

2. Upper tail probabilities. The main result of this section is

THEOREM 2.1. *If* $0 \leqq p \leqq \frac{1}{4}$, $np \leqq k \leqq n$, *or* $np \leqq k \leqq n(1 - p)$, *then*

$$(*) \qquad \bar{B}(k, n, p) \geqq 1 - \Phi((k - np)/(npq)^{\frac{1}{2}}), \qquad \textit{where} \quad q = 1 - p .$$

We break the proof of this theorem into cases.

CASE 1. $p \leqq \frac{1}{4}$, $k \geqq nq$. For fixed k, n, the difference $\bar{B}(k, n, p) - 1 + \Phi((k - np)/(npq)^{\frac{1}{2}})$ of binominal and normal tails goes to 0 as $p \downarrow 0$, so to prove Case 1 we show that the derivative of the difference is nonnegative for $(n - k)/n \leqq p \leqq \frac{1}{4}$. Direct evaluation reveals this derivative as nonnegative if and only if

$$G(k, n, p) = [1 - (k - np)/(2kq)]^{-1} b(k, n, p)(npq)^{\frac{1}{2}}/\phi((k - np)/(npq)^{\frac{1}{2}})$$

is at least 1, where ϕ is the normal density. Now $G(k, n, p) = [G(k, n, p)/G(k, n, k/n)]\, G(k, n, k/n)$. If $k < n$, then direct evaluation of the term in square brackets shows it to be of the form $(1 - (k - np)/(2kq))^{-1} \exp(M(n, p, k/n))$, where for $p \leqq u < 1$ we define $M(n, p, u) = (nu + \frac{1}{2}) \log(p/u) + (n - nu + \frac{1}{2}) \log(q/(1 - u)) + n(u - p)^2/(2pq) = \frac{1}{2} \log(pq/(u(1 - u))) + n(F(u, p) - G(u, p))$. The functions $F(u, p)$, $G(u, p)$ are as in the introduction, where it was shown that for $p \leqq u \leqq 1$, $F(u, p) \geqq G(u, p)$. Since for $u \geqq q$, $\log(pq/(u(1 - u))) \geqq 0$, it follows that $M(n, p, u) \geqq 0$, and in particular for $nq \leqq k < n$, $M(n, p, k/n) \geqq 0$.

LEMMA 2.1. *For all* $0 < k < n$, $\log G(k, n, k/n) \geqq -(8 \min(k, n - k))^{-1}$.

PROOF. $\log G(k, n, k/n) = \log[\binom{n}{k}(2\pi)^{\frac{1}{2}} k^{k+\frac{1}{2}}(n - k)^{n-k+\frac{1}{2}} n^{-n-\frac{1}{2}}]$. By the discussion of Stirling's formula for binomial coefficients in Feller (1968), pages 53–54, $\log G(k, n, k/n) > (12n)^{-1} - (12k)^{-1} - (12(n - k))^{-1}$. By symmetry in k and $n - k$, we may assume $k \leqq n/2$, in which case direct calculation yields $(8k)^{-1} + \log G(k, n, k/n) > (n^2 - nk - 2k^2)/(24nk(n - k)) \geqq 0$, proving the lemma.

To complete the proof of (*) in Case 1 for $k < n$, it suffices to show that $(1 - (k - np)/(2kq))^{-1} \exp(-(8(n - k))^{-1}) \geqq 1$, or even that $1 - (8(n - k))^{-1} \geqq 1 - (k - np)/(2kq)$, i.e., $kq \leqq 4(n - k)(k - np)$. When $k \geqq nq$, $p \leqq \frac{1}{4}$ this follows easily, as $4(n - k)(k - np) \geqq 4(n/2 - np) > k(2 - 4p) > qk$.

Finally, when $k = n$, (*) becomes $p^n \geqq 1 - \Phi((nq/p)^{\frac{1}{2}})$. But by the well-known inequality $1 - \Phi(x) < x^{-1}\phi(x)$ for $x > 0$, we have $1 - \Phi((nq/p)^{\frac{1}{2}}) < (2\pi nq/p)^{-\frac{1}{2}} \exp(-nq/(2p)) < \exp(-nq/(2p))$, so it suffices to prove $p \geqq \exp(-q/(2p))$ for $p \leqq \frac{1}{4}$ which was done in the introduction in the form $F(1, p) \geqq G(1, p)$).

CASE 2. $p < \frac{1}{2}$, $n/2 < k \leqq nq$. Again we prove (*) by showing $G(k, n, p) \geqq 1$ for $p \leqq 1 - k/n$ when $k > n/2$ is fixed. Differentiating $\log G(k, n, p)$ with respect to p, we see that $G(k, n, p)$ is decreasing in p whenever $p \leqq \frac{1}{2}$, $k \geqq np + \frac{1}{2}$, so in particular whenever $p \leqq \frac{1}{2}$, $k > n/2$. To prove this case, it suffices to check $G(k, n, 1 - k/n) \geqq 1$.

LEMMA 2.2. *If* $0 < p \leqq \frac{1}{2}$, *then* $p \exp((q - p)/(2pq))/q \geqq 1$.

PROOF. True for $p = \frac{1}{2}$, and the left-hand side decreases in p.

LEMMA 2.3. *If* $k > n/2$, *then* $G(k, n, 1 - k/n) \geqq 1$.

PROOF. By Lemma 2.1, $G(n - k, n, 1 - k/n) \geqq \exp(-(8(n - k))^{-1})$. But $G(k, n, 1 - k/n)/G(n - k, n, 1 - k/n) = (2k^2/(k^2 + (n - k^2))) \cdot [((n - k)/k) \exp(n(2k - n)/(2k(n - k)))]^{2k-n}$, and Lemma 2.2 with $p = (n - k)/n$ shows the square-bracketed term $\geqq 1$. Since also $2k > n$, the entire expression above is $\geqq (2k^2/(k^2 + (n - k)^2)) \cdot [((n - k)/k) \exp(n(2k - n)/(2k(n - k)))]$, and $G(k, n, 1 - k/n) \geqq (2k(n - k)/(k^2 + (n - k)^2)) \exp(n(2k - n)/(2k(n - k)) - 1/(8(n - k))) = [1 + (n - 2k)^2/(2k(n - k))]^{-1} \exp(n(2k - n)/(2k(n - k)) - 1/(8(n - k))) > \exp(-(n - 2k)^2/(2k(n - k)) + n(2k - n)/2k(n - k)) - 1/(8(n - k)))$, and $\log G(k, n, 1 - k/n) > (2k - n)/k - 1/(8(n - k))$.

Now since $2k - n \geqq 1$, if $k \leqq 8(n - k)$ then $\log G(k, n, 1 - k|n) \geqq 0$. If $9k > 8n$, then $(2k - n)/k > \frac{7}{8} > 1/(8(n - k))$, and the lemma and Case 2 are proved.

The remaining cases needed to prove Theorem 2.1 are

CASE 3a. $np \leqq k \leqq nq$, $p < \frac{1}{2}$, and
CASE 3b. $p = \frac{1}{2}$, $k = n/2$.

Case 3a follows immediately, for $k \geqq 2$, from Cases 1 and 2 and the local refinement (Theorem 3.1) to be proved in the next section. When $k = 1 \geqq np$, $\bar{B}(1, n, p) = 1 - q^n \geqq 1 - \exp(-np) \geqq 1 - \Phi((1 - np)/(np)^{\frac{1}{2}})$ by Bohman's inequality ((i) in the introduction), and since $1 \geqq np$ this is $\geqq 1 - \Phi((1 - np)/(npq)^{\frac{1}{2}})$, as in (v) in the introduction, proving (*). Case 3b is also contained in (v) of the introduction. All the cases of Theorem 2.1 are now proven.

The above results on binomial tails actually followed from information about $G(k, n, p)$, i.e., about individual binomial probabilities. We collect these local results in

THEOREM 2.2. (i) *If* $nq \geqq k > n/2$, *then* $G(k, n, p) > 1$.

(ii) *If* $1 - k/n \leqq p \leqq \frac{1}{4}$, *then* $G(k, n, p) > 1$.
(iii) *If* $p \leqq \frac{1}{4}$, $k \geqq n/3$, *and* $n \geqq 25$, *then* $G(k, n, p) > 1$.

In each case, $G(k, n, p) > 1$ is equivalent to the inequality $b(k, n, p) > (npq)^{-\frac{1}{2}}\phi((k-np)/(npq)^{\frac{1}{2}})(1 - (k - np)/(2kq))$.

PROOF. (i) and (ii) were respectively proved in Case 2 and Case 1 of Theorem 2.1 above. For (iii) we may assume $p < \frac{1}{3} \leqq k/n \leqq q$. Since $k - np > \frac{1}{2}$, $G(k, n, \cdot)$ is again decreasing, and it suffices to show $G(k, n, \frac{1}{4}) > 1$ for $k \leqq n/2$ (when $k \geqq 3n/4$ we are done by (ii), and when $\frac{1}{2} < k/n \leqq \frac{3}{4}$ (i) proves our claim).

Just as before, $G(k, n, \frac{1}{4}) = [G(k, n, \frac{1}{4})/G(k, n, k/n)]G(k, n, k/n)$, and the square-bracketed term is $(1 - (k - n/4)/(3k/2))^{-1} \exp(M(n, \frac{1}{4}, k/n))$, while for $k \leqq n/2$, $G(k, n, k/n) \geqq \exp(-(8k)^{-1})$ by Lemma 2.1.

Now $\exp(-(8k)^{-1}) > 1 - (8k)^{-1} \geqq 1 - (k - n/4)/(3k/2)$ because $k - n/4 > 1$ in case (iii), hence $G(k, n, \frac{1}{4}) > \exp(M(n, \frac{1}{4}, k/n))$. But differentiating twice with respect to u shows that $M(n, \frac{1}{4}, u)$ is convex on $[\frac{1}{4}, \frac{3}{4}]$, so it is enough to verify that $M(n, \frac{1}{4}, \frac{1}{3}) > 0$ and $\partial/\partial u M(n, \frac{1}{4}, \frac{1}{3}) > 0$. We have $M(n, \frac{1}{4}, \frac{1}{3}) = (n/3 + \frac{1}{2}) \log(\frac{3}{4}) + (2n/3 + \frac{1}{2}) \log(\frac{9}{8}) + n/54 > 0$ and $\partial/\partial u M(n, \frac{1}{4}, \frac{1}{3}) = n \log(\frac{2}{3}) + 4n/9 - \frac{3}{4} > 0$ for $n \geqq 25$, finishing the theorem.

3. Local refinements. We turn now to local theorems, refining Theorem 2.1 to an inequality for individual binomial probabilities in

THEOREM 3.1. *If* $np \leqq k \leqq n(1 - p)$ *and* $k \geqq 2$, *then*

$$(**) \qquad b(k, n, p) \geqq \Phi((k - np + 1)/(npq)^{\frac{1}{2}}) - \Phi((k - np)/(npq)^{\frac{1}{2}}) .$$

We remark that (**) holds if and only if $H(k, n, p) \equiv b(k, n, p)(npq)^{\frac{1}{2}}/\phi((k - np)/(npq)^{\frac{1}{2}}) \geqq \int_{k-np}^{k-np+1} \exp((k-np)^2/(2npq) - u^2/(2npq))\, du = \int_0^1 \exp(-(2(k-np)t + t^2)/(2npq))\, dt$, and we call this last expression $D(k, n, p)$. Differentiation with respect to p shows in the range $0 \leqq p \leqq \frac{1}{2}$, $k \geqq np$ that $D(k, n, \cdot)$ increases, while $H(k, n, p)$ increases in p precisely when $k \leqq np + (npq)^{\frac{1}{2}}$.

For fixed k, n with $k \leqq n$, we define $p_0(k, n)$ as the root $\leqq k/n$ of $(k - np)^2 = np(1 - p)$, so that $p_0(k, n) = (k + \frac{1}{2} - (k + \frac{1}{4} - k^2/n)^{\frac{1}{2}})/(n + 1)$ and $k - np_0 = (np_0 q_0)^{\frac{1}{2}}$. Moreover, since $np + (npq)^{\frac{1}{2}}$ increases in p on $[0, \frac{1}{2}]$, whenever $np \leqq k \leqq n/2$ the inequalities $p \leqq p_0(k, n)$ and $k \geqq np + (npq)^{\frac{1}{2}}$ are equivalent. Therefore, by the previous paragraph, if $k \leqq n/2$ and (**) holds for $p = p_0(k, n)$, then (**) holds as well for all $0 \leqq p \leqq p_0(k, n)$. We proceed first to prove Theorem 3.1 for $p \geqq p_0(k, n)$ in the special case $k \leqq n/2$.

For the next lemmas, we introduce the notations $x = k - np$, $s = (npq)^{\frac{1}{2}}$, in order to express $\log H(k, n, p)$ and $\log D(k, n, p)$ as power series in $k - np$.

LEMMA 3.1. *If* $0 < p \leqq \frac{1}{2}$, $np \leqq k \leqq n/2$, *and* $p \geqq p_0(k, n)$, *then* $\log H(k, n, p) \geqq -x/(2kq) - (8k)^{-1} - x^2/(4k^2)$.

PROOF. By Lemma 2.1, $\log H(k, n, p) = \log[b(k, n, p)s^{-1}/\phi(x/s)] \geqq (x/s)^2/2 + (k + \frac{1}{2}) \log(np/k) + (n - k + \frac{1}{2}) \log(nq/(n - k)) - (8k)^{-1}$. Writing $\log(np/k) = \log(1 - x/k)$ and $\log(nq/(n - k)) = -\log(1 - x/(nq))$, expanding in powers of

$x \geqq 0$ and collecting terms, we have

$$\begin{aligned}\log H(k, n, p) \geqq & -(8k)^{-1} + (x/s)^2/2 + x^2/(2nq) + x^2/(2nq)^2 - x^2/(2k) \\ & - x^2/(2k)^2 - x/(2k) - x^2/(nq) + x/(2nq) \\ & + \textstyle\sum_{r=3}^{\infty} x^r[(nq + \frac{1}{2})/(r(nq)^r) - (k + \frac{1}{2})/(rk^r) \\ & - ((r-1)(nq)^{r-1})^{-1}] .\end{aligned}$$

The terms of degree 1 and 2 in x are easily seen to be $\geqq xp/(kq) - x/(2kq) - x^2/(4k^2) + x^3/(2nkp)$, while $\sum_{r=3}^{\infty} \geqq -\sum_{r=3}^{\infty} x^r[(k+\frac{1}{2})/(3k^r) + (2(nq)^2)^{-1}] = -(k+\frac{1}{2})x^3np/(3k^4) - (n-k)x^3/(2(nq)^3)$.

But $x \geqq 0$, and since $\frac{1}{2} \geqq p \geqq p_0(k, n)$, also $x^2 \leqq npq$. Therefore $x^3/(2nkp) \geqq (k + \frac{1}{2})x^3np/(3k^4)$ for $k \geqq 1$, and $xp/(kq) \geqq (n-k)x^3/(2(nq)^3)$. Altogether $\log H(k, n, p) \geqq -(8k)^{-1} - x/(2kq) - x^2/(4k^2)$, and the lemma is proved.

We observe that for $k \leqq n/2$, $p_0(k, n)$ is an increasing function of k, and $np_0(1 - p_0) = np_0q_0$ is an increasing function of both n and k. Hence as k and n range over positive values with $2 \leqq k \leqq n/2$, $np_0q_0 \geqq 4p_0(2, 4)(1-p_0(2, 4)) = .8$. So for all $\frac{1}{2} \geqq p \geqq p_0(k, n)$, $npq \geqq .8$.

LEMMA 3.2. *If* $2 \leqq k \leqq n/2$, $np \leqq k \leqq np + (npq)^{\frac{1}{2}}$, *then*

$$\log D(k, n, p) < -(6npq)^{-1} - x/(2npq) + (25(npq)^2)^{-1} + 11x^2/(108(npq)^2) .$$

PROOF. Using the Cauchy–Schwarz inequality on the integral defining $D)k, n, p)$, we find

$$\begin{aligned}\log D(k, n, p) &\leqq (\tfrac{1}{2}) \log \textstyle\int_0^1 \exp(-t^2/(npq))\, dt + (\tfrac{1}{2}) \log \int_0^1 \exp(-2tx/(npq))\, dt \\ &= -x/(2npq) + (\tfrac{1}{2}) \log \textstyle\int_0^1 \exp(-t^2/(npq))\, dt \\ &\quad + (\tfrac{1}{2}) \log \textstyle\int_0^1 \exp((1 - 2t)x/(npq))\, dt .\end{aligned}$$

Our hypotheses imply $p \geqq p_0(k, n)$, so $npq \geqq .8$ and $\int_0^1 \exp(-t^2/(npq))\, dt < 1 - (3npq)^{-1} + (10(npq)^2)^{-1} < 1$. Taking logarithms and again using $npq \geqq .8$, we find

$$\begin{aligned}&\log \textstyle\int_0^1 \exp(-t^2/(npq))\, dt \\ &\qquad < -(3npq)^{-1} + (10(npq)^2)^{-1} - ((3npq)^{-1} - (10(npq)^2)^{-1})^2/2 \\ &\qquad < -(3npq)^{-1} + 2/(25(npq)^2) .\end{aligned}$$

Now

$$\begin{aligned}\textstyle\int_0^1 \exp((1 - 2t)x/(npq))\, dt &= \textstyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \exp(-2ux/(npq))\, du \\ &< 2 \textstyle\int_0^{\frac{1}{2}} \exp(2(ux/(npq))^2)\, du\end{aligned}$$

by the inequality $\cosh(t) \leqq \exp(t^2/2)$, valid for all real t. Expanding in a series for x and using $x^2 \leqq npq$, $npq \geqq .8$, shows this expression $< 1 + x^2/(6(npq)^2) + (x/(npq))^4 \cdot (\frac{1}{40} + \frac{1}{280} + (\frac{1}{40})(\frac{2}{13})^2 + \cdots) < 1 + (\frac{22}{108})x^2/(npq)^2$. Finally, $(\frac{1}{2}) \log \int_0^1 \exp((1 - 2t)x/(npq))\, dt < (\frac{11}{108})x^2/(npq)^2$, finishing the lemma.

To conclude the proof of (**) in case $2 \leqq k \leqq n/2$, $k/n \geqq p \geqq p_0(k, n)$, we combine Lemmas 3.1 and 3.2 to find

$$\log H(k, n, p) - \log D(k, n, p) > [(2nkpq)^{-1} - (4k^2)^{-1} - 11/(108(npq)^2]x^2 + [(6npq)^{-1} - (8k)^{-1} - (25(npq)^2)^{-1}] .$$

Since $k \geqq 2$ and $npq \geqq .8$, $(8k)^{-1} \leqq \frac{1}{16} < (25(npq) - 6)/(150(npq)^2) = (6npq)^{-1} - (25(npq)^2)^{-1}$. Also $npq \leqq k \leqq np + (npq)^{\frac{1}{2}} = npq(q^{-1} + (npq)^{-\frac{1}{2}}) < 3.2npq$, and the quadratic expression $k/(2npq) - \frac{1}{4} - (\frac{11}{108})(k/(npq))^2$ is positive for $k/(npq)$ between the roots $(\frac{108}{44})(1 \pm (1 - \frac{44}{108})^{\frac{1}{2}})$, which are .56 and 4.3. Hence $\log H(k, n, p) - \log D(k, n, p) > 0$, and (**) holds.

As remarked above, (**) for $\frac{1}{2} \geqq p \geqq p_0(k, n)$ also implies (**) for $p \leqq p_0(k, n)$, so all that remains in Theorem 3.1 is to remove the restriction $k \leqq n/2$. But when $k \leqq n/2$,

$$\begin{aligned} \log H(n - k, n, p) &- \log H(k, n, p) \\ &= (n - 2k) \log (p/q) + ((n - k - np)^2 - (k - np)^2)/(2npq) \\ &= (n - 2k)[\log (p/q) + (q - p)/(2pq)] \geqq 0 \end{aligned}$$

by Lemma 2.2. Also $D(k, n, p)$ decreases in k by inspection, hence $D(n - k, n, p) < D(k, n, p)$. Therefore (**) for $k \leqq n/2$ immediately implies (**) for $k \geqq n/2$.

Our final result strengthens inequality (**) in certain cases by giving lower bounds for the error.

THEOREM 3.2. *If* $p \leqq \frac{1}{4}$ *and either* (i) $k \geqq nq$ *or* (ii) $k \geqq n/3$, $n \geqq 27$, *then*

$$\begin{aligned} b(k, n, p) - \Phi((k - np + 1)/(npq)^{\frac{1}{2}}) + \Phi((k - np)/(npq)^{\frac{1}{2}}) \\ > \gamma(npq)^{-\frac{1}{2}}\phi((k - np)/(npq)^{\frac{1}{2}}) , \end{aligned}$$

where

$$\gamma = \alpha \equiv ((k - np)^2/(2npq))(k^{-1} - (3npq)^{-1})$$

in case (ii), *and* $\gamma = \max (\alpha, 0.16)$ *in case* (i).

PROOF. By Theorem 2.2, $G(k, n, p) > 1$, i.e., $H(k, n, p) > 1 - (k - np)/(2kq)$. In either of our cases $k > n/4 + (3n/16)^{\frac{1}{2}} \geqq np + (npq)^{\frac{1}{2}}$, hence $H(k, n, p)$ decreases in p, and $H(k, n, p) \geqq H(k, n, \frac{1}{4}) > 1 - (k - n/4)/(3k/2) = (k + n/2)/(3k) \geqq \frac{1}{2}$. Now $D(k, n, p)$ is $< \int_0^1 \exp(-t(k - np)/(npq))\, dt$, increases in p, and decreases in k. So for $k \geqq nq$, $D(k, n, p) \leqq D(3n/4, n, \frac{1}{4}) = \int_0^1 \exp(-8t/3)\, dt = (\frac{3}{8})(1 - \exp(-\frac{8}{3})) < .34$, and in case (i) $H(k, n, p) - D(k, n, p) > \frac{1}{2} - .34 = .16$. In either case $H(k, n, p) - D(k, n, p) > 1 - (k - np)/(2kq) - (npq/(k - np)) \cdot (1 - \exp(-(k - np)/(npq))) > 1 - (k - np)/(2kq) - 1 + (k - np)/(2npq) - (k - np)^2/(6(npq)^2) = (k - np)^2(2npq)^{-1}(k^{-1} - (3npq)^{-1})$. Therefore $H(k, n, p) - D(k, n, p) > \gamma$, and multiplying through by $(npq)^{-\frac{1}{2}}\phi(k - np)/(npq)^{\frac{1}{2}})$ proves the theorem.

4. Applications. We conclude with a short discussion of statistical applications of inequalities involving tail and individual binomial probabilities. Foremost among these is the recognition of systematic errors in the common approximations used in tests of significance. For example, suppose one wanted to test the null hypothesis H_0 that event E occurs with probability p_1 against the alternative H_1 that it occurs with probability $> p_1$. Given the results in n independent trials of whether E occurred, the size α of the (likelihood ratio) test of H_0 versus H_1 is $\bar{B}(k, n, p_1)$, where we accept H_0 iff E has occurred $< k$ times. Since this and any other reasonable test will reject H_0 for large n if the number of occurrences is $> n(1 - p_1)$, where $p_1 < \frac{1}{2}$, we apply Theorem 3.14 (or, for small samples, assume $p_1 < \frac{1}{4}$ and apply Theorem 3.12) to find $\alpha = \bar{B}(k, n, p_1) \geqq 1 - \Phi((k - np_1)/(np_1 q_1)^{\frac{1}{2}})$. So in this case the binomial tail test is conservative compared to the normal tail test. Alternatively, the normal-tail test of H_0 versus H_1 with size α, which is precisely the chi-squared test of H_0 at level α, is conservative with respect to errors of the second kind.

Binomially distributed statistics arise as well in more general tests of goodness of fit, and the following example provided one motivation for the work in Sections 2 and 3. We suppose the sample space partitioned into mutually exclusive events $E_1, \cdots, E_m$, and we wish to test the results of N independent trials against the null hypothesis that event E_i occurs with probability p_i, $i = 1, 2, \cdots, m$, $p_1 + \cdots + p_m = 1$. The N trials will give r_i occurrences of E_i, where $r_1 + \cdots + r_m = N$, and we define $Z = \min_j \{B(r_j, N, p_j)\}$, where $B(r_j, N, p_j) \equiv 1 - \bar{B}(r_j + 1, N, p_j)$. Then the n-min test (a "slippage" test) at level α consists in finding γ such that the probability $\Pr\{Z \leqq \gamma\} \doteq \alpha$ and in rejecting the null-hypothetical distribution if in a given realization of N trials it happens that $\min\{B(r_j, N, p_j)\} \leqq \gamma$. This test is defined and implemented in Dudley, Perkins, and Giné (1975) and developed in unpublished preprints of Dudley. We remark that for large sample size N, it will be no loss in generality to assume that the r_j which minimizes $B(r_j, N, p_j)$ is $\leqq Np_j - 1$. Then by inequalities (i), (ii), (v) above,

$$\alpha \geqq \Pr\{\min_j \textstyle\sum_{i=0}^{r_j} p(i, Np_j) \leqq \gamma\} \geqq \Pr\{\min_j \Phi((r_j - Np_j)/(Np_j)^{\frac{1}{2}}) \leqq \gamma\}\,.$$

But the $(r_j - Np_j)/(Np_j)^{\frac{1}{2}}$ are asymptotically jointly normal with means 0 and variances $1 - p_j$ under the hypothetical distribution, and covariance $E\{(r_j - Np_j)(r_i - Np_i)/(N^2 p_i p_j)^{\frac{1}{2}}\} = -p_i p_j (1 - p_i)(1 - p_j)$, so we may calculate the distribution function of $Z^* = \min_j ((r_j - Np_j)/(Np_j)^{\frac{1}{2}})$ in terms of incomplete gamma integrals and use Z^* as test-statistic in a conservative approximation to n-min with respect to type II errors.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
COLLEGE PARK, MARYLAND 20742