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CONVERGENCE RATES AND r-QUICK VERSIONS OF THE STRONG LAW FOR STATIONARY MIXING SEQUENCES[1]

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In this paper we prove a theorem on the convergence rate in the Marcinkiewicz-Zygmund strong law for stationary mixing sequences. Our result gives the r-quick strong law and the finiteness of moments of the largest excess of boundary crossings for such sequences.

1. Introduction. In [1], Baum and Katz have proved the following well-known theorem for partial sums S_n of i.i.d. random variables $X_1, X_2, \cdots$: Let $\alpha > \frac{1}{2}$, $p\alpha > 1$, and assume that $EX_1 = 0$ if $\alpha \leqq 1$. Then

$$(1.1) \qquad E|X_1|^p < \infty \Leftrightarrow \sum_1^\infty n^{p\alpha-2} P[\sup_{j \geqq n} j^{-\alpha}|S_j| \geqq \varepsilon] < \infty \quad \text{for all} \quad \varepsilon > 0$$
$$\Leftrightarrow \sum_1^\infty n^{p\alpha-2} P[|S_n| \geqq \varepsilon n^\alpha] < \infty \quad \text{for some} \quad \varepsilon > 0\,.$$

This result is related to the convergence rate in the Marcinkiewicz-Zygmund strong law of large numbers and generalizes an earlier theorem of Hsu and Robbins [5] and Erdös [4] who considered the special case $\alpha = 1$ and $p = 2$.

Series of the type considered in (1.1) are related to the complete convergence criterion for sample sums of Hsu and Robbins [5] and the r-quick convergence criterion of Strassen [15] and Lai [11]. We say that a sequence Z_n of random variables converges to 0 r-quickly ($r > 0$) if

$$(1.2) \qquad E(\sup\{n \geqq 1 : |Z_n| \geqq \varepsilon\})^r < \infty \quad \text{for all} \quad \varepsilon > 0 \quad (\sup \varnothing = 0)\,.$$

Obviously $Z_n \to 0$ r-quickly for some $r > 0$ implies that $Z_n \to 0$ a.s. The equivalence (1.1) leads to the following r-quick version of the Marcinkiewicz-Zygmund strong law (cf. [3] and [11]):

$$(1.3) \qquad E|X_1|^p < \infty \Leftrightarrow n^{-\alpha}S_n \to 0 \quad (p\alpha - 1)\text{-quickly}$$
$$\Leftrightarrow n^{-\alpha}X_n \to 0 \quad (p\alpha - 1)\text{-quickly}\,.$$

The above strengthened form of the law of large numbers has useful applications in the field of sequential analysis in statistics (cf. [9], [10], [11]). In view of such applications, it is desirable to extend (1.1) and (1.3) from the i.i.d. case to other dependent cases. Motivated by applications to sequential analysis of time series, we shall study in this paper the important case where $X_1, X_2, \cdots$ form a stationary sequence satisfying certain mixing conditions. We are able to

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prove an analogue of (1.1) and thereby extend the r-quick strong law (1.3) to such sequences. Our main results are stated in Theorem 1 of Section 2 and Theorem 2 of Section 4. It is interesting to note that in the absence of mixing conditions, even when the zero-mean stationary sequence X_n is assumed to be ergodic and uniformly bounded, the ordinary Marcinkiewicz–Zygmund strong law need not even be true and one may have $P[\lim_{n\to\infty} n^{-\alpha}S_n = 0] = 0$ for all $\alpha < 1$ in spite of the uniform boundedness of the random variables (cf. [7, pages 135–137]). Some applications of the r-quick strong law for stationary mixing sequences will be given in Section 5.

2. The moment criterion and other equivalent statements of the r-quick version of the strong law for stationary mixing sequences. Let $X_1, X_2, \cdots$ be a stationary sequence of random variables. We shall assume that the sequence X_n satisfies the classical mixing conditions of Ibragimov (i.e., φ-mixing) or of Rosenblatt (i.e., strong mixing). The sequence X_n is said to be φ-mixing if φ is a nonnegative function of the positive integers such that $\lim_{n\to\infty} \varphi(n) = 0$ and for each k and n,

$$|P(E_1 \cap E_2) - P(E_1)P(E_2)| \leqq \varphi(n)P(E_1) \tag{2.1}$$

for all $E_1 \in \mathscr{B}(X_1, \cdots, X_k)$ and $E_2 \in \mathscr{B}(X_{k+n}, X_{k+n+1}, \cdots)$. Without loss of generality (cf. [2], page 166), we can assume that

$$1 \geqq \varphi(1) \geqq \varphi(2) \geqq \cdots . \tag{2.2}$$

The sequence X_n is said to be strong mixing if

$$\sup |P(E_1 \cap E_2) - P(E_1)P(E_2)| = \rho(n) \downarrow 0 \quad \text{as} \quad n \to \infty , \tag{2.3}$$

where the supremum in (2.3) is taken over all $E_1 \in \mathscr{B}(X_1, \cdots, X_k)$, $E_2 \in \mathscr{B}(X_{k+n}, X_{k+n+1}, \cdots)$ and over all $k = 1, 2, \cdots$. The function $\rho(n)$ in (2.3) is called the mixing coefficient.

Theorem 1 below gives the analogues of (1.1) and (1.3) for φ-mixing and strong mixing sequences. As we are primarily interested in events of large deviations, the mixing condition (2.1) or (2.3) can help estimate their joint probabilities only when such events are remotely separated. To handle events of large deviations which are not too far apart, we need to reinforce the mixing condition by requiring that the bivariate probabilities of random variables which are sufficiently far apart decrease faster than the univariate tail probabilities, i.e.,

(2.4) There exists $\beta > 1$ and a positive integer m such that as $x \to \infty$,

$$\sup_{i>m} P[|X_1| > x, |X_i| > x] = O(P^\beta[|X_1| > x]) .$$

Condition (2.4) is obviously satisfied with $\beta = 2$ when the random variables X_n are independent, or more generally are m-dependent, and so (2.4) can be regarded as a property of a sequence which is asymptotically independent in some sense. This condition, however, is also satisfied by sequences which are not asymptotically independent. For example, if X_n is a stationary Gaussian sequence with zero mean and unit variance and $\limsup_{n\to\infty} |\mathrm{Cov}\,(X_1, X_n)| < 1$, then it is easy to see that (2.4) holds. We shall discuss more about this condition in Section 4 and also drop it by imposing instead a moment condition (on X_1) which is just slightly stronger than the weakest possible moment condition under (2.4). When α is sufficiently large, the condition (2.4) alone in fact suffices to give the desired conclusions without the mixing condition (2.1) or (2.3).

THEOREM 1. *Let $X_1, X_2, \cdots$ be a stationary sequence such that* (2.4) *holds. Let* $S_n = X_1 + \cdots + X_n (S_0 = X_0 = 0)$.

(i) *Suppose $p > 0$ and $p\alpha \geqq \max\{(\beta p + 1), \beta\}/(\beta - 1)$, where β is as given in* (2.4). *Then $\alpha > 1$ and the following statements are equivalent*:

(2.5) $$E|X_1|^p < \infty\,;$$

(2.6) $$\textstyle\sum_1^\infty n^{p\alpha-2} P[\max_{j\leqq n} |S_j| \geqq \varepsilon n^\alpha] < \infty \quad \text{for all (or equivalently for some)} \quad \varepsilon > 0\,;$$

(2.7) $$\textstyle\sum_1^\infty n^{p\alpha-2} P[\sup_{j\geqq n} j^{-\alpha}|S_j| \geqq \varepsilon] < \infty \quad \text{for all (or equivalently for some)} \quad \varepsilon > 0\,;$$

(2.8) $$n^{-\alpha}S_n \to 0 \quad (p\alpha - 1)\text{-quickly};$$

(2.9) $$n^{-\alpha}X_n \to 0 \quad (p\alpha - 1)\text{-quickly};$$

(2.10) $$E\{\sup_{n\geqq 0} (|S_n| - \varepsilon n^\alpha)\}^{(p\alpha-1)/\alpha} < \infty \quad \text{for all (or equivalently for some)} \quad \varepsilon > 0\,;$$

(2.11) $$E\{\sup_{n\geqq 0} (|X_n| - \varepsilon n^\alpha)\}^{(p\alpha-1)/\alpha} < \infty \quad \text{for all (or equivalently for some)} \quad \varepsilon > 0\,;$$

(2.12) $$\textstyle\sum_1^\infty n^{p\alpha-2} P[\sup_{j\geqq n} j^{-\alpha}|X_j| \geqq \varepsilon] < \infty \quad \text{for all (or equivalently for some)} \quad \varepsilon > 0\,;$$

(2.13) $$\textstyle\sum_1^\infty n^{p\alpha-2} P[\max_{j\leqq n} |X_j| \geqq \varepsilon n^\alpha] < \infty \quad \text{for all (or equivalently for some)} \quad \varepsilon > 0\,.$$

(ii) *Suppose $p \geqq 2$, $\alpha > \frac{1}{2}$ and $p\alpha \leqq (\beta p + 1)/(\beta - 1)$. Assume further that the sequence X_n is φ-mixing such that φ satisfies*

(2.14) $$\varphi(n) = O(n^{-\theta}) \quad \text{as} \quad n \to \infty \quad \text{for some} \quad \theta > \max\{p/(p-1), (\beta p + 2)(p-2)/(2\alpha p(\beta - 1))\}\,.$$

Moreover, when $\alpha \leqq 1$, assume that $EX_1 = 0$. Then the statements (2.5), (2.6), (2.7), (2.8), (2.9), (2.10), (2.11), (2.12), *and* (2.13) *are still equivalent.*

(iii) *Suppose $p > 2$, $\alpha > \frac{1}{2}$ and $p\alpha \leqq (\beta p + 1)/(\beta - 1)$. Assume further that the sequence X_n is strong mixing with mixing coefficient $\rho(n)$ which satisfies*

(2.15) $$\rho(n) = O(n^{-\theta}) \quad \text{as} \quad n \to \infty \quad \text{for some} \quad \theta > \max\{p/(p-2), (\beta p + 2)/(\beta - 1), (p\alpha + 1)/(\beta - 1)\}\,.$$

Moreover, when $\alpha \leqq 1$, *assume that* $EX_1 = 0$. *Then the conclusion of* (ii) *still holds.*

3. **Proof of Theorem 1.** The most difficult part of the proof of Theorem 1 lies in showing (2.5) $\Rightarrow$ (2.6) under the assumptions of (ii) and (iii). This will be given in Lemmas 1 and 2. In the i.i.d. case, the proof of this implication by Baum and Katz is based on symmetrization and the truncation method of Erdös (cf. [1], [4], [8]) and effectively exploits the higher moments of the partial sums of the truncated random variables. For example, in Erdös's original proof for the case $\alpha = 1$ and $p = 2$, he computes $E\tilde{S}_n^4$, where $\tilde{S}_n$ is the nth partial sum of the appropriately truncated variables *with the truncation depending on n*. This leads to rather involved estimates for higher p's (cf. [8]), and carrying these delicate estimates over to the general mixing case of Theorem 1 is a formidable task. By a different approach based on a stopping time technique, Y. S. Chow and I have recently found a considerably simpler proof in [3] and our method does not involve any complicated moment estimates. Unforutnately our stopping time technique depends very heavily on the i.i.d. structure. While each of these approaches separately does not seem to work for the general mixing case, a combination of certain refinements of both methods works and it constitutes the main idea of the proof of Lemmas 1 and 2. The proof of (2.5) $\Rightarrow$ (2.6) under the assumption of Theorem 1 (i) is much easier and is given in Lemma 3.

LEMMA 1. *Under the assumptions of Theorem* 1 (ii), *if* (2.5) *holds, then* (2.6) *holds for all* $\varepsilon > 0$.

PROOF. Assume that (2.5) holds. Without loss of generality, we can assume that φ satisfies (2.2). We shall first consider the case $p > 2$. In view of (2.14), we can choose $1 > \delta > (p + 2)/(\beta p + 2)$ such that

$$\sum_1^\infty n^{(p-2)/2-1}\varphi([n^{\alpha(1-\delta)}]) < \infty . \tag{3.1}$$

We note that since $p\alpha \leqq (\beta p + 1)/(\beta - 1)$, $\alpha(1 - \delta) < 1$. Take a positive integer k such that

$$kp(\alpha - \tfrac{1}{2}) > p\alpha - 1 . \tag{3.2}$$

Let $\varepsilon > 0$. We shall now estimate $P[\max_{j\leqq n} |S_j| \geqq \varepsilon n^\alpha]$.

For fixed n, define $\tau = \inf\{j \geqq 1 : |S_j| \geqq \varepsilon n^\alpha/(2k)\}$. Let $S_{i,j} = X_{i+1} + \cdots + X_{i+j}$ denote the delayed sum. We note that for $i = 1, \cdots, n$, on the event $[\max_{j\leqq n} |S_j| \geqq \varepsilon n^\alpha] \cap [\tau = i]$, if $\max_{1\leqq j\leqq[n^{\alpha(1-\delta)}]} |S_{i-1,j}| \leqq \varepsilon n^\alpha/(2k)$, then since $\max_{j\leqq i-1} |S_j| \leqq \varepsilon n^\alpha/(2k)$, we must have $\max_{1\leqq j\leqq n} |S_{i-1+[n^{\alpha(1-\delta)}],j}| \geqq \varepsilon n^\alpha(1 - k^{-1})$. Hence

$$\begin{aligned} &P[\max_{j\leqq n} |S_j| \geqq \varepsilon n^\alpha] \\ &\quad\leqq \sum_{i=1}^n P[\tau = i, \max_{1\leqq j\leqq[n^{\alpha(1-\delta)}]} |S_{i-1,j}| > \varepsilon n^\alpha/(2k)] \\ &\qquad + \sum_{i=1}^n P[\tau = i, \max_{1\leqq j\leqq n} |S_{i-1+[n^{\alpha(1-\delta)}],j}| \geqq \varepsilon n^\alpha(1 - k^{-1})] . \end{aligned} \tag{3.3}$$

By the φ-mixing condition (2.1) and stationarity,

$$\sum_{i=1}^{n} P[\tau = i, \max_{1\leq j\leq n} |S_{i-1+[n^{\alpha(1-\delta)}],j}| \geqq \varepsilon n^{\alpha}(1-k^{-1})] \leqq \sum_{i=1}^{n} P[\tau = i]\{P[\max_{j\leq n} |S_j| \geqq \varepsilon n^{\alpha}(1-k^{-1})] + \varphi([n^{\alpha(1-\delta)}])\}. \tag{3.4}$$

Let $X_j' = X_j I_{[|X_j|<\varepsilon n^{\alpha}/(4km)]}$, where m is as given in (2.4), and let $S'_{i,j} = X'_{i+1} + \cdots + X'_{i+j}$, $S_j' = X_1' + \cdots + X_j'$ and $S_j'(m) = X_1' + X'_{m+1} + X'_{2m+1} + \cdots + X'_{(j-1)m+1}$. We note that since $\alpha(1-\delta) < 1$,

$$\begin{aligned}\sum_{i=1}^{n} P[\tau = i, \max_{1\leq j\leq [n^{\alpha(1-\delta)}]} |S_{i-1,j}| > \varepsilon n^{\alpha}/(2k)] &\leqq P[\max_{j\leq 2n} |X_j| \geqq \varepsilon n^{\alpha}/(4km)] \\ &\quad + \sum_{i=1}^{n} P[\max_{1\leq j\leq [n^{\alpha(1-\delta)}]} |S'_{i-1,j}| > \varepsilon n^{\alpha}/(2k)] \\ &\leqq 2nP[|X_1| \geqq \varepsilon n^{\alpha}/(4km)] \\ &\quad + nmP[\max_{j\leq [n^{\alpha(1-\delta)}]} |S_j'(m)| > \varepsilon n^{\alpha}/(2km)], \quad \text{by stationarity} \\ &= 2nP[|X_1| \geqq \varepsilon n^{\alpha}/(4km)] + O(n^{1+2\alpha(1-\delta)-\beta p\alpha\delta}).\end{aligned} \tag{3.5}$$

To see the last relation above, we note that in order that $\max_{j\leq [n^{\alpha(1-\delta)}]} |S_j'(m)| > \varepsilon n^{\alpha}/(2km)$, at least two of the summands X_i' must exceed $\varepsilon n^{\alpha\delta}/(4km)$ in absolute value since the absolute value of each summand X_i' is less than $\varepsilon n^{\alpha}/(4km)$ and there are only $[n^{\alpha(1-\delta)}]$ such summands. This type of agrument is due to Erdös [4]. Hence

$$\begin{aligned}P[\max_{j\leq [n^{\alpha(1-\delta)}]} |S_j'(m)| > \varepsilon n^{\alpha}/(2km)] &\leqq \binom{[n^{\alpha(1-\delta)}]}{2} \sup_{i>m} P[|X_1| > \varepsilon n^{\alpha\delta}/(4km), |X_i| > \varepsilon n^{\alpha\delta}/(4km)] \\ &= O(n^{2\alpha(1-\delta)} P^{\beta}[|X_1| > \varepsilon n^{\alpha\delta}/(4km)]) \quad \text{by (2.4)} \\ &= O(n^{2\alpha(1-\delta)-\beta p\alpha\delta}) \quad \text{by the Markov inequality.}\end{aligned} \tag{3.6}$$

From (3.3), (3.4) and (3.5), it follows that

$$\begin{aligned}P[\max_{j\leq n} |S_j| \geqq \varepsilon n^{\alpha}] &\leqq P[\tau \leqq n]\{P[\max_{j\leq n} |S_j| \geqq \varepsilon n^{\alpha}(1-k^{-1})] + \varphi([n^{\alpha(1-\delta)}])\} \\ &\quad + 2nP[|X_1| \geqq \varepsilon n^{\alpha}/(4km)] + O(n^{1+2\alpha(1-\delta)-\beta p\alpha\delta}).\end{aligned} \tag{3.7}$$

Repeating the same argument as before, we obtain that for $\nu = 1, \cdots, k-2$,

$$\begin{aligned}P[\max_{j\leq n} |S_j| \geqq \varepsilon n^{\alpha}(1-\nu k^{-1})] &\leqq P[\tau \leqq n]\{P[\max_{j\leq n} |S_j| \geqq \varepsilon n^{\alpha}(1-(\nu+1)k^{-1})] + \varphi([n^{\alpha(1-\delta)}])\} \\ &\quad + 2nP[|X_1| \geqq \varepsilon n^{\alpha}/(4km)] + O(n^{1+2\alpha(1-\delta)-\beta p\alpha\delta}).\end{aligned} \tag{3.8}$$

Since $P[\max_{j\leq n} |S_j| \geqq \varepsilon n^{\alpha}k^{-1}] \leqq P[\tau \leqq n]$, it follows from (3.7) and (3.8) that

$$\begin{aligned}P[\max_{j\leq n} |S_j| \geqq \varepsilon n^{\alpha}] &\leqq P^{k}[\tau \leqq n] + kP[\tau \leqq n]\varphi([n^{\alpha(1-\delta)}]) \\ &\quad + 2knP[|X_1| \geqq \varepsilon n^{\alpha}/(4km)] + O(n^{1+2\alpha(1-\delta)-\beta p\alpha\delta}).\end{aligned} \tag{3.9}$$

We can assume that $EX_1 = 0$ without loss of generality since $E|S_n| = o(n^{\alpha})$ when $\alpha > 1$ and by hypothesis $EX_1 = 0$ when $\alpha \leqq 1$. Since $|EX_1X_k| \leqq 2(E|X_1|^p)^{2/p}\{\varphi(k-1)\}^{(p-1)/p}$ for $k > 1$ (cf. [2], page 170), $\sum_{k=2}^{\infty} |EX_1X_k| < \infty$ by

(2.14), and so

(3.10) $$ES_n^2 \sim \sigma^2 n \quad \text{where} \quad \sigma^2 = EX_1^2 + 2\sum_{k=2}^\infty EX_1X_k < \infty$$

(cf. [2], page 172). By a theorem of Ibragimov ([6], page 361), the finiteness of $E|X_1|^p$ then implies that $E|S_n|^p \leqq Mn^{p/2}$ for some $M > 0$, and so by stationarity,

(3.11) $$E|S_{i,n}|^p \leqq Mn^{p/2} \quad \text{for all} \quad i = 1, 2, \cdots \quad \text{and} \quad n = 1, 2, \cdots.$$

Since $p > 2$, it follows from (3.11) and a theorem of Serfling (cf. [14], Corollary B1) that

(3.12) $$E\max_{j\leqq n}|S_j|^p = O(n^{p/2}).$$

Therefore by the Markov inequality,

(3.13) $$\begin{aligned} P[\tau \leqq n] &= P[\max_{j\leqq n}|S_j| \geqq \varepsilon n^\alpha/(2k)] \\ &\leqq (2k/(\varepsilon n^\alpha))^p E\max_{j\leqq n}|S_j|^p = O(n^{p/2-\alpha p}). \end{aligned}$$

In view of (3.2), it follows that $\sum_1^\infty n^{p\alpha-2}P^k[\tau \leqq n] < \infty$. Furthermore, by (3.1) and (3.13),

$$\sum_1^\infty n^{p\alpha-2}P[\tau \leqq n]\varphi([n^{\alpha(1-\delta)}]) < \infty.$$

Noting that $\delta > (p+2)/(\beta p+2)$, $\sum_1^\infty n^{(p\alpha-2)+(1+2\alpha(1-\delta)-\beta p\alpha\delta)} < \infty$. Since $E|X_1|^p < \infty$, $\sum_1^\infty n^{p\alpha-1}P[|X_1| \geqq \varepsilon n^\alpha/(4km)] < \infty$. Hence the inequality (3.9) implies that (2.6) holds.

We now consider the case $p = 2$. Let $1 > \delta > (p+2)/(\beta p+2)$ as before. By (2.14),

(3.14) $$\sum_1^\infty n^{-1}(\log n)^2\varphi([n^{\alpha(1-\delta)}]) < \infty.$$

By stationarity and a theorem of Serfling (cf. [14], Corollary A3.1), (3.10) implies that

(3.15) $$E\max_{j\leqq n}|S_j|^2 = O(n(\log n)^2).$$

The rest of the proof is then similar to that before.

LEMMA 2. *Under the assumptions of Theorem* 1 (iii), *if* (2.5) *holds, then* (2.6) *holds for all* $\varepsilon > 0$.

PROOF. Assume that (2.5) holds. To prove (2.6), we can assume as in Lemma 1 that $EX_1 = 0$ without loss of generality since $E|S_n| = o(n^\alpha)$ when $\alpha > 1$. We first note that

(3.16) $$ES_n^2 \sim \sigma^2 n \quad \text{where} \quad \sigma^2 = EX_1^2 + 2\sum_{k=2}^\infty EX_1X_k < \infty.$$

To see this, since $|EX_1X_k| \leqq 10(E|X_1|^p)^{2/p}\{\rho(k-1)\}^{(p-2)/p}$ for $k > 1$ (cf. [13], page 82) and $\rho(n) = O(n^{-\theta})$ with $\theta > p/(p-2)$ by (2.15), $\sum_{k=2}^\infty |EX_1X_k| < \infty$. Hence (3.16) follows (cf. [2], page 172).

We now modify the proof of Lemma 1 to the present strong mixing case. By (2.15), $\rho(n) = O(n^{-\theta})$ with $\theta > (\beta p+2)/(\beta-1)$ and therefore we can choose δ

such that $1 > \delta > (p + 2)/(\beta p + 2)$ and

$$\sum_1^\infty n^{p\alpha-1}\rho([n^{\alpha(1-\delta)}]) < \infty \,. \tag{3.17}$$

Since $p\alpha \leqq (\beta p + 1)/(\beta - 1)$, $\alpha(1 - \delta) < 1$. Take a positive integer k such that

$$k(2\alpha - 1) > p\alpha - 1 \,. \tag{3.18}$$

Let $\varepsilon > 0$ and define τ and $S_{i,j}$ as in the proof of Lemma 1. We note that relation (3.3) still holds, and in place of (3.4), we now have

$$\begin{aligned}\sum_{i=1}^n P[\tau = i,\ \max_{1\leqq j\leqq n} |S_{i-1+[n^{\alpha(1-\delta)}],j}| \geqq \varepsilon n^\alpha(1 - k^{-1})] \\ \leqq \sum_{i=1}^n P[\tau = i]P[\max_{1\leqq j\leqq n}|S_j| \geqq \varepsilon n^\alpha(1 - k^{-1})] + n\rho([n^{\alpha(1-\delta)}]) \,.\end{aligned} \tag{3.19}$$

Also relation (3.5) still holds. By (3.3), (3.5) and (3.19), we obtain in place of (3.7) the following inequality:

$$\begin{aligned}P[\max_{j\leqq n} |S_j| \geqq \varepsilon n^\alpha] \\ \leqq P[\tau \leqq n]P[\max_{j\leqq n} |S_j| \geqq \varepsilon n^\alpha(1 - k^{-1})] \\ + n\rho([n^{\alpha(1-\delta)}]) + 2nP[|X_1| \geqq \varepsilon n^\alpha/(4km)] + O(n^{1+2\alpha(1-\delta)-\beta p\alpha\delta}) \,.\end{aligned} \tag{3.20}$$

Repeating the same argument as in Lemma 1, this leads to

$$\begin{aligned}P[\max_{j\leqq n} |S_j| \geqq \varepsilon n^\alpha] \\ \leqq P^k[\tau \leqq n] + kn\rho([n^{\alpha(1-\delta)}]) \\ + 2knP[|X_1| \geqq \varepsilon n^\alpha/(4km)] + O(n^{1+2\alpha(1-\delta)-\beta p\alpha\delta}) \,.\end{aligned} \tag{3.21}$$

By stationarity and Serfling's theorem (cf. [14], Corollary A3.1), (3.16) implies that $E\max_{j\leqq n} S_j^2 = O(n(\log n)^2)$. Hence by the Chebyshev inequality,

$$\begin{aligned}P^k[\tau \leqq n] &= P^k[\max_{j\leqq n} |S_j| \geqq \varepsilon n^\alpha/(2k)] \\ &= O(\{n^{-2\alpha+1}(\log n)^2\}^k) \,.\end{aligned} \tag{3.22}$$

From (2.5), (3.17), (3.18), (3.21) and (3.22), (2.6) follows easily.

LEMMA 3. *Let $p > 0$, $\alpha > 1$. Let $X_1, X_2, \cdots$ be a stationary sequence such that* (2.4) *holds with β satisfying $p\alpha \geqq (\beta p + 1)/(\beta - 1)$. If* (2.5) *holds, then* (2.6) *holds for all $\varepsilon > 0$.*

PROOF. Let $\delta = (p\alpha + 1)/(\beta p\alpha)$. Since $p\alpha \geqq (\beta p + 1)/(\beta - 1)$, $(\alpha - 1)/\alpha \geqq \delta$. Let $X_j' = X_j I_{[|X_j|<\varepsilon n^\alpha/(2m)]}$, $S_j' = X_1' + \cdots + X_j'$ and $S_j'(m) = X_1' + X'_{m+1} + \cdots + X'_{(j-1)m+1}$, where m is as given by (2.4). By stationarity, we have

$$\begin{aligned}P[\max_{j\leqq n} |S_j| \geqq \varepsilon n^\alpha] \\ \leqq P[\max_{j\leqq n} |X_j| \geqq \varepsilon n^\alpha/(2m)] + P[\max_{j\leqq n} |S_j'| \geqq \varepsilon n^\alpha] \\ \leqq nP[|X_1| \geqq \varepsilon n^\alpha/(2m)] + mP[\max_{j\leqq n} |S_j'(m)| \geqq \varepsilon n^\alpha/m] \,.\end{aligned} \tag{3.23}$$

We now apply an Erdös-type argument as in (3.6). Since $|X_j'| < \varepsilon n^\alpha/(2m)$ and $n^{1+\alpha\delta} \leqq n^\alpha$, in order that $\max_{j\leqq n} |S_j'(m)| \geqq \varepsilon n^\alpha/m$, at least two of the summands

X_j' must exceed $\varepsilon n^{\alpha\delta}/(2m)$ in absolute value. Hence

$$\begin{aligned}(3.24)\qquad & P[\max_{j\leqq n}|S_j'(m)|\geqq \varepsilon n^{\alpha}/m] \\ & \leqq \binom{n}{2}\sup_{i>m} P[|X_1|>\varepsilon n^{\alpha\delta}/(2m),\ |X_i|>\varepsilon n^{\alpha\delta}/(2m)] \\ & = O(n^2P^{\beta}[|X_1|>\varepsilon n^{\alpha\delta}/(2m)]) \quad \text{by (2.4)} \\ & = O(n^{p\alpha\delta+1-p\alpha}P[|X_1|>\varepsilon n^{\alpha\delta}/(2m)])\,,\end{aligned}$$

since $P^{\beta-1}[|X_1|>\varepsilon n^{\alpha\delta}/(2m)] = O(n^{-(\beta-1)p\alpha\delta})$ by the Markov inequality and $\beta p\alpha\delta = p\alpha+1$. Obviously the finiteness of $E|X_1|^p$ is equivalent to $\sum_1^\infty n^{p\alpha-1}P[|X_1|>\varepsilon n^{\alpha}/(2m)]<\infty$ and also to $\sum_1^\infty n^{p\alpha\delta-1}P[|X_1|>\varepsilon n^{\alpha\delta}/(2m)]<\infty$. Therefore from (3.23) and (3.24), (2.6) follows.

LEMMA 4. *Let $Z_1, Z_2, \cdots$ be an arbitrary sequence of random variables and let $Z_0 = 0$. Then for any positive constants ε, p, α with $p\alpha>1$,*

$$\begin{aligned}(p\alpha-1)&\varepsilon^{(p\alpha-1)/\alpha}\textstyle\int_0^\infty t^{p\alpha-2}P[\sup_{k\geqq t}k^{-\alpha}|Z_k|\geqq 2\varepsilon]\,dt \\ &\leqq E\{\sup_{n\geqq 0}(|Z_n|-\varepsilon n^{\alpha})\}^{(p\alpha-1)/\alpha} \\ &\leqq (2^{(p\alpha-1)/\alpha}-1)^{-1}(p\alpha-1)\varepsilon^{(p\alpha-1)/\alpha}\textstyle\int_0^\infty t^{p\alpha-2}P[\max_{k\leqq t}|Z_k|\geqq \frac{1}{2}\varepsilon t^{\alpha}]\,dt\,;\end{aligned}$$

$$E(\sup\{n: |Z_n|\geqq \varepsilon n^{\alpha}\})^{p\alpha-1}\leqq (p\alpha-1)\textstyle\int_0^\infty t^{p\alpha-2}P[\sup_{k\geqq t}k^{-\alpha}|Z_k|\geqq\varepsilon]\,dt\,.$$

PROOF. See [3], page 63.

From Lemma 4, it follows that (2.6) $\Rightarrow$ (2.10) $\Rightarrow$ (2.7) $\Rightarrow$ (2.8) and (2.13) $\Rightarrow$ (2.11) $\Rightarrow$ (2.12) $\Rightarrow$ (2.9). Since $P[\max_{j\leqq n}|X_j|\geqq 2\varepsilon n^{\alpha}]\leqq P[\max_{j\leqq n}|S_j|\geqq \varepsilon n^{\alpha}]$ and $P[\sup_{j\geqq n+1} j^{-\alpha}|X_j|\geqq 2\varepsilon]\leqq P[\sup_{j\geqq n} j^{-\alpha}|S_j|\geqq\varepsilon]$, it is obvious that (2.6) $\Rightarrow$ (2.13), (2.7) $\Rightarrow$ (2.12) and (2.8) $\Rightarrow$ (2.9). Since (2.5) $\Rightarrow$ (2.6) by Lemmas 1, 2, and 3, it remains to show that (2.9) $\Rightarrow$ (2.13) $\Rightarrow$ (2.5). This is done in the following lemma.

LEMMA 5. *Let $X_1, X_2, \cdots$ be a stationary sequence. Let $\alpha>0$ and $p>1/\alpha$. Set $X_0=0$.*

(i) *For all $\varepsilon>0$, $E(\sup\{n: |X_n|\geqq\varepsilon(n/2)^{\alpha}\})^{p\alpha-1}\geqq(p\alpha-1)\int_0^\infty t^{p\alpha-2}P[\max_{j\leqq t}|X_j|\geqq \varepsilon t^{\alpha}]\,dt$.*

(ii) *If the sequence X_n is φ-mixing, then*

$$(3.25)\qquad \textstyle\sum_1^\infty n^{p\alpha-2}P[\max_{j\leqq n}|X_j|\geqq\varepsilon n^{\alpha}]<\infty \quad \text{for some}\quad \varepsilon>0 \Rightarrow E|X_1|^p<\infty\,.$$

(iii) *If (2.4) holds with $\beta/(\beta-1)\leqq p\alpha$, then the implication (3.25) still holds.*

(iv) *If (2.4) holds and the sequence X_n is strong mixing with mixing coefficient $\rho(n)=O(n^{-\theta})$ for some $\theta>\max\{1/(p\alpha-1), (p\alpha+1)/(\beta-1)\}$, where β is given by (2.4), then the implication (3.25) still holds.*

REMARK. In Theorem 1 (iii), since $\theta>\max\{p/(p-2), (p\alpha+1)/(\beta-1)\}$ and $p/(p-2)>p/(p-1/\alpha)>1/(p\alpha-1)$ for $\alpha>\frac{1}{2}$, θ also satisfies the assumptions of Lemma 5 (iv).

PROOF. To prove (i), letting $L = \sup\{n : |X_n| \geqq \varepsilon(n/2)^\alpha\}$, we have

$$\begin{aligned} EL^{p\alpha-1} &= (p\alpha - 1)\int_0^\infty t^{p\alpha-2}P[L \geqq t]\,dt \\ &= (p\alpha - 1)\int_0^\infty t^{p\alpha-2}P[|X_n| \geqq \varepsilon(n/2)^\alpha \text{ for some } n \geqq t]\,dt \\ &\geqq (p\alpha - 1)\int_0^\infty t^{p\alpha-2}P[\max_{t<n\leqq 2t}|X_n| \geqq \varepsilon t^\alpha]\,dt \\ &\geqq (p\alpha - 1)\int_0^\infty t^{p\alpha-2}P[\max_{n\leqq t}|X_n| \geqq \varepsilon t^\alpha]\,dt\,, \quad \text{by stationarity.} \end{aligned}$$

We now prove (ii). Assume that for some $\varepsilon > 0$,

$$\sum_1^\infty n^{p\alpha-2}P[\max_{j\leqq n}|X_j| \geqq \varepsilon(n/2)^\alpha] < \infty\,. \tag{3.26}$$

Then as $n \to \infty$,

$$\begin{aligned} n^{p\alpha-1}&P[\max_{j\leqq n}|X_j| \geqq \varepsilon n^\alpha] \\ &= O(\textstyle\sum_{k=n}^{2n} k^{p\alpha-2}P[\max_{j\leqq k}|X_j| \geqq \varepsilon(k/2)^\alpha]) \to 0\,. \end{aligned} \tag{3.27}$$

Hence we can choose positive integers n_0 and ν such that

$$P[\max_{j\leqq n}|X_j| < \varepsilon n^\alpha] - \varphi(\nu - 1) \geqq \tfrac{1}{2} \qquad \text{for all} \quad n \geqq n_0\,.$$

We note that for $n \geqq n_0$,

$$\begin{aligned} P[\max_{j\leqq n}&|X_j| \geqq \varepsilon n^\alpha] \\ &\geqq P[|X_1| \geqq \varepsilon n^\alpha, \max_{\nu\leqq j\leqq n}|X_j| < \varepsilon n^\alpha] \\ &\quad + P[|X_\nu| \geqq \varepsilon n^\alpha, \max_{2\nu\leqq j\leqq n}|X_j| < \varepsilon n^\alpha] + \cdots \\ &\quad + P[|X_{\nu[n/\nu]}| \geqq \varepsilon n^\alpha] \\ &\geqq [n/\nu]P[|X_1| \geqq \varepsilon n^\alpha]\{P[\max_{j\leqq n}|X_j| < \varepsilon n^\alpha] - \varphi(\nu - 1)\} \\ &\geqq \tfrac{1}{2}[n/\nu]P[|X_1| \geqq \varepsilon n^\alpha]\,. \end{aligned} \tag{3.28}$$

Therefore from (3.26), $\sum_1^\infty n^{p\alpha-1}P[|X_1| \geqq \varepsilon n^\alpha] < \infty$ and so $E|X_1|^p < \infty$.

We now prove (iii). Let m be as given by (2.4). Then

$$\begin{aligned} P[\max_{j\leqq n}&|X_j| \geqq \varepsilon n^\alpha] \\ &\geqq P(\textstyle\bigcup_{1\leqq i\leqq[n/m]}[|X_{mi}| \geqq \varepsilon n^\alpha]) \\ &\geqq [n/m]P[|X_1| \geqq \varepsilon n^\alpha] \\ &\qquad - \textstyle\sum_{1\leqq j<i\leqq[n/m]} P[|X_{mi}| \geqq \varepsilon n^\alpha, |X_{mj}| \geqq \varepsilon n^\alpha] \\ &\geqq [n/m]P[|X_1| \geqq \varepsilon n^\alpha]\{1 - O(nP^{\beta-1}[|X_1| \geqq \varepsilon n^\alpha])\}\,. \end{aligned} \tag{3.29}$$

The last relation follows from (2.4). Assume that (3.26) holds for some $\varepsilon > 0$. Then by (3.27), $nP^{\beta-1}[|X_1| \geqq \varepsilon n^\alpha] = o(n^{1-(\beta-1)(p\alpha-1)}) = o(1)$ since $p\alpha \geqq \beta/(\beta - 1)$. Therefore from (3.26) and (3.29), $E|X_1|^p < \infty$.

We now use a combination of the argument for (ii) and (iii) to prove (iv). Assume that (3.26) holds for some $\varepsilon > 0$. Then (3.27) holds. If $p\alpha \geqq 2$, then obviously (3.27) implies that

$$nP[|X_1| \geqq \varepsilon n^\alpha] \to 0\,. \tag{3.30}$$

We now show that (3.30) still holds when $p\alpha < 2$. Since $1 < p\alpha < 2$, if we set $r = p\alpha - 1$, then $0 < r < 1$. Letting $\nu = [n^r]$, we obtain by an argument like

(3.28) using strong mixing instead of φ-mixing that

$$P[\max_{j\leqq n}|X_j|\geqq \varepsilon n^\alpha] \geqq [n/\nu]\{P[|X_1|\geqq \varepsilon n^\alpha]P[\max_{j\leqq n}|X_j|<\varepsilon n^\alpha]-\rho(\nu-1)\} \geqq \tfrac{1}{2}n^{1-r}P[|X_1|\geqq \varepsilon n^\alpha]-2n^{1-r}\rho([n^r]-1) \quad \text{for all large } n. \tag{3.31}$$

Since $r=p\alpha-1$ and $\rho(n)=O(n^{-\theta})$ for some $\theta>1/(p\alpha-1)$, (3.30) follows easily from (3.27) and (3.31).

Clearly (3.29) still holds. If $\beta\geqq 2$, then (3.29) and (3.30) imply that $P[\max_{j\leqq n}|X_j|\geqq \varepsilon n^\alpha]\geqq(1+o(1))[n/m]P[|X_1|\geqq\varepsilon n^\alpha]$, where m is given by (2.4). Now assume that $\beta<2$ and let $s=\beta-1$ so that $0<s<1$. Then as in (3.29),

$$P[\max_{j\leqq n}|X_j|\geqq \varepsilon n^\alpha]\geqq[n/m]P[|X_1|\geqq\varepsilon n^\alpha] - \sum_{1\leqq j<i\leqq[n/m]}P[|X_{mi}|\geqq\varepsilon n^\alpha, |X_{mj}|\geqq\varepsilon n^\alpha]. \tag{3.32}$$

By the strong mixing condition,

$$\sum_{j=1}^{[n/m]}\sum_{i=j+[n^s]}^{2[n/m]}P[|X_{mi}|\geqq\varepsilon n^\alpha, |X_{mj}|\geqq\varepsilon n^\alpha] \leqq 2n^2P^2[|X_1|\geqq\varepsilon n^\alpha]+2n^2\rho([n^s]) = o(nP[|X_1|\geqq\varepsilon n^\alpha])+O(n^{2-\theta s}) \quad \text{by (3.30)}. \tag{3.33}$$

Since $s=\beta-1$ and (3.30) holds, it follows from (2.4) that

$$\sum_{j=1}^{[n/m]}\sum_{i=j+1}^{j+[n^s]}P[|X_{mi}|\geqq\varepsilon n^\alpha, |X_{mj}|\geqq\varepsilon n^\alpha] \leqq n^{1+s}P^\beta[|X_1|\geqq\varepsilon n^\alpha]=o(nP[|X_1|\geqq\varepsilon n^\alpha]). \tag{3.34}$$

From (3.32), (3.33) and (3.34), we obtain

$$P[\max_{j\leqq n}|X_j|\geqq\varepsilon n^\alpha]\geqq(1+o(1))[n/m]P[|X_1|\geqq\varepsilon n^\alpha]+O(n^{2-\theta s}). \tag{3.35}$$

Since $\theta>(p\alpha+1)/(\beta-1)$, $\sum_1^\infty n^{p\alpha-\theta s}<\infty$ and so from (3.26) and (3.35), the implication (3.25) follows.

4. Remarks and a variant of Theorem 1. One of the beauties of the Hsu–Robbins–Erdös–Baum–Katz theorem (1.1) lies in the fact that the simple moment condition $E|X_1|^p<\infty$ is not only sufficient to guarantee the convergence of the series $\sum_1^\infty n^{p\alpha-2}P[|S_n|\geqq\varepsilon n^\alpha]$, but it is actually necessary as well. Thus in (1.3), a simple moment condition on X_1 is both necessary and sufficient for the r-quick strong law in the i.i.d. case and so the form of the result bears a close resemblance to the usual strong law. In Theorem 1, we have been able to generalize this definitive result to stationary mixing sequences by not only exploiting the mixing property but also using the assumption (2.4) on bivariate tail probabilities. Assumption (2.4) first was needed in our proof of the necessity of the moment condition $E|X_1|^p<\infty$ in Lemma 5 (iii) and (iv) where we had to bound $P[\max_{j\leqq n}|X_j|\geqq\varepsilon n^\alpha]$ below by $(1+o(1))nP[|X_1|\geqq\varepsilon n^\alpha]$. It is somewhat reminiscent of the assumptions on bivariate tail probabilities that one usually finds in the literature on the asymptotic distribution of the maximum of a stationary mixing sequence (see, for example, [12] and [16]). Also in our proofs of the sufficiency of the moment condition in Lemmas 1, 2, 3, the assumption (2.4) was crucial to the Erdös-type argument in (3.6) and (3.24) involving the bivariate tail probabilities. In the following theorem, we shall drop assumption (2.4) and impose instead a slightly stronger moment condition, i.e., $E|X_1|^q < \infty$ for some $q > p$. With this moment condition, we do not need the delicate Erdös-type argument and can therefore drop assumption (2.4) but still get the $(p\alpha - 1)$-quick convergence of $n^{-\alpha}S_n$.

THEOREM 2. *Let $\alpha > \frac{1}{2}$ and $p > 1/\alpha$. Let $X_1, X_2, \cdots$ be a stationary sequence and let $S_n = X_1 + \cdots + X_n$. Assume that $E|X_1|^q < \infty$ for some $q > \max\{p, 2\}$, and in the case $\alpha \leqq 1$, assume that $EX_1 = 0$.*

(i) *Suppose the sequence X_n is strong mixing with mixing coefficient $\rho(n) = O(n^{-\theta})$ for some $\theta > \max\{q/(q-2), pq/(q-p)\}$. Then as $n \to \infty$, $n^{-\alpha}X_n \to 0$ and $n^{-\alpha}S_n \to 0$ $(p\alpha - 1)$-quickly. Furthermore the statements* (2.6), (2.7), (2.10), (2.11), (2.12) *and* (2.13) *hold (for all $\varepsilon > 0$).*

(ii) *Suppose the sequence X_n is φ-mixing such that $\varphi(n) = O(n^{-\theta})$ for some $\theta > \max\{q/(q-1), q(p-2)/(4\alpha(q-p)), (p-2)/(2\alpha)\}$. Then the conclusions of* (i) *still hold.*

PROOF. To prove (i), we need only show that (2.6) holds for all $\varepsilon > 0$. (See Lemma 4 and the discussion following it.) Without loss of generality, we can assume that $EX_1 = 0$. Since $\theta > q/(q-2)$ and $\rho(n) = O(n^{-\theta})$, $\sum_1^\infty (\rho(n))^{(q-2)/q} < \infty$ and so (3.16) still holds. Since $\theta > p$ and $p\theta/(\theta - p) < q$, we can choose $\delta > 0$ such that $q\delta > p$ and $\delta < (\theta - p)/\theta (< 1)$. Therefore

$$\sum_1^\infty n^{p\alpha - 1 - q\alpha\delta} < \infty . \tag{4.1}$$

Also since $\rho(n) = O(n^{-\theta})$ and $1 - \delta > p/\theta$,

$$\sum_1^\infty n^{p\alpha - 1}\rho([n^{\alpha(1-\delta)}]) < \infty . \tag{4.2}$$

We now argue as in the proof of Lemma 2. Take a positive integer k such that (3.18) holds. Define τ and $S_{i,j}$ as in Lemma 1 and note that the relations (3.3) and (3.19) still hold. Instead of (3.5), we now have

$$\begin{aligned} \sum_{i=1}^n P[\tau = i, \max\nolimits_{1 \leqq j \leqq [n^{\alpha(1-\delta)}]} |S_{i-1,j}| > \varepsilon n^\alpha/(2k)] & \\ \leqq P[\max\nolimits_{1 \leqq j \leqq 2n} |X_j| \geqq \varepsilon n^{\alpha\delta}/(2k)] & \\ \leqq 2nP[|X_1| \geqq \varepsilon n^{\alpha\delta}/(2k)] = O(n^{1-q\alpha\delta}) . & \end{aligned} \tag{4.3}$$

From (3.3), (3.19) and (4.3), we obtain

$$\begin{aligned} P[\max\nolimits_{j \leqq n} |S_j| \geqq \varepsilon n^\alpha] \leqq P[\tau \leqq n]P[\max\nolimits_{j \leqq n} |S_j| \geqq \varepsilon n^\alpha(1 - k^{-1})] & \\ + n\rho([n^{\alpha(1-\delta)}]) + O(n^{1-q\alpha\delta}) . & \end{aligned} \tag{4.4}$$

In view of (4.1), (4.2) and (4.4), the rest of the proof that (2.6) holds is exactly analogous to that of Lemma 2.

We now prove (2.6) for the φ-mixing case in (ii). Again assume that $EX_1 = 0$. Since $\varphi(n) = O(n^{-\theta})$ and $\theta > q/(q-1)$, $\sum_1^\infty (\varphi(n))^{(q-1)/q} < \infty$. Noting that

$q > 2$, we obtain by an argument as in Lemma 1 that

$$E \max_{j \leqq n} |S_j|^q = O(n^{q/2}) . \tag{4.5}$$

This result and the φ-mixing condition itself enable us to sharpen the argument we just used for the strong mixing case. Since $(q - p)/q > (p - 2)/(4\theta\alpha)$ and $p - 2 < 2\theta\alpha$, we can choose $0 < \delta < 1$ such that

$$1 - \delta > (p - 2)/(2\theta\alpha) \quad \text{and} \quad 1 + \delta > 2p/q . \tag{4.6}$$

Take a positive integer k such that (3.2) holds and define τ and $S_{i,j}$ as in Lemma 1. In place of (4.3), we now make use of (4.5) and the Markov inequality to obtain that

$$\begin{aligned} &\sum_{i=1}^{n} P[\tau = i, \max_{1 \leqq j \leqq [n^{\alpha(1-\delta)}]} |S_{i-1,j}| > \varepsilon n^{\alpha}/(2k)] \\ &\qquad \leqq nP[\max_{1 \leqq j \leqq [n^{\alpha(1-\delta)}]} |S_j| > \varepsilon n^{\alpha}/(2k)] = O(n^{1+\frac{1}{2}q\alpha(1-\delta)-q\alpha}) . \end{aligned} \tag{4.7}$$

We note that relations (3.3) and (3.4) still hold. From (3.3), (3.4), (4.7), we have

$$\begin{aligned} P[\max_{j \leqq n} |S_j| \geqq \varepsilon n^{\alpha}] \leqq P[\tau \leqq n]P[\max_{j \leqq n} |S_j| \geqq \varepsilon n^{\alpha}(1 - k^{-1})] \\ + P[\tau \leqq n]\varphi([n^{\alpha(1-\delta)}]) + O(n^{1-\frac{1}{2}q\alpha(1+\delta)}) . \end{aligned} \tag{4.8}$$

Clearly (3.13) still holds, so $P[\tau \leqq n] = O(n^{p/2-\alpha p})$. Therefore in view of (4.6) and (4.8), we can use an analogous argument as that of Lemma 1 to show (2.6) for the φ-mixing case.

5. Applications to renewal theory and first passage times for stationary mixing sequences. Let $X_1, X_2, \cdots$ be i.i.d. random variables with $EX_1 = \mu > 0$ and $E(X_1^-)^2 < \infty$. Let $S_n = X_1 + \cdots + X_n (S_0 = 0)$. The elementary renewal theorem states that

$$\sum_{n=0}^{\infty} P[S_n \leqq c] \sim \mu^{-1}c \quad \text{as} \quad c \to \infty . \tag{5.1}$$

Making use of Theorem 2, we can extend (5.1) to stationary mixing sequences.

THEOREM 3. *Let $X_1, X_2, \cdots$ be a stationary sequence with $EX_1 = \mu > 0$ and $E(X_1^-)^q < \infty$ for some $q > 2$. Assume that the sequence X_n is strong mixing with mixing coefficient $\rho(n) = O(n^{-\theta})$ for some $\theta > 2q/(q - 2)$. Then* (5.1) *still holds.*

PROOF. Since the strong mixing property implies ergodicity,

$$\lim_{n\to\infty} n^{-1}S_n = \mu \quad \text{a.s.} \tag{5.2}$$

Let $N(c) = \sum_{n=0}^{\infty} I_{[S_n \leqq c]}$. It follows easily from (5.2) that

$$\lim_{c\to\infty} \mu N(c)/c = 1 \quad \text{a.s.} \tag{5.3}$$

Therefore to prove (5.1), we need only show that the dominated convergence theorem is applicable. Choose b large enough so that $EX_1' \geqq \frac{1}{2}\mu$, where we define $X_i' = X_i I_{[X_i \leqq b]}$ and $S_n' = X_1' + \cdots + X_n'$. Let $L = \sup\{n \geqq 1 : S_n' \leqq \mu n/3\}$. We note that if $n \geqq \max(L + 1, 3\mu^{-1}c)$, then $S_n \geqq S_n' > \mu n/3 \geqq c$, and so

$$N(c) \leqq L + 3\mu^{-1}c \,. \tag{5.4}$$

Since $E|X_1'|^q < \infty$, it follows from Theorem 2 (where we set $p = 2$ and $\alpha = 1$) that $n^{-1}S_n' \to EX_1'$ (1-quickly). Hence $EL < \infty$ and (5.1) follows from (5.2), (5.4) and the dominated convergence theorem.

For another related application of Theorem 2, consider the first passage time $T(c) = \inf\{n \geqq 1 : S_n \geqq c\}$. From Theorem 2, (5.2), (5.4) and the obvious inequality $T(c) \leqq N(c) + 1$, it follows that

$$ET(c) \sim \mu^{-1}c \qquad \text{as} \quad c \to \infty \tag{5.5}$$

under the assumptions of Theorem 3. A straightforward modification of Theorem 3 also yields the asymptotic behavior of the moments $ET^r(c)$ and $EN^r(c)$. Furthermore by using similar ideas, one can obtain asymptotic approximations for the expected sample sizes of certain sequential procedures in time series analysis. Details of these statistical problems will be presented elsewhere.

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