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A NOTE ON PLANAR BROWNIAN MOTION

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The well-known fact that planar Brownian motion does not hit points is proved anew by a stopping-time argument.

Let $Z(t)$ be a Brownian motion in the plane. A familiar and elegant application of potential theory shows that for any initial point z_0 and any $z_1 \neq z_0$

$$P_{z_0}\{Z(t) = z_1 \text{ for some } t \geqq 0\} = 0 . \tag{1}$$

Here we give another proof of (1) using stopping times.

Without loss of generality we can take $z_0 = (1, 0)$, $z_1 = (0, 0)$. Let $U_1 < U_2 < U_3 < \cdots$ be, in alternation, the successive times when $Z(t)$ first hits the y-axis, then the x-axis, then the y-axis, then $\cdots$. It will be seen below that each $Z(U_n)$ has an absolutely continuous distribution on the appropriate axis, so that with probability one none of them coincides with the origin, and therefore the sequence $\{U_n\}$ is actually well defined. Then in order to establish (1) it will be enough to prove

$$U_n \to \infty \quad P_{z_0}\text{-almost surely.} \tag{2}$$

We nead some standard facts about one-dimensional Brownian motions. Let $X(t)$ be one, starting at zero, and let $b \neq 0$; let T be the first time that $X(t) = b$. Then T has a stable distribution of index $\frac{1}{2}$; more precisely, $T \sim b^2 S$, where $E(\exp(-rS)) = \exp(-(2r)^{\frac{1}{2}})$, $r > 0$, and the sign "$\sim$" means "is distributed as." Next, let $Y(t)$ be another linear Brownian motion, independent of $X(t)$. Then $Y(T) \sim |b|C$, where C is Cauchy, $E(\exp(iuC)) = \exp(-|u|)$. (Potential theorists will recognize this as giving the Poisson integral for the Dirichlet problem in a half-plane.) In what follows we will introduce a mutually independent sequence of pairs (S_n, C_n), distributed like $(T, Y(T))$ above when $b = 1$.

Returning to the two-dimensional case we see at once that $U_1 \sim S_1$ and $Z(U_1) = (0, Y_1)$, say, with $Y_1 \sim C_1$; thus $Z(U_1) \neq (0, 0)$ almost surely. Next, by the strong Markov property, we have $U_2 - U_1 \sim C_1^2 S_2$, and $Z(U_2) = (X_2, 0)$, with $X_2 \sim |C_1| C_2$. Clearly (S_2, C_2) is independent of (S_1, C_1). Continuing in this fashion we find that $U_{n+1} - U_n \sim (C_1 C_2 \cdots C_n)^2 S_{n+1}$, and that on the appropriate axis $Z(U_{n+1})$ has the coordinate $|C_1 C_2 \cdots C_n| C_{n+1}$; the pair (S_{n+1}, C_{n+1}) is independent of the earlier ones, and in particular, $C_1, C_2, \cdots, C_n$ are independent.

In order to prove (2) we must show that $\sum (C_1 C_2 \cdots C_n)^2 S_{n+1}$ is divergent with probability one. This is true with room to spare, since, as we now prove, the terms of the series almost surely fail to approach zero. In fact, since

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$E(\log|C|) = 0$ and $\mathrm{Var}(\log|C|)$ is finite we can apply the law of the iterated logarithm to the partial sums of the sequence $\{\log|C_n|\}$. A weak consequence is that $C_1 C_2 \cdots C_n \geqq \exp n^{\frac{1}{3}}$ i.o. almost surely. Since S has a bounded density function the series $\sum P\{S_{n+1} < \exp(-(n)^{\frac{1}{3}})\}$ converges, so that $S_{n+1} \geqq \exp(-(n)^{\frac{1}{3}})$ for all sufficiently large n, with probability 1. Hence lim sup $(C_1 C_2 \cdots C_n)^2 S_{n+1}$ is infinite, almost surely. This completes the proof.

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