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ON SEMI-MARKOV AND SEMIREGENERATIVE PROCESSES II[1]

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An ergodic theorem is given for the age process $(I(t), Z(t))$ associated with a (possibly transient) semi-Markov chain $(I_n, X_n)_{n=0}^{\infty}$ whose sojourn times are not exclusively integer valued. Asymptotically the Markov part ($I(t)$: the state occupied at time t) and the renewal part ($Z(t)$: the age in $I(t)$ at time t) split into independent parts. This yields the following ergodic result for a semiregenerative process V_t with embedded semi-Markov chain $(I_n, X_n)_{n=0}^{\infty}$:

$$\lim_{t\to\infty}\left|\text{Prob}\{V_t \in A\} - \int_{\pi} \frac{A_\pi}{\mu_\pi}\,\text{Prob}\{I(t) = d\pi\}\right| = 0$$

where π is in the state space of I_n, μ_π is the mean sojourn time in π and A_π is the mean time V_t is in a set A during a sojourn in π.

0. Introduction. This paper is a continuation of [6] dealing with semi-Markov processes whose sojourn times are not exclusively integer valued, the so-called continuous case. Most results in this paper have an analogue in [6]. Certain measurability difficulties crop up, however, in the continuous case. Specifically, strong mixing implies only almost sure (not everywhere) results on the space-time harmonic functions of the age process (compare Lemma 1 here and Lemma 1 in [6]). The same difficulty is manifest in [5]. ([5] would serve as an easy introduction to this paper.) It is consequently impossible to apply Theorems 1 and 2 in [6] directly and this leads to weak convergence analogues of the results in [6] (compare Theorems 3 and 4 here with those in [6]). Proofs follow in Section 2.

Unless otherwise specified notation and definitions are carried over unchanged from [6]; however, since we deal only with the continuous case henceforth R, (R_+) represents $(-\infty, \infty)$, $([0, \infty))$. B, (B_+) represents the Borel sets on R, (R_+) and m is Lebesgue measure.

As in Section 2 in [6], we consider a semi-Markov kernel $\mathbf{\Pi}$ defined on $(E, \mathcal{E}) = (\Pi \times R, \mathcal{G} \otimes B)$ where $(\Pi, \mathcal{G})$ is a measurable space. That is,

$$\mathbf{\Pi}(\pi, x; d\pi', dx') = \mathbf{\Pi}(\pi, 0, d\pi', dx') \qquad \text{for all} \quad (\pi, x) \in E;$$

that is, the transition is independent of x.

Given an initial probability measure $\delta_{(\pi, x)}$ on $(E, \mathcal{E})$ we may construct a probability space $\{\Lambda, \mathcal{Q}, \mathbf{\Pi}^{(\pi, x)}\}$ on which a Markov chain $(I_n, X_n)_{n=0}^{\infty}$ (called a semi-Markov chain) is defined having initial distribution $\delta_{(\pi, x)}$ and transition

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probability kernel Π. If, moreover, $S_n = \Sigma_{k=0}^n X_k$ then $(I_n)_{n=0}^\infty$ and $(I_n, S_n)_{n=0}^\infty$ are both Markov chains defined on $\{\Lambda, \mathcal{Q}, \Pi^{(\pi, x)}\}$. Let $\mathfrak{R}$ be the transition kernel of $(I_n)_{n=0}^\infty$. Therefore

$$\mathfrak{R}(\pi; d\pi') = \Pi(\pi, 0; d\pi' \times R).$$

Let

$$\mathfrak{R}^\pi\{I_n = d\pi'\} = \Pi^{(\pi,0)}\{(I_n, X_n) \in d\pi' \times R\}$$
$$= \Pi^{(\pi,s)}\{(I_n, X_n) \in d\pi' \times R\}.$$

Hence we may consider $(I_n)_{n=0}^\infty$ to be defined on $\{\Lambda, \mathcal{Q}, \mathfrak{R}^\pi\}$. Let Q be the transition kernel of $(I_n, S_n)_{n=0}^\infty$. Hence

$$Q(\pi, s; d\pi', s + dx') = \Pi(\pi, s; d\pi', dx').$$

Let $F^\pi(x) = \Pi^{(\pi,0)}\{X_1 \leqslant x\}$.

It is useful to recall the strong mixing condition imposed in [6]. Let $\{T_n\}_{n=0}^\infty$ be the coordinate functions defined on the probability space $\{R^\infty, B^\infty, \Gamma\}$ ($R^\infty = R \times R \times \cdots$, $B^\infty = B \times B \times \cdots$) and suppose the $\{T_n\}_{n=0}^\infty$ are independent. Below we define when this product probability measure Γ is strongly d-mixing (strongly mixing). Following [6] let K_n be a partition of $\{0, 1, 2, \cdots\}$ of the form $K_n = \{i | i_n < i \leqslant i_{n+1}\}$. Given K_n define $Y_n = \Sigma_{i \in K_n} T_i$. For $d > 0$ and $\varepsilon = 1/2r$, r an integer, set

$$B_k(\varepsilon) = \{x | -\varepsilon < x - 2k\varepsilon \leqslant \varepsilon\}$$
$$q_{nk}(\varepsilon, d) = \min[\text{Prob}\{Y_n \in B_k(\varepsilon)\}, \text{Prob}\{Y_n - d \in B_k(\varepsilon)\}]$$
$$q_n(\varepsilon, d) = \Sigma_{k=-\infty}^\infty q_{nk}(\varepsilon, d).$$

DEFINITION 0.a. The sequence $\{T_n\}_{n=0}^\infty$ is called strongly d-mixing if $\forall \varepsilon$ there exists a sequence K_n such that

$$\Sigma_{n=0}^\infty q_n(\varepsilon, d) = \infty.$$

Furthermore the sequence $\{T_n\}_{n=0}^\infty$ is called strongly mixing if the closure of the smallest subgroup containing

$$\{d | \{T_n\}_{n=0}^\infty \text{ is strongly } d\text{-mixing}\}$$

is R. The measure Γ on $\{R^\infty, B^\infty\}$ is called strongly d-mixing (strongly mixing) if the coordinate functions $\{T_n\}_{n=0}^\infty$ are strongly d-mixing (strongly mixing).

A sufficient condition for the sequence $\{T_n\}_{n=0}^\infty$ to be strongly d-mixing is the following: there exists a sequence of real numbers t_n and for all $\varepsilon > 0$ a $\delta = \delta(\varepsilon) > 0$ such that

$$\min(\text{Prob}\{T_n \in [t_n, t_n + \varepsilon)\}, \text{Prob}\{T_n \in [t_n + d, t_n + d + \varepsilon)\}) > \delta(\varepsilon)$$

DEFINITION 0.b. The semi-Markov kernel Π is called strongly d-mixing (respectively strongly mixing) if $\forall (\pi, s) \in E$ the probability $\Gamma^I_{(\pi,s)}$ defined on cylinder sets of $\{R^\infty, B^\infty\}$ by

$$\Gamma^I_{(\pi,s)}\{B_0 \times B_1 \times \cdots \times B_n \times R \times R \times \cdots\}$$
$$= \Pi^{(\pi,s)}\{X_0 \in B_0, X_1 \in B_1, \cdots, X_n \in B_n | I_0, I_1, \cdots\}$$

for each trajectory $(I_0, I_1, \cdots)$ is strongly d-mixing (respectively strongly mixing) a.s.-$\mathfrak{R}^\pi$. This means the sojourn times $(X_n)_{n=0}^\infty$ conditioned on knowing the trajectory of the semi-Markov process (i.e., $(I_n)_{n=0}^\infty$) are strongly d-mixing (respectively strongly mixing). This is meaningful since $(X_n)_{n=0}^\infty$ are independent under the measure $\Gamma^I_{\pi, s}$.

We remark that a semi-Markov kernel $\mathbf{\Pi}$ is strongly mixing if, for instance, for each $\pi \in \Pi$,

$$\mathbf{\Pi}^{(\pi, 0)}(X_1 = dx, I_1 = d\rho) = f^{(\pi, \rho)}(x)m(dx)\mathbf{\Pi}^{(\pi, 0)}(I_1 = d\rho)$$

where $f^{(\pi, \rho)}$ is a density bounded below on some fixed interval, uniformly in (π, ρ).

ASSUMPTION. *Positivity and no explosions.* With initial measure $\delta_{(\pi, 0)}$ construct $\mathbf{\Pi}^{(\pi, 0)}$, $\{\Lambda, \mathcal{C}\}$, $(I_n, X_n)_{n=0}^\infty$ and $(I_n, S_n)_{n=0}^\infty$. We suppose henceforth that the chain is positive, that is, $X_n > 0$ $\forall n$. We may therefore consider $\mathbf{\Pi}$ to be defined on $(E_+, \mathcal{E}_+) = (\Pi \times R_+, \mathcal{G} \otimes B_+)$. Moreover we demand that $\mathbf{\Pi}^{(\pi, 0)}\{\lim_{n\to\infty} S_n = \infty\} = 1$. It is clear that this condition is automatically fulfilled if $(I_n, X_n)_{n=0}^\infty$ is strongly mixing.

As in [6] we may now construct the age process starting at $(\pi, 0)$ by defining

$$(I(t), Z(t)) = (I_{n-1}, t - S_{n-1}) \; S_{n-1} \leqslant t < S_n$$

where I_n and S_n are defined on $\{\Lambda, \mathcal{C}, \mathbf{\Pi}^{(\pi, 0)}\}$. This is just the last state entered before time t coupled with the time elapsed while in this state up to time t. The age process starting at (π, x) may be viewed as already being in progress for a time x so in general we define the transition kernel H_t of the age process by

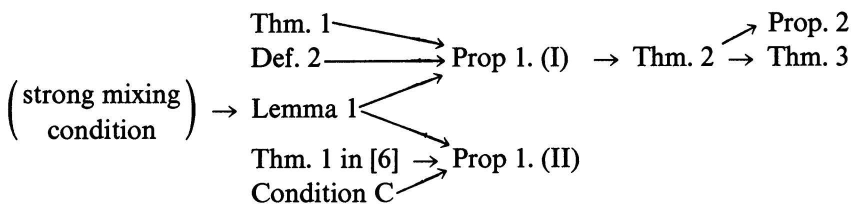
$$H_t(\pi, x; d\pi', dx') = \mathbf{\Pi}^{(\pi, 0)}\{(I(t + x), Z(t + x)) = (d\pi', dx')|X_1 > x\}.$$

As in [6] we may also define $H_t(\pi, x; d\pi', dx')$ when $1 - F^\pi(x) = 0$. Starting from (π, x) at $t = 0$, $(I(t), Z(t)) = (\pi, x + t)$ for $0 \leqslant t < [x] + 1 - x$. When $t = [x] + 1 - x$, $(I(t), Z(t)) = (\pi, 0)$ and the process can now be defined as usual. In effect we have let the age process $Z(t)$ increase from x to the next highest integer. It is clear that for $t > 1$ the support of $H_t(\pi, x; d\pi', dx')$ is contained in $\{(\pi', x'); 1 - F^{\pi'}(x') > 0\}$ for all (π, x). From H_t and any initial measure α on $(E_+, \mathcal{E}_+)$ we may finally construct (see [6]) the Markov process

$$\{\Omega, \mathfrak{F}, H^\alpha, (W_t)_{t\in R_+}, (\mathfrak{F}_t)_{t\in R_+}, (\theta_t)_{t\in R_+}\}$$

where W_t has right continuous paths a.s.-H^α and W_t defined on $\{\Omega, \mathfrak{F}, H^{(\pi, 0)}\}$ has the same distribution as $(I(t), Z(t))$ defined on $\{\Lambda, \mathcal{C}, \mathbf{\Pi}^{(\pi, 0)}\}$.

The logical plan of the paper is as follows:



For a first reading of this paper, the following special case might be kept in mind. Consider a semi-Markov chain $(n+1, X_n)_{n=0}^{\infty}$ with state space $\Pi = \{1, 2, 3, \cdots\}$ which starts out in state 1 ($I_0 = 1$, $X_0 = 0$) and jumps successively to state 2 ($I_1 = 2$), to state 3 ($I_2 = 3$), and so on with sojourn times $X_1, X_2, \cdots$, having distributions $F^1, F^2, \cdots$. A bounded function h defined on $\Pi \times R_+$ which is harmonic for the Markov chain $(n+1, S_n)_{n=0}^{\infty}$, satisfies:

$$\text{(H)} \qquad h(n, x) = \int_0^{\infty} h(n+1, x+y) F^n(dy) \qquad x \in R_+, \quad n \in \Pi.$$

A bounded function h satisfying (H) may be viewed as a space-time harmonic function for the Markov chain S_n. If $|X_n| \leqslant L$, Orey [9] has given necessary and sufficient conditions ensuring that solutions of (H) which are continuous in x are constant. Mineka [8] has given conditions on $\{F^n\}_{n=1}^{\infty}$ sufficient for ensuring that all bounded, continuous (in x) solutions of (H) are constant ($|X_n| \leqslant L$ is not required). The general strong mixing condition for semi-Markov chains given in Definition 0 becomes Mineka's condition in our special case. That is, $(n+1, X_n)_{n=0}^{\infty}$ is strongly mixing if the sequence $\{X_n\}_{n=0}^{\infty}$ is.

A regularization argument shows that if all bounded continuous solutions of (H) are constant then all bounded solutions of (H) are constant a.s.-m. Hence the strong mixing condition implies that if h is a bounded, harmonic function for the chain $(n+1, S_n)_{n=0}^{\infty}$ then $h(n, x) = c$ (c a constant) for all n a.s.-m in x.

The age process associated with $(n+1, X_n)_{n=0}^{\infty}$ becomes:

$$(I(t), Z(t)) = (n, t - S_{n-1}) \qquad \text{if} \quad S_{n-1} \leqslant t < S_n.$$

By the technique used in Lemma 1 in [6] the study of the space-time harmonic functions for $(I(t), Z(t))$ may be reduced to studying (H). This is the content of Lemma 1 which proves that the strong mixing condition implies the bounded space-time harmonic functions of $(I(t), Z(t))$ are almost surely constant.

Some of the general results may be reinterpreted for our special case. Proposition 2 becomes: Let $\{X_n\}_{n=1}^{\infty}$ be a strongly mixing sequence of independent random variables having distributions $\{F^n\}_{n=1}^{\infty}$. Suppose that for all n, $F^n(0) = 0$, and that there exists a distribution G with finite mean such that $F^n(x) \geqslant G(x)$ for $x \geqslant 0$ and for all n. If, moreover, $\inf \mu_n > 0$ where μ_n is the mean of F^n, then

$$\lim_{t\to\infty}\Sigma_{n=1}^{\infty}\left|\operatorname{Prob}\{S_n \in [t-h, t]\} - \frac{h}{\mu_n}\operatorname{Prob}\{S_{n-1} \leqslant t < S_n\}\right| = 0.$$

This gives Blackwell's renewal theorem for nonidentical random variables.

Theorem 4 applied to our special case enables us to do the following example:

EXAMPLE. *The two-stage renewal process.* Following the example in [2], page 380, we consider alternate sojourns in two states E_1 and E_2. Denote the length of the nth sojourn in E_1 (E_2 respectively) by U_n (D_n). Suppose U_n (D_n) has mean ${}^1\mu_n$ (${}^2\mu_n$). Further suppose $(U_n, D_n)_{n=1}^{\infty}$, forms an independent sequence (U_n and D_n are not necessarily independent). Next suppose $(U_n + D_n)_{n=1}^{\infty}$ is a strongly mixing sequence whose distributions $(F_n)_{n=1}^{\infty}$ are bounded by a distribution G having finite mean as follows: $F^n(x) \geqslant G(x)\ \forall x \in R_+$. If finally $\inf_n({}^1\mu_n + {}^2\mu_n) > 0$, then

$$\lim_{t\to\infty}\left|\operatorname{Prob}\{\text{in state } E_1 \text{ at time } t\} - \Sigma_{n=1}^{\infty}\frac{{}^1\mu_n}{{}^1\mu_n + {}^2\mu_n}\cdot\operatorname{Prob}\{S_{n-1} \leqslant t < S_n\}\right| = 0$$

where $S_n = \Sigma_{j=1}^{n}(U_j + D_j)$.

Note that if ${}^1\mu_n = {}^1\mu$ and ${}^2\mu_n = {}^2\mu$ for all n then

$$\lim_{t\to\infty}\operatorname{Prob}\{\text{in state } E_1 \text{ at time } t\} = \frac{{}^1\mu}{{}^1\mu + {}^2\mu}.$$

This is an example of a regenerative process whose stochastic mechanism changes with time. The proof is at the end of Section 2. General conditions for the existence of the limit

$$\lim_{t\to\infty}\Sigma_{n=1}^{\infty}\frac{\alpha_n}{\mu_n}\operatorname{Prob}\{S_{n-1} \leqslant t < S_n\}$$

are given in [7].

1. Statement of results. We keep the definitions of Section 1 of [6]. The following paragraph gives the general construction of the space-time process associated with an arbitrary transition kernel P.

Let $(P_{t,t+s})_{t,s\in R_+}$ be a probability transition semigroup defined on a measure space $(S, \mathscr{Q})$ admitting for any initial probability α on $(S, \mathscr{Q})$ the construction of a Markov process on S:

$$\{\Omega, \mathscr{F}, (P^\alpha), (X_t)_{t\in R_+}, (\mathscr{F}_t)_{t\in R_+}, (\theta_t)_{t\in R_+}\}.$$

$(X_t)_{t\in R_+}$ is defined on $\{\Omega, \mathscr{F}, P^\alpha\}$. $(\theta_t)_{t\in R_+}$ is the shift operator. $\mathscr{F}_t = \sigma(X_s)_{s\leqslant t}$ (the σ-field generated by X_s, $s \leqslant t$), $\mathscr{F}^t = \sigma(X_s)_{s\geqslant t}$, $\mathscr{F}^\infty = \cap_{t=0}^{\infty}\mathscr{F}^t$. $\mathscr{I}$ is the (shift) invariant σ-field.

Throughout we will denote by the same symbol both the probability measure and the expectation operator derived from a probability transition kernel. Henceforth denote P^{δ_x} by P^x.

Defining $\tilde{S} = S \times R_+$, $B(\tilde{S}) = B(S) \times B(R_+)$ and

$$\tilde{P}_t((x, s), (B \times \{t + s\})) = P_{t, t+s}(x, B)$$

where $x \in S$, t, $s \in R_+$ and $B \in \mathcal{Q}$ gives the space-time transition semigroup, which admits the construction of a Markov process

$$\{\tilde{\Omega}, \tilde{\mathfrak{F}}, (\tilde{P}^{(x,t)})_{(x,t)\in\tilde{S}}, (\tilde{X}_t)_{t\in R_+}, (\tilde{\mathfrak{F}}_t)_{t\in R_+}, (\tilde{\theta}_t)_{t\in R_+}\}.$$

A function $\tilde{h}$ defined on $(\tilde{S}, B(\tilde{S}))$ is called space-time harmonic for $\tilde{P}_t$ if $\tilde{h}$ is jointly measurable on $S \times R_+$ and $\tilde{h} = \tilde{P}_t\tilde{h}$.

Let α and β be two probability measures on $(S, \mathcal{Q})$. We have

THEOREM 1. *If for any bounded, space-time harmonic function $\tilde{h}$ (harmonic for $\tilde{P}_t$) there is a constant $C_{\tilde{h}}$ such that*

$$\lim_{t\to\infty}\left(\frac{\alpha + \beta}{2}\right)P_{0,t}\{x|\tilde{h}(x, t) \neq C_{\tilde{h}}\} = 0,$$

then

$$\lim_{t\to\infty}\|\alpha P_{0,t} - \beta P_{0,t}\| = 0.$$

Henceforth we specialize to the stationary kernel H_t defined on $(E_+, \mathcal{E}_+)$. The construction of $\tilde{H}_t$, $\tilde{H}^\alpha$ and $\tilde{E}_+$ is like that for the general space-time process (see [6] for details).

In analogy with the condition imposed in Theorem 1(b) in [5] we define:

DEFINITION 1. The semi-Markov chain $(I_n, X_n)_{n=0}^{\infty}$ satisfies condition C if $\forall(\pi, s) \in E_+$ the singular part w.r.t. m of the measure

$$\Pi^{(\pi,s)}\{S_n \in dx|I_0, I_1, \cdots\} \quad \text{on} \quad (R, B)$$

tends to 0 as n tends to ∞ a.s.-$\mathfrak{R}^\pi$.

In practice condition C would hold if all the sojourn times had distributions which were absolutely continuous w.r.t. Lebesgue measure or at least which uniformly had an absolutely continuous part.

In analogy with Lemma 1 in [6]:

LEMMA 1.

(a) *If the semi-Markov kernel Π is strongly mixing and if $\tilde{h}$ is a bounded, space-time harmonic function on $\tilde{E}_+$ then there exists a measurable, bounded function h on $(E_+, \mathcal{E}_+)$ such that for $\pi \in \Pi$ and $t \in R_+$ $\tilde{h}(\pi, x, t) = h(\pi, x)$ a.s.-m (in x). If, moreover, the only bounded, measurable solutions to $k = \mathfrak{R}k$ are constants, then for all (π, t), $\tilde{h}(\pi, x, t) = C_{\tilde{h}}$ a.s.-m (in x) for some constant $C_{\tilde{h}}$.*

(b) *If* $(I_n, X_n)_{n=0}^{\infty}$ *also satisfies condition C then* $\tilde{h}(\pi, x, t) = h(\pi, x)$ *for all t. If, moreover, the only bounded, measurable solutions to* $k = \mathfrak{R}k$ *are constants then* $\tilde{h}$ *is constant.*

In order to apply Theorem 1 or Theorem 1 in [6] we must show that the $\tilde{H}_t$ measure of points where $\tilde{h}$ is not well behaved tends to 0 as t tends to ∞. Imposing condition C eliminates the problem. However, without this assumption we need:

DEFINITION 2. A measure α on $(E_+, \mathfrak{S}_+)$ is called absolutely continuous w.r.t. m (a.c.-m) if for all $A \in \mathfrak{S}_+$ such that $m\{x|(\pi, x) \in A\} = 0\ \forall\pi$ we have $\alpha(A) = 0$.

LEMMA 2. *If a probability measure* α *on* $(E_+, \mathfrak{S}_+)$ *is* a.c.-m *then* αH_t *is* a.c.-m *for all* $t > 1$.

The continuous analogue of Proposition 4 in [6] now follows easily:

PROPOSITION 1.

I(a). *If the semi-Markov kernel* Π *is strongly mixing then* $\forall s \in R_+$ *and for any* a.c.-m *initial distribution* α

$$\lim_{t\to\infty}\|\alpha H_t - \alpha H_{t+s}\| = 0.$$

I(b). *If, moreover, the only bounded measurable solutions to* $k = \mathfrak{R}k$ *are constants, then for any* a.c.-m *initial distributions* α *and* β

$$\lim_{t\to\infty}\|\alpha H_t - \beta H_t\| = 0.$$

II(a). *If the semi-Markov kernel* Π *is strongly mixing and satisfies condition C then* $\forall s \in R_+$ *and for any initial distribution* α *on* E_+

$$\lim_{t\to\infty}\|\alpha H_t - \alpha H_{t+s}\| = 0.$$

II(b). *If, moreover, the only bounded measurable solutions to* $k = \mathfrak{R}k$ *are constants, then for any initial distributions* α *and* β

$$\lim_{t\to\infty}\|\alpha H_t - \beta H_t\| = 0.$$

The proof of Theorem 3 in [6] does not work since $\alpha H_{t-x}(d\pi, 0)m(dx) = 0$; that is, the measure αH_t is diffuse. This may be remedied by defining the potential U of (I_n, S_n):

DEFINITION 3. For $A \in \mathfrak{S}_+$ let $A + s_0 = \{(\pi, s + s_0)|(\pi, s) \in A\}$. Define

$$U(\pi_0, s_0; A) = \Pi^{(\pi_0, 0)}\left\{\textstyle\sum_{n=0}^{\infty}\chi_{A+s_0}(I_n, S_n)|X_1 > s_0\right\},$$

and in general for any probability distribution α on $(E_+, \mathfrak{S}_+)$ define $\alpha U(A) = \int U(\pi, s; A)\alpha(d\pi, ds)$. αU induces the measure $\alpha U(d\pi, ds)$ on $(E_+, \mathfrak{S}_+)$. By construction it is clear that $\alpha H_t(d\pi, ds) = \alpha U(d\pi, t - ds) \cdot (1 - F^{\pi}(s))$.

Another difficulty arises from the fact that $\inf_{\pi}\mu_{\pi}$ may be 0 (in the lattice case $\mu_{\pi} \geqslant 1$ for all π). This results in the following weakened analogue of Theorem 3 in [6].

THEOREM 2. *If* α *satisfies* $\lim_{t\to\infty}\|\alpha H_t - \alpha H_{t+s}\| = 0\ \forall s$ *and if* $\forall\pi\ F^{\pi}(s) \geqslant G(s)$ *and* $\int sG(ds) < \infty$ *then*

$$\lim_{t\to\infty}\|\mu_{\pi}\cdot\alpha H_t(d\pi, dx) - (1 - F^{\pi}(x))\cdot m(dx)\cdot\alpha H_t(d\pi)\| = 0.$$

COROLLARY 1. *If* $\inf_{\pi}\mu_{\pi} > 0$ *and the hypotheses of Theorem* 2 *hold then*

$$\lim_{t\to\infty;\ t\in R_+}\|\alpha H_t - \alpha_t\| = 0 \tag{1}$$

where

$$\alpha_t(d\pi, dx) = \frac{(1 - F^{\pi}(x))}{\mu_{\pi}}\cdot m(dx)\cdot\alpha H_t(d\pi).$$

Corollary 1 may be rephrased by saying that asymptotically the measure $\Pi^{\alpha}\{I(t) \in d\pi, Z(t) \in dx\}$ splits into a Markov part $\Pi^{\alpha}\{I(t) \in d\pi\}$ and the renewal part $((1 - F^{\pi}(x))/\mu_{\pi})m(dx)$.

It seems unlikely that one may suppress the condition $\inf_{\pi}\mu_{\pi} > 0$ in Corollary 1 but it would be more natural. If $\lim_{t\to\infty}\|\alpha H_t - \beta H_t\| = 0$ and if (1) holds then clearly $\lim_{t\to\infty}\|\beta H_t - \alpha_t\| = 0$. Hence Propositions 1.I(b) and 1.II(b) give sufficient conditions for ignoring the initial distribution. We will not bother to formulate theorems for different starting distributions. If the hypotheses of Proposition 1.II(a) are satisfied then Theorem 2 and Corollary 1 work for all starting distributions α. We may then proceed as in [6] to establish results analogous to Theorem 4 in [6]. On the other hand if condition C does not hold we obtain weaker results. Henceforth we consider only this latter situation.

If α is a probability measure on $(E_+, \mathcal{E}_+)$ a.c.-m, then if $(I_n, X_n)_{n=0}^{\infty}$ is regular (that is, Π is strongly mixing and for all $\pi \in \Pi$, $F^{\pi}(s) \geqslant G(s)$ where $\int sG(ds) < \infty$), and if $\inf_{\pi}\mu_{\pi} > 0$, then by Proposition 1.I(a) and Corollary 1, (1) holds. By using the technique of starting with a smooth initial measure, such that (1) holds, and shrinking it to a point (as in Corollary 2 in [5]) we have:

PROPOSITION 2 (Blackwell's theorem). *If the semi-Markov chain* $(I_n, X_n)_{n=0}^{\infty}$ *is regular, if* $\inf_{\pi}\mu_{\pi} > 0$ *and if* $\delta^0 = \delta_{(\pi_0, 0)}$ *then*

$$\lim_{t\to\infty;\ t\in R_+}\|\delta^0 U(d\pi, [t-h, t]) - \frac{h}{\mu_{\pi}}\delta^0 H_t(d\pi)\| = 0.$$

This may easily be generalized to

COROLLARY 2. *Under the hypotheses of Proposition* 2

$$\lim_{t\to\infty;\ t\in R_+}\|\alpha U(d\pi, [t-h, t]) - \frac{h}{\mu_{\pi}}\cdot\alpha H_t(d\pi)\| = 0$$

for all starting measures α.

DEFINITION 4. Let $z(\pi, s)$ be a bounded, positive measurable function on E_+. Let

$$\bar{z}_k^\pi = \sup_{s \in [(k-1)h, kh)} \underline{z}(\pi, s)$$

$$\underline{z}_k^\pi = \inf_{s \in [(k-1)h, kh)} z(\pi, s)$$

$$\bar{z}(\pi, s) = \Sigma_{k=1}^\infty \bar{z}_k^\pi \chi_{\{[(k-1)h, kh)\}}(s) \quad \text{and}$$

$$\underline{z}(\pi, s) = \Sigma_{k=1}^\infty \underline{z}_k^\pi \chi_{\{[(k-1)h, kh)\}}(s).$$

We say $z(\pi, s)$ is uniformly directly Riemann integrable (u.d.r.) if there exists an h such that for $0 \leqslant h \leqslant \bar{h}$ $h\Sigma_{k=1}^\infty \sup_\pi \bar{z}_k^\pi < \infty$ and

$$\lim_{h \to 0} h \Sigma_{k=1}^\infty \bar{z}_k^\pi = \lim_{h \to 0} h \Sigma_{k=1}^\infty \underline{z}_k^\pi = \int_{0 \leqslant s < \infty} z(\pi, s) \cdot m(ds)$$

uniformly in π.

THEOREM 3 (key renewal theorem). *If the semi-Markov chain $(I_n, X_n)_{n=0}^\infty$ is regular, if* $\inf_\pi \mu_\pi > 0$ *and if z is* u.d.r. *then*

$$\lim_{t \to \infty} \left\| \int_{0 \leqslant s < \infty} z(\pi, t-s) \cdot \alpha U(d\pi, ds) - \frac{1}{\mu_\pi} \int_{0 \leqslant s < \infty} z(\pi, s) m(ds) \cdot \alpha H_t(d\pi) \right\| = 0$$

for all initial probability measures α. (Here $\| \quad \|$ is the total variation on $\{\Pi, \mathcal{G}\}$.) Hence uniformly for $G \in \mathcal{G}$

$$\lim_{t \to \infty} \left| \int_{G \times [0, \infty]} z(\pi, t-s) \cdot \alpha U(d\pi, ds) - \int_G \alpha H_t(d\pi) \frac{1}{\mu_\pi} \int_{0 \leqslant s < \infty} z(\pi, s) m(ds) \right| = 0.$$

Using the fact that $\alpha H_t(d\pi, ds) = \alpha U(d\pi, t - ds) \cdot (1 - F^\pi(s))$ we have:

COROLLARY 3. *If $f(\pi, s) \cdot (1 - F^\pi(s))$ is* u.d.r., *if $(I_n, X_n)_{n=0}^\infty$ is regular and if* $\inf_\pi \mu_\pi > 0$ *then*

$$\lim_{t \to \infty} \left\| \int_{0 \leqslant s < \infty} f(\pi, s) \cdot \alpha H_t(d\pi, ds) - \int_{0 \leqslant s < \infty} f(\pi, s) \cdot \alpha_t(d\pi, ds) \right\| = 0.$$

We now turn to applications to semiregenerative processes (see [1]). Let $\{\Omega, \mathcal{F}, \mathbf{P}^\alpha, (V_t)_{t \in R_+}\}$ be a delayed, semiregenerative process with embedded semi-Markov chain $(I_n, X_n)_{n=0}^\infty$; that is, $\mathbf{P}^\alpha\{I_0 = d\pi, X_0 = ds\} = \alpha(d\pi, ds)$. Let A be a measurable set in the range of $(V_t)_{t \in R_+}$.

THEOREM 4. *If $(I_n, X_n)_{n=0}^\infty$ is regular, if* $\inf_\pi \mu_\pi > 0$ *and if $K(\pi, s; A) = \mathbf{P}^{\delta(\pi, 0)}\{V_s \in A, X_1 > s\}$ is* u.d.r. *then*

$$\lim_{t \to \infty} \left\| \mathbf{P}^\alpha\{V_t \in A, I(t) \in d\pi\} - \frac{A_\pi}{\mu_\pi} \cdot \alpha H_t(d\pi) \right\| = 0$$

where $A_\pi = \int_{0 \leqslant s < \infty} K(\pi, s; A) m(ds)$. (Hence A_π is the mean time V_t is in A during a sojourn in the state π.)

We include the following special result for future reference:

REMARK 1. If the semi-Markov chain $(I_n, X_n)_{n=0}^{\infty}$ is regular, if $\inf_\pi \mu_\pi > 0$ and if the only bounded measurable solutions k of $k = \mathfrak{R}k$ are constants then

$$\lim_{t\to\infty}\|\alpha H_t(d\pi) - \beta H_t(d\pi)\| = 0$$

for all probability measures α and β on $(E_+, \mathcal{E}_+)$.

2. Proofs.

Theorem 1. [4] and [5] provide proofs; however, for completeness, a quick proof is given.

By the technique used in Proposition 2 in [5] it is clear that the tail field of $(X_t)_{t\in R_+}$ is trivial w.r.t. P^γ where $\gamma = (\alpha + \beta)/2$. Now $\alpha \ll \gamma$ so set $f = d\alpha/d\gamma$. Thus

$$\lim_{t\to\infty;\, t\in R_+} \sup_{F\in\mathcal{F}^t}|P^\gamma\{f(X_0)\cdot\chi_F\} - P^\gamma f(X_0)\cdot P^\gamma F| = 0;$$

that is,

$$\lim_{t\to\infty} \sup_{G\in\mathcal{Q}}|\alpha P_{0,t}(G) - \gamma P_{0,t}(G)| = 0.$$

Hence

$$\lim_{t\to\infty}\|\alpha P_{0,t} - \beta P_{0,t}\| = 0. \qquad \square$$

Lemma 1.

(a) Following the proof of Lemma 1 in [5], $\{\tilde{h}(\tilde{W}_t), \tilde{\mathcal{F}}_t\}$ is a bounded martingale. Moreover, due to the path structure of $(\tilde{W}_s)_{s\in R_+}$ (from a point (π, x, t) $\tilde{W}_s$ advances linearly to $(\pi, x+s, t+s)$ before jumping out of state π) it is clear that

$$\lim_{s\to 0^+}\tilde{h}(\pi, x+s, t+s) = \tilde{h}(\pi, x, t).$$

This follows since

$$\begin{aligned}\tilde{h}(\pi, x, t) &= \tilde{H}^{(\pi,x,t)}\tilde{h}(\tilde{W}_s)\\ &= \tilde{h}(\pi, x+s, t+s)\Pi^{(\pi,0)}\{X_1 > x+s|X_1 > x\}\\ &\quad+\Pi^{(\pi,0)}\{\tilde{h}(\tilde{W}_s)\cdot\{X_1 \leqslant x+s\}|X_1 > x\}.\end{aligned}$$

Hence

$$\begin{aligned}|\tilde{h}(\pi, x, t) - \tilde{h}(\pi, x+s, t+s)| &\leqslant \tilde{h}(\pi, x+s, t+s)\left|\frac{1-F^\pi(x+s)}{1-F^\pi(x)} - 1\right|\\ &\quad+\sup\tilde{h}\cdot\left|\frac{F^\pi(x+s) - F^\pi(x)}{1-F^\pi(x)}\right|.\end{aligned}$$

That is, $\lim_{s\to 0^+}\tilde{h}(\pi, x+s, t+s) = \tilde{h}(\pi, x, t)$. If $\Pi^{(\pi,0)}\{X_1 > x\} = 0$, by definition $\tilde{W}_s = (\pi, x+s, t+s)$ a.s.-$\Pi^{(\pi,x,t)}$ for $0 \leqslant s < [x+1] - x$. It again follows easily that

$$\lim_{s\to 0^+}\tilde{h}(\pi, x+s, t+s) = \tilde{h}(\pi, x, t).$$

The above continuity and the path structure of $(\tilde{W}_s)_{s\in R_+}$ also imply that $\tilde{h}(\tilde{X}_s)$ has, almost surely, right continuous trajectories. Thus, with τ as defined in Lemma 1 in [6],

$$\tilde{h}(\pi, 0, t) = \tilde{H}^{(\pi, 0, t)}\tilde{h}(\tilde{W}_\tau)$$
$$= \int \tilde{h}(\pi', 0, s')Q(\pi, t; d\pi', ds').$$

Again letting $h^*(\pi, s) = h(\pi, 0, s)$ we have $h^* = Qh^*$. By regularization there are equi-uniformly (in π) continuous (in s) functions $h^\varepsilon(\pi, s)$ such that $h^\varepsilon = Qh^\varepsilon$ and $\lim_{\varepsilon\to\infty} h^\varepsilon(\pi, s) = h^*(\pi, s)$ a.s.-m for each π. By Proposition 3 in [6] $h^\varepsilon(\pi, s)$ is constant in s; hence letting ε tend to 0, we see that $h^*(\pi, s)$ is constant in s a.s.-m, equal to some $k(\pi)$, a measurable function on $\{\Pi, \mathcal{G}\}$. Moreover,

$$\tilde{h}(\pi, x, t) = \int_\Pi \int_{0\leqslant s<\infty} \tilde{h}(\pi', 0, t + s)\Pi^{(\pi, x)}\{I(\tilde{W}_\tau) \in d\pi', \tau \in ds\}.$$

Let

$$h(\pi, x) = \int_\Pi k(\pi')\Pi^{(\pi, x)}\{I(\tilde{W}_\tau) \in d\pi'\}.$$

We note that by construction $\lim_{s\to 0^+} h(\pi, x + s) = h(\pi, x)$. This follows from the path structure of $\tilde{W}_t$. Let

$$A_{(\pi, x)} = \{t \mid h(\pi, x) > \tilde{h}(\pi, x, t)\}.$$

If $mA_{(\pi, x)} > 0$ we have

$$\int_{A_{(\pi, x)}} h(\pi, x)m(dt) > \int_{A_{(\pi, x)}} \tilde{h}(\pi, x, t)m(dt).$$

However,

$$\int_{A_{(\pi, x)}} \tilde{h}(\pi, x, t)m(dt)$$
$$= \int_{A_{(\pi, x)}} m(dt)\int_\Pi \int_{0\leqslant s<\infty} \tilde{h}(\pi', 0, t + s)\Pi^{(\pi, x)}\{I(\tilde{W}_\tau) \in d\pi', \tau \in ds\}$$
$$= \int_\Pi \int_{0\leqslant s<\infty} \int_{A_{(\pi, x)}} m(dt)\tilde{h}(\pi', 0, t + s)\Pi^{(\pi, x)}\{I(\tilde{W}_\tau) = d\pi', \tau \in ds\}$$
$$= \int_\Pi \int_{0\leqslant s<\infty} \int_{A_{(\pi, x)}} m(dt)k(\pi')\Pi^{(\pi, x)}\{I(\tilde{W}_\tau) = d\pi', \tau \in ds\}$$
$$= \int_{A_{(\pi, x)}} h(\pi, x)m(dt).$$

Thus $mA_{(\pi, x)} = 0$. By symmetry then $m\{t \mid h(\pi, x) \neq \tilde{h}(\pi, x, t)\} = 0$. By Fubini's theorem for π fixed, $m \otimes m\{(x, t) \mid h(\pi, x) \neq \tilde{h}(\pi, x, t)\} = 0$. Now for π and t fixed let $B_{(\pi, t)} = \{x \mid \tilde{h}(\pi, x, t) \neq h(\pi, x)\}$. If $m(B_{(\pi, t)}) > 0$ then by the fact that $\lim_{s\to 0^+} \tilde{h}(\pi, x + s, t + s) = \tilde{h}(\pi, x, t)$ and $\lim_{s\to 0^+} h(\pi, x + s) = h(\pi, x)$ we see $m \otimes m\{(x, t) \mid h(\pi, x) \neq \tilde{h}(\pi, x, t)\} > 0$, a contradiction. Therefore for all π, t

$$m\{x \mid \tilde{h}(\pi, x, t) \neq h(\pi, x)\} = 0.$$

If, moreover, the only bounded, measurable solutions to $k = \Re k$ are constant, then clearly h^ε defined above is a constant, say $C_{\tilde{h}}$. Hence $h^*(\pi, s) = C_{\tilde{h}}$ a.s.-m for each π fixed. By the above argument then $m\{t \mid \tilde{h}(\pi, x, t) \neq C_{\tilde{h}}\} = 0$ for all π, x. As before this gives (for π and t fixed) $m\{x \mid \tilde{h}(\pi, x, t) \neq C_{\tilde{h}}\} = 0$.

(b) Proceeding as in (a) we have for π fixed $h^*(\pi, s) = k(\pi)$ a.s.-m (in s). Now

$$h^*(\pi, s) = \mathfrak{R}^\pi\{\Pi^{(\pi, s)}\{h^*(I_n, S_n)|I_0, I_1, \cdots, \}\}.$$

However, by condition C

$$\lim_{n\to\infty}|\Pi^{(\pi, s)}\{h^*(I_n, S_n) - k(I_n)|I_0, I_1, \cdots \}| = 0$$

since for any π $h^*(\pi, s)$ and $k(\pi)$ differ only on a null set. Hence by dominated convergence $h^*(\pi, s) = k(\pi)$ for all s. Hence $\tilde{h}(\pi, x, t) = h(\pi, x)$ everywhere. Moreover, if k is constant then so is $\tilde{h}$. □

Lemma 2. Let $A \in \mathcal{E}_+$ be such that $m\{x|(\pi, x) \in A\} = 0$ for all π. Now we wish to prove $\alpha H_t(A) = 0$. It suffices to consider α of the form $\alpha\{\pi_0 \times R_+\} = 1$. Write α of this form as $\alpha^{\pi_0}(dx)$. α^{π_0} is a measure on R_+. It is sufficient to study two cases:

(i) $\alpha^{\pi_0}\{x|1 - F^{\pi_0}(x) = 0\} = 1$, (ii) $\alpha^{\pi_0}\{x|1 - F^{\pi_0}(x) = 0\} = 0$. We start with case (ii).

$$\begin{aligned}\alpha^{\pi_0}H_t(A) &= \int_{0\leqslant x<\infty}\alpha^{\pi_0}(dx)\Pi^{(\pi_0, 0)}\{(I(t + x), Z(t + x)) \in A|X_1 > x\}\\ &\leqslant \int_{0\leqslant x<\infty}\frac{\alpha^{\pi_0}(dx)}{(1 - F^{\pi_0}(x))}\Pi^{(\pi_0, 0)}\{(I(t + x), Z(t + x)) \in A\}\\ &= \Pi^{(\pi_0, 0)}\left\{\int_{0\leqslant x<\infty}\frac{\alpha^{\pi_0}(dx)}{(1 - F^{\pi_0}(x))}\cdot \chi_A\{(I(t + x), Z(t + x))\}\right\}.\end{aligned}$$

As x increases from 0 to ∞, along any trajectory of $(I(t + x), Z(t + x))$ there are a countable number of sojourns. Consider one of these sojourns in state π. There is a maximal interval $[t + t_0, t + t_1]$ such that $I(t + x) = \pi$ for $t_0 \leqslant x < t_1$. During this sojourn $Z(t + x)$ increases linearly in x from 0 at $x = t_0$ to $t_1 - t_0$ at $x = t_1^-$ before there is a jump to a new state. During this sojourn

$$\chi_A\{(I(t + x), Z(t + x))\} = 1$$

on $\{x|t_0 \leqslant x < t_1, x - t_0 \in A^\pi\} \subset A^\pi + t_0$, where $A^\pi = \{s|(\pi, s) \in A\}$ and $A^\pi + t_0 = \{s + t_0|s \in A^\pi\}$. By hypothesis $m(A^\pi) = 0$. Hence $m(A^\pi + t_0) = 0$, and for this sojourn $m\{x|(I(t + x), Z(t + x)) \in A\} = 0$. As remarked, any trajectory is composed of a countable number of sojourns. Hence

$$m\{x|(I(t + x), Z(t + x)) \in A\} = 0$$

for all trajectories. It follows that $\alpha^{\pi_0}H(A) = 0$ since α^{π_0} is absolutely continuous w.r.t. Lebesgue measure.

For case (i) let $\beta^{\pi_0}\{B\} = \alpha^{\pi_0}\{\cup_{k=0}^{\infty}\{k + B\}\}$ for B a Borel set in [0, 1], where $\{k + B\} = \{x|x - k \in B\}$. β^{π_0} is absolutely continuous w.r.t. Lebesgue measure. By the definition of H_t

$$\alpha^{\pi_0}H_t(A) = \int_0^1\Pi^{(\pi_0, 0)}\{(I_{(t-1+s)}, Z_{(t-1+s)}) \in A\}\,\beta^{\pi_0}(ds)$$

for $t > 1$. We conclude $\alpha^{\pi_0}H_t(A) = 0$ by the same technique used in case (ii).

Proposition 1.

I(a). Proceed as in Proposition 4 in [6]. Consider $(I(t), Z(t))_{t \in R_+}$ to be the coordinate process defined on the canonical probability space of trajectories $\{\Omega, \mathcal{F}, H^\alpha\}$. The shift operator θ_s defined on $\omega \in \Omega$ is such that

$$(I(t), Z(t))(\theta_s \omega) = (I(t + s), Z(t + s))(\omega).$$

For any tail event T of $(I(t), Z(t))_{t=0}^\infty$ there is a bounded space-time harmonic function $\tilde{h}$ such that

$$\lim_{t \to \infty} \tilde{h}(\tilde{W}_t) = \lim_{t \to \infty} \tilde{h}(I(t), Z(t), t) = T \quad \text{a.s.-}H^\alpha.$$

(Note that $\tilde{h}(\tilde{W}_t)$ is right continuous by the argument given in Lemma 1(a)). By Lemma 1(a) there exists a measurable bounded function h on E_+ such that for all π, t $\tilde{h}(x, \pi, t) = h(\pi, x)$ a.s.-m in x. However, by Lemma 2, αH_t is a.c.-m so for fixed s, t

$$\tilde{h}(I(t), Z(t), t - s) = h(I(t), Z(t)) \quad \text{a.s.-}H^\alpha.$$

Let $\{t_n\}_{n=1}^\infty$ be an increasing sequence tending to ∞. Then

$$T = \{\omega | \omega \in \Omega, \tilde{h}(I(t_k), Z(t_k), t_k)(\omega) \to 1\} \quad \text{a.s.-}H^\alpha.$$

Now

$$\begin{aligned} \theta_s T &= \{\theta_s \omega | \omega \in \Omega, \tilde{h}(I(t_k), Z(t_k), t_k)(\omega) \to 1\} \qquad \text{a.s.} - H^\alpha \\ &= \{\omega | \omega \in \Omega, \tilde{h}(I(t_k - s), Z(t_k - s), t_k)(\omega) \to 1\} \\ &= \{\omega | \omega \in \Omega, h(I(t_k - s), Z(t_k - s))(\omega) \to 1\} \qquad \text{a.s.} - H^\alpha \\ &= \{\omega | \omega \in \Omega, \tilde{h}(I(t_k - s), Z(t_k - s), t_k - s)(\omega) \to 1\} \qquad \text{a.s.} - H^\alpha \\ &= T \qquad \text{a.s.} - H^\alpha. \end{aligned}$$

T is therefore invariant. The rest of the proof is as in Proposition 4 in [6].

I(b). By Lemma 2, $((\alpha + \beta)/2)H_t$ is a.c.-m. By Lemma 1(a) any space-time harmonic function $\tilde{h}(\pi, x, t)$ is constant a.s.-m in x for all π, t. The result follows from Theorem 1.

II(a) follows from Lemma 1(b) and Theorem 2 in [6].

II(b) follows from Lemma 1(b) and Theorem 1 in [6]. □

Theorem 2.

(2)

$$\begin{aligned} &\left\| \int_{0 \leqslant x < \infty} \alpha H_{t-x}(d\pi, ds) \cdot (1 - F^\pi(x)) m(dx) - \mu_\pi \cdot \alpha H_t(d\pi, ds) \right\| \\ &\qquad = \left\| \int_{0 \leqslant x < \infty} (\alpha H_{t-x}(d\pi, ds) - \alpha H_t(d\pi, ds)) \cdot (1 - F^\pi(x)) m(dx) \right\|. \end{aligned}$$

Now let A be the set on which the signed measure

$$\int_{0 \leqslant x < \infty} (\alpha H_{t-x}(d\pi, ds) - \alpha H_t(d\pi, ds)) \cdot (1 - F^\pi(x)) m(dx)$$

is positive. Then

$$\|\int_{0\leqslant x<\infty}(\alpha H_{t-x}(d\pi, ds) - \alpha H_t(d\pi, ds))\cdot(1 - F^{\pi}(x))m(dx)\|$$
$$= \int_{\Pi}\big[\int_{0\leqslant x<\infty}(\alpha H_{t-x}(d\pi, A^{\pi}) - \alpha H_t(d\pi, A^{\pi})\cdot(1 - F^{\pi}(x))m(dx)\big]$$
$$= \int_{0\leqslant x<\infty}m(dx)\big[\int_{\Pi}(\alpha H_{t-x}(d\pi, A^{\pi}) - \alpha H_t(d\pi, A^{\pi}))\cdot(1 - F^{\pi}(x))\big]$$

Now for x fixed,

$$\sup_{A\in\mathcal{E}_+}\big[\int_{\Pi}(\alpha H_{t-x}(d\pi, A^{\pi})\text{-}\alpha H_t(d\pi, A^{\pi}))\cdot(1 - F^{\pi}(x))\big]$$
$$\leqslant \|\alpha H_{t-x}(d\pi, ds) - \alpha H_t(d\pi, ds)\|\cdot(1 - G(x)).$$

Hence, substituting the above bound into (2) gives:

$$\|\int_{0\leqslant x<\infty}\alpha H_{t-x}(d\pi, ds)\cdot(1 - F^{\pi}(x))m(dx) - \mu_{\pi}\cdot\alpha H_t(d\pi, ds)\|$$
$$\leqslant \int_{0\leqslant x<\infty}\|\alpha H_{t-x}(d\pi, ds) - \alpha H_t(d\pi, ds)\|\cdot(1 - G(x))m(dx).$$

Therefore (2) tends to 0 as $t\to\infty$ by hypothesis, and dominated convergence. Next

$$\alpha U(d\pi, x + ds)m(dx) = \alpha U(d\pi, s + dx)m(ds).$$

Thus

$$\alpha H_{t-x}(d\pi, ds)\cdot(1 - F^{\pi}(x))m(dx)$$
$$= \alpha U(d\pi, t - x - ds)\cdot(1 - F^{\pi}(s))\cdot(1 - F^{\pi}(x))m(dx)$$
$$= \alpha U(d\pi, t - s - dx)\cdot(1 - F^{\pi}(s))\cdot(1 - F^{\pi}(x))m(ds)$$
$$= \alpha H_{t-s}(d\pi, dx)\cdot(1 - F^{\pi}(s))\cdot m(ds).$$

Therefore

$$(3)\quad \|\int_{0\leqslant x<\infty}\alpha H_{t-x}(d\pi, ds)\cdot(1 - F^{\pi}(x))m(dx)$$
$$- (1 - F^{\pi}(s))m(ds)\cdot\alpha H_t(d\pi)\|$$
$$= \|[\alpha H_{t-s}(d\pi) - \alpha H_t(d\pi)]\cdot(1 - F^{\pi}(s))\cdot m(ds)\|.$$

Let $A\in\mathcal{E}_+$ be the set on which the signed measure $[\alpha H_{t-s}(d\pi) - \alpha H_t(d\pi)]\cdot(1 - F^{\pi}(s))m(ds)$ is positive. Thus

$$\|[\alpha H_{t-s}(d\pi) - \alpha H_t(d\pi)]\cdot(1 - F^{\pi}(s))\cdot m(ds)\|$$
$$= \int_{0\leqslant s<\infty}m(ds)\int_{A^s}[\alpha H_{t-s}(d\pi) - \alpha H_t(d\pi)]\cdot(1 - F^{\pi}(s))$$

where $A^s = \{\pi|(\pi, s)\in A\}$. Now for s fixed

$$\sup_{G\in\mathcal{G}}\int_G[\alpha H_{t-s}(d\pi) - \alpha H_t(d\pi)]\cdot(1 - F^{\pi}(s))$$
$$\leqslant \|\alpha H_{t-s}(d\pi) - \alpha H_t(d\pi)\|\cdot(1 - G(s)).$$

Hence

$$\|\int_{0\leqslant x<\infty}\alpha H_{t-x}(d\pi, ds)\cdot(1 - F^{\pi}(x))m(dx) - (1 - F^{\pi}(s))m(ds)\cdot\alpha H_t(d\pi)\|$$
$$\leqslant \int_{0\leqslant s<\infty}\|\alpha H_{t-s}(d\pi) - \alpha H_t(d\pi)\|\cdot(1 - G(s)m(ds).$$

By hypothesis, $\lim_{t\to\infty}\|\alpha H_{t-s}(d\pi) - \alpha H_t(d\pi)\| = 0$ so by dominated convergence (3) tends to 0 as t tends to ∞. The result follows. □

Corollary 1. Divide by μ_π. □

Proposition 2. Let η^ε be the uniform probability measure on $\pi_0 \times [0, \varepsilon]$ for any $\varepsilon > 0$. Since $\inf_\pi \mu_\pi > 0$ and $F^\pi(x) \geqslant G(x)\ \forall \pi, x$ there exists an $\tilde{l} > 0$ such that $1 - F^\pi(\tilde{l}) > \alpha > 0\ \forall \pi$. Hence by Corollary 1 for $0 \leqslant l \leqslant \tilde{l}$

$$\lim_{t\to\infty}\left\| \int_{0\leqslant s\leqslant l} \frac{1}{(1 - F^\pi(s))} \cdot \left[\eta^\varepsilon H_t(d\pi, ds) - \eta_t^\varepsilon(d\pi, ds)\right]\right\| = 0$$

since $(1/(1 - F^\pi(s)))\chi_{[0,l]}(s)$ is bounded. Using

$$\eta^\varepsilon U(d\pi, t - ds) \cdot (1 - F^\pi(s)) = \eta^\varepsilon H_t(d\pi, ds)$$

we have

$$\lim_{t\to\infty}\left\|\eta^\varepsilon U(d\pi, [t - l, t]) - \frac{l}{\mu_\pi}\eta^\varepsilon H_t(d\pi)\right\| = 0. \tag{4}$$

Now

$$\begin{aligned}\delta^0 U(G \times [t - l, t]) &= \Pi^{(\pi_0, 0)}\{\textstyle\sum_{n=0}^\infty \chi_{G\times[t-l,t]}(I_n, S_n)\}\\ &= \Pi^{(\pi_0, 0)}\{\textstyle\sum_{n=0}^\infty \chi_{G\times[t-l,t]}(I_n, S_n)|X_1 > s\} \cdot P^{(\pi_0,0)}\{X_1 > s\}\\ &\quad + \Pi^{(\pi_0, 0)}\{\textstyle\sum_{n=0}^\infty \chi_{G\times[t-l,t]}(I_n, S_n)|X_1 \leqslant s\} \cdot P^{(\pi_0,0)}\{X_1 \leqslant s\}.\end{aligned}$$

Clearly,

$$\begin{aligned}\Pi^{(\pi_0, 0)}\{\textstyle\sum_{n=0}^\infty \chi_{Gx[t-l,t]}(I_n, X_n)|X_1\} &\leqslant 1 + 2\alpha + 3(1 - \alpha)\alpha\\ &\quad + \cdots + n(1 - \alpha)^{n-2}\alpha + \cdots \quad \text{a.s.-}\Pi^{(\pi_0, 0)}\end{aligned}$$

and hence is uniformly bounded, by M say, for all t and all $0 \leqslant l \leqslant \tilde{l}$. Pick $\varepsilon > 0$ sufficiently small that $(1 + M) \cdot P^{(\pi_0,0)}\{X_1 \leqslant \varepsilon\} < \varepsilon_1$ where ε_1 is arbitrarily small. Then

$$|\delta^0 U(G \times [t - l, t]) - \Pi^{(\pi_0, 0)}\{\textstyle\sum_{n=0}^\infty \chi_{G\times[t-l,t]}(I_n, S_n)|X_1 > s\}| \leqslant 2\varepsilon_1$$

uniformly in $G \in \mathcal{G}$, t, $0 \leqslant l \leqslant \tilde{l}$ and $0 \leqslant s \leqslant \varepsilon$. Since

$$\delta_{(\pi_0, s)} U(G \times [t - l - \varepsilon, t]) = \Pi^{(\pi_0, 0)}\{\textstyle\sum_{n=0}^\infty \chi_{G\times[t+s-l-\varepsilon, t+s]}(I_n, S_n)|X_1 > s\}$$

and

$$\delta_{(\pi_0, s)} U(G \times [t - l, t - \varepsilon]) = \Pi^{(\pi_0, 0)}\{\textstyle\sum_{n=0}^\infty \chi_{G\times[t+s-l, t+s-\varepsilon]}(I_n, S_n)|X_1 > s\}$$

we have for $0 \leqslant s \leqslant \varepsilon$

$$\begin{aligned}\delta^0 U(G \times [t - l, t]) &\leqslant \delta_{(\pi_0, s)} U(G \times [t - l - \varepsilon, t]) + 2\varepsilon_1\\ \delta^0 U(G \times [t - l, t]) &\geqslant \delta_{(\pi_0, s)} U(G \times [t - l, t - \varepsilon]) - 2\varepsilon_1\end{aligned}$$

by an analysis of trajectories. Hence

$$\eta^{\varepsilon}U(G \times [t-l, t-\varepsilon]) - 2\varepsilon_1 \leqslant \delta^0 U(G \times [t-l, t]) \leqslant \eta^{\varepsilon}U(G \times [t-l-\varepsilon, t]) + 2\varepsilon_1.$$

So

$$\left\| \delta^0 U(d\pi, [t-l, t]) - \frac{l}{\mu_\pi}\cdot \eta^{\varepsilon}H_t(d\pi) \right\| \leqslant \left\| \eta^{\varepsilon}U(d\pi, [t-l-\varepsilon, t]) - \frac{l}{\mu_\pi}\cdot \eta^{\varepsilon}H_t(d\pi) \right\| + 2\varepsilon_1 + \left\| \eta^{\varepsilon}U(d\pi, [t-l, t-\varepsilon]) - \frac{l}{\mu_\pi}\cdot \eta^{\varepsilon}H_t(d\pi) \right\| + 2\varepsilon_1.$$

Thus by (4)

$$\limsup_{t\to\infty} \left\| \delta^0 U(d\pi, [t-l, t]) - \frac{l}{\mu_\pi}\cdot \eta^{\varepsilon}H_t(d\pi) \right\| \leqslant \limsup_{t\to\infty} \left\| \frac{\varepsilon}{\mu_\pi}\cdot \eta^{\varepsilon}H_t(d\pi) \right\| + 2\varepsilon_1 + \limsup_{t\to\infty} \left\| \frac{\varepsilon}{\mu_\pi}\cdot \eta^{\varepsilon}H_t(d\pi) \right\| + 2\varepsilon_1 \leqslant \frac{2\varepsilon}{\inf_\pi \mu_\pi} + 4\varepsilon_1$$

uniformly in $0 \leqslant l \leqslant \tilde{l} - \varepsilon$. A similar analysis of $\delta^0 H_t(\mathrm{d}\pi)$ gives

$$|\delta^0 H_t(G \times R_+) - \delta_{(\pi_0, s)}H_t(G \times R_+)| \leqslant \delta_{(\pi_0, s)}H_t(\Pi \times [0, \varepsilon]) + 2\varepsilon_1 \quad \forall G \in \mathcal{G}, \quad 0 \leqslant s \leqslant \varepsilon.$$

Hence

$$\limsup_{t\to\infty} \left\| \frac{l}{\mu_\pi}\cdot \eta^{\varepsilon}H_t(d\pi) - \frac{l}{\mu_\pi}\delta^0 H_t(d\pi) \right\| \leqslant \limsup_{t\to\infty} \frac{\tilde{l}}{\inf_\pi \mu_\pi}[\eta^{\varepsilon}H_t(\Pi \times [0, \varepsilon])] + 2\varepsilon_1 \leqslant \frac{\tilde{l}}{\inf_\pi \mu_\pi} \limsup_{t\to\infty} \left[\int_\Pi \int_0^\varepsilon \frac{(1 - F^\pi(s))}{\mu_\pi} m(ds)\cdot \eta^{\varepsilon}H_t(d\pi) \right] + 2\varepsilon_1$$

(see Corollary 1)

$$\leqslant \frac{\tilde{l}}{(\inf_\pi \mu_\pi)^2}\varepsilon + 2\varepsilon_1.$$

Gathering our results we have

$$\limsup_{t\to\infty} \left\| \delta^0 U(d\pi, [t-l, t]) - \frac{l}{\mu_\pi}\delta^0 H_t(d\pi) \right\|$$

is arbitrarily small for $0 \leqslant l \leqslant \tilde{l} - \varepsilon$. Thus

$$\lim_{t\to\infty}\left\|\delta^0 U(d\pi, [t-l, t]) - \frac{l}{\mu_\pi}\delta^0 H_t(d\pi)\right\| = 0,$$

for $0 \leqslant l \leqslant \tilde{l} - \varepsilon$. The desired result now follows by linearity. □

Corollary 2. We may discount the case where the support of α is not in $\{(\pi, s) \in E \mid 1 - F^\pi(s) > 0\}$ by waiting a time $\delta > 1$ and redefining α as αH_δ. Next by the proof of Proposition 2 the case where $\alpha = \delta_{(\pi_0, s_0)}$ follows. However

$$\left\|\alpha U(d\pi, [t-h, t]) - \frac{h}{\mu_\pi}\alpha H_t(d\pi)\right\|$$
$$\leqslant \int \alpha(d\pi_0, ds_0)\left\|\delta_{(\pi_0, s_0)}U(d\pi, [t-h, t]) - \frac{h}{\mu_\pi}\delta_{(\pi_0, s_0)}H_t(d\pi)\right\|.$$

The result follows by dominated convergence. □

Theorem 3. Let $\bar{h}$ be as in Definition 4. For $0 < h < \bar{h}$

$$\left\|\int_{0\leqslant s<\infty}\bar{z}(\pi, t-s)\alpha U(d\pi, ds) - \frac{1}{\mu_\pi}\int_{0\leqslant s<\infty}\bar{z}(\pi, s)\cdot m(ds)\cdot\alpha H_t(d\pi)\right\|$$
$$\leqslant \left\|\Sigma_{k=1}^{\infty}\left[\bar{z}_k^\pi\cdot\left(\alpha U\left(d\pi, \left[t-kh, t-(k-1)h\right)\right) - \frac{h}{\mu_\pi}\cdot\alpha H_t(d\pi)\right)\right]\right\|$$
$$\leqslant \Sigma_{k=1}^{\infty}\sup_\pi \bar{z}_k^\pi\cdot\left\|\alpha U(d\pi, [t-kh, t-(k-1)h)) - \frac{h}{\mu_\pi}\cdot\alpha H(d\pi)\right\|.$$

By Corollary 2

$$\lim_{t\to\infty}\left\|\alpha U\left(d\pi, [t-kh, t-(k-1)h\right) - \frac{h}{\mu_\pi}\cdot\alpha H_t(d\pi)\right\| = 0$$

for all k. Moreover, by the proof of Proposition 1 it is clear that for $0 \leqslant h \leqslant \tilde{l}$

$$\left\|\alpha U\left(d\pi, ([t-kh, t-(k-1)h]) - \frac{h}{\mu_\pi}\cdot\alpha H_t(d\pi)\right)\right\|$$
$$\leqslant \alpha U(\Pi\times([t-kh, t-(k-1)h])) + \frac{h}{\inf_\pi \mu_\pi}$$

is uniformly bounded for all t. Therefore by dominated convergence

$$(5)\quad \lim_{t\to\infty}\left\|\int_{0\leqslant s<\infty}\bar{z}(\pi, t-s)\cdot\alpha U(d\pi, ds) - \frac{1}{\mu_\pi}\int_{0\leqslant s<\infty}\bar{z}(\pi, s)\cdot m(ds)\cdot\alpha H_t(d\pi)\right\|$$
$$= 0 \quad \text{if } h \text{ is sufficiently small.}$$

Similarly

$$(6)\quad \lim_{t\to\infty}\left\|\int_{0\leqslant s<\infty}\underline{z}(\pi, t-s)\alpha U(d\pi, ds) - \frac{1}{\mu_\pi}\int_{0\leqslant s<\infty}\underline{z}(\pi, s)m(ds)\cdot\alpha H_t(d\pi)\right\|$$
$$= 0.$$

Now $\underline{z}(\pi, s) \leqslant z(\pi, s) \leqslant \bar{z}(\pi, s)$ so

$$\left\| \int_{0 \leqslant s < \infty} z(\pi, t-s) \cdot \alpha U(d\pi, ds) - \frac{1}{\mu_\pi} \int_{0 \leqslant s < \infty} z(\pi, s) \cdot m(ds) \cdot \alpha H_t(d\pi) \right\|$$

$$(7) \quad \leqslant \left\| \int_{0 \leqslant s < \infty} \bar{z}(\pi, t-s) \cdot \alpha U(d\pi, ds) - \frac{1}{\mu_\pi} \int_{0 \leqslant s < \infty} z(\pi, s) \cdot m(ds) \cdot \alpha H_t(d\pi) \right\|$$

$$(8) \quad + \left\| \int_{0 \leqslant s < \infty} \underline{z}(\pi, t-s) \cdot \alpha U(d\pi, ds) - \frac{1}{\mu_\pi} \int_{0 \leqslant s < \infty} z(\pi, s) \cdot m(ds) \cdot \alpha H_t(d\pi) \right\|.$$

Now

$$\left\| \int_{0 \leqslant s < \infty} \bar{z}(\pi, t-s) \cdot \alpha U(d\pi, ds) - \frac{1}{\mu_\pi} \int_{0 \leqslant s < \infty} z(\pi, s) \cdot m(ds) \cdot \alpha H_t(d\pi) \right\|$$
$$\leqslant \left\| \int_{0 \leqslant s < \infty} \bar{z}(\pi, t-s) \cdot \alpha U(d\pi, ds) - \frac{1}{\mu_\pi} \int_{0 \leqslant s < \infty} \bar{z}(\pi, s) \cdot m(ds) \cdot \alpha H_t(d\pi) \right\|$$
$$+ \left\| \frac{1}{\mu_\pi} \int_{0 \leqslant s < \infty} \bar{z}(\pi, s) \cdot m(ds) \cdot \alpha H_t(d\pi) - \frac{1}{\mu_\pi} \int_{0 \leqslant s < \infty} z(\pi, s) m(ds) \cdot \alpha H_t(d\pi) \right\|.$$

The first quantity is (5). The second may be made arbitrarily small since $\int_{0 \leqslant s < \infty} \bar{z}(\pi, s) \cdot m(ds)$ tends to $\int_{0 \leqslant s < \infty} z(\pi, s) \cdot m(ds)$ uniformly in π as h tends to 0. Therefore $\lim_{t \to \infty}(7)$ is arbitrarily small as h tends to 0. Similarly $\lim_{t \to \infty}(8)$ is arbitrarily small as h tends to 0. The result follows. □

Corollary 3. Apply Theorem 3.

Theorem 4.

$$\mathbf{P}^{\delta(\pi, 0)}\{V_s \in A | X_1 > s\} \cdot (1 - F^\pi(s)) \quad \text{is u.d.r.}$$

By Corollary 3

$$\lim_{t \to \infty} \left\| \int_{0 \leqslant s < \infty} \mathbf{P}^{\delta(\pi, 0)}\{V_s \in A | X_1 > s\} \cdot [\alpha H_t(d\pi, ds) - \alpha_t(d\pi, ds)] \right\| = 0.$$

However

$$\mathbf{P}^\alpha\{V_t \in A, I(t) \in d\pi\} = \mathbf{P}^\alpha\{V_t \in A | (I(t), Z(t)) = (\pi, s)\} \cdot \alpha H_t(d\pi, ds)$$
$$= \mathbf{P}^{\delta(\pi, 0)}\{V_s \in A | X_1 > s\} \cdot \alpha H_t(d\pi, ds)$$

and

$$\int_{0 \leqslant s < \infty} \mathbf{P}^{\delta(\pi, 0)}\{V_s \in A | X_1 > s\} \cdot \alpha_t(d\pi, ds)$$
$$= \int_{0 \leqslant s < \infty} \mathbf{P}^{\delta(\pi, 0)}\{V_s \in A; X_1 > s\} \frac{1}{\mu_\pi} \cdot \alpha H_t(d\pi) \cdot m(ds)$$
$$= \frac{A_\pi}{\mu_\pi} \alpha H_t(d\pi).$$

Gathering up these results gives the theorem. □

REMARK 1. First consider $\alpha = \delta^0$, $\beta = \eta^\varepsilon$ where δ^0 and η^ε are as in the proof of Proposition 2. From the proof of Proposition 2 we have

$$\|\delta^0 H_t(d\pi) - \eta^\varepsilon H_t(d\pi)\| \leqslant \eta^\varepsilon H_t(\Pi \times [0, \varepsilon]) + 2\varepsilon_1$$

uniformly in t. Hence

$$\limsup_{t\to\infty}\|\delta^0 H_t(d\pi) - \eta^\varepsilon H_t(d\pi)\| \leqslant \int_\Pi \int_{0\leqslant s\leqslant\varepsilon} \frac{1 - F^\pi(s)}{\mu_\pi} \cdot m(ds) \cdot \eta^\varepsilon H_t(d\pi) + 2\varepsilon_1$$

$$\leqslant \frac{\varepsilon}{\inf_\pi \mu_\pi} + 2\varepsilon_1.$$

Now if γ is a.c.-m then by Proposition 1.I(b)

$$\lim_{t\to\infty}\|\eta^\varepsilon H_t(d\pi) - \gamma H_t(d\pi)\| = 0.$$

Hence

$$\limsup_{t\to\infty}\|\delta^0 H_t(d\pi) - \gamma H_t(d\pi)\| \leqslant \frac{\varepsilon}{\inf_\pi \mu_\pi} + 2\varepsilon_1.$$

Therefore $\lim_{t\to\infty}\|\delta^0 H_t(d\pi) - \gamma H_t(d\pi)\| = 0$. Proceeding as in Corollary 2 we have $\lim_{t\to\infty}\|\alpha H_t(d\pi) - \gamma H_t(d\pi)\| = 0$ for any probability measure α on $(E_+, \mathcal{E}_+)$. The result now follows easily. □

EXAMPLE. Let $\Pi = \{1, 2, \cdots\}$, $\mathcal{G}$ be the set of subsets of Π. The $\{U_n\}_{n=1}^\infty$, $\{D_n\}_{n=1}^\infty$ are defined on some probability space $\{\Omega', \mathcal{F}', P'\}$. We remark that $(I_n, X_n) = (n_0 + n, (U_{n_0+n-1} + D_{n_0+n-1}))$ $(= (n_0, x)$ for $n = 0)$ defines a semi-Markov chain with initial distribution $\delta_{(n_0, x)}$. $(I_n, X_n)_{n=0}^\infty$ is defined on $\{\Omega', \mathcal{F}', P'\}$ but if the initial distribution is $\delta_{(n_0, x)}$ denote P' by $E^{(n_0, x)}$. Now on $\{\Omega', \mathcal{F}', E^{(n_0, 0)}\}$ define

$$\begin{aligned} V_t &= 1 \quad \text{if } S'_{n-1} \leqslant t < S'_{n-1} + U_{n_0+n} \quad \text{for some } n \\ &= 0 \quad \text{otherwise} \quad (S'_n = \Sigma_{k=1}^n X_k). \end{aligned}$$

V_t is a regenerative process with embedded semi-Markov chain $(I_n, X_n)_{n=0}^\infty$ defined on $\{\Omega', \mathcal{F}', E^{(n_0, 0)}\}$. Also setting $A = \{1\}$,

$$E^{(n_0, 0)}\{V_s \in A, X_1 > s\} = P'\{U_{n_0} > s\}.$$

By monotonicity $K(n, s; A) = P'\{U_n > s\}$ is u.d.r. The hypotheses of Theorem 4 are satisfied hence

$$\lim_{t\to\infty}\left|E^{(1,0)}\{V_t \in A\} - \Sigma_{k=1}^\infty \frac{{}^1\mu_k}{{}^1\mu_k + {}^2\mu_k} \cdot P'\{S_{k-1} \leqslant t < S_k\}\right|$$

$$= 0, \quad \text{since } {}^1\mu_k = \int_{0\leqslant s<\infty} P'\{U_k > s\}\, ds.$$

Finally

$$\text{Prob}\{\text{in state } E_1 \text{ at time } t\} = E^{(1,0)}\{V_t \in A\}.$$

This gives the result. □

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