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A LOCAL LIMIT THEOREM FOR LARGE DEVIATIONS OF SUMS OF INDEPENDENT, NONIDENTICALLY DISTRIBUTED RANDOM VARIABLES

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A local limit theorem is given for large deviations of sums of independent, nonidentically distributed, integer valued random variables.

Introduction. Let $\xi_{n1}, \xi_{n2}, \cdots, \xi_{nn}$ $(n = 1, 2, \cdots)$ be an array of integer valued random variables such that for each n, $\xi_{n1}, \cdots, \xi_{nn}$ are independent. The local limit theorem for large deviations deals with the asymptotic behaviour of

$$p_n(x) = P(\xi_{n1} + \xi_{n2} + \cdots + \xi_{nn} = x)$$

as $n \to \infty$, when the integer x increases with n. For nonidentically distributed ξ_{nk}, a local limit theorem for large deviations is given in [2]. Here we give conditions which are easier to check and which yield a simpler proof using the "Bernoulli part" decomposition introduced in [1].

Results. Let $\mu_{nk} = E\xi_{nk}$, $B_n^2 = \Sigma_{k=1}^n E(\xi_{nk} - \mu_{nk})^2$ and $A_n = \Sigma_{k=1}^n \mu_{nk}$ (all notation is as in [2]). Define the following conditions:

(I) $\limsup_{n\to\infty} \frac{1}{n} \Sigma_{k=1}^n E \exp a|\xi_{nk}| < \infty$ for some positive constant a.

(II) There exists a constant $c > 0$ such that

$$\liminf_{n\to\infty} \frac{1}{n} B_n^2 \geqslant c.$$

(III′) $\liminf_{n\to\infty} \frac{1}{n} \Sigma_{k=1}^n [\Sigma_{j=-\infty}^{\infty} \min\{P(\xi_{nk} = j), P(\xi_{nk} = j + 1)\}] > 0.$

Note that (I) and (II) are as in [2]. Condition (III′) here replaces (III) in [2]. We show

THEOREM 1. *Suppose conditions* (I), (II) *and* (III′) *are fulfilled and let* $\omega(n)$ *be a sequence such that* $\lim_{n\to\infty}\omega(n) = \infty$; *then*

$$p_n(x) = \frac{1}{(2\pi)^{\frac{1}{2}} B_n} \exp\left(\frac{-(x - A_n)^2}{2B_n^2} + \frac{(x - A_n)^3}{n^2}\lambda_n\left(\frac{x - A_n}{n}\right)\right)\left(1 + O\left(\frac{x - A_n}{n}\right)\right),$$

uniformity for x *in* $1 \leqslant |x - A_n| \leqslant n/\omega(n)$, *where for each* n, $\lambda_n(\tau)$ *is a special power series converging uniformly with respect to* n *for sufficiently small* τ.

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Lemmas and proofs. For complex z define, as in [2],

$$M_{n,k}(z) = E \exp z(\xi_{nk} - \mu_{nk}) \text{ and } M_n(z) = \Pi_{k=1}^{n} M_{n,k}(z).$$

The proof in [2] essentially involves finding a bound for

$$\int_{\varepsilon \leqslant |t| \leqslant \pi} \left| \frac{M_n(z_o + it)}{M_n(z_o)} \right| dt \tag{1}$$

where z_o is a positive, sufficiently small, real number. Here we obtain a bound for (1) more easily using the following decomposition.

DEFINITION 1. If ξ_{nk} is expressed as $\xi_{nk} = Y_{nk} + \varepsilon_{nk} L_{nk}$ where ε_{nk} and L_{nk} are Bernoulli random variables, such that $P(L_{nk} = 0) = P(L_{nk} = 1) = \frac{1}{2}$ and L_{nk} is independent of $(Y_{nk}, \varepsilon_{nk})$, then $\varepsilon_{nk} L_{nk}$ is called a Bernoulli part of ξ_{nk} (the trivial representation $\varepsilon_{nk} = 0$ and $Y_{nk} = \xi_{nk}$ is always possible).

LEMMA 1. *Let ξ_{nk} be represented as in Definition 1. If $\varepsilon > 0$ then there exists a $\beta > 0$ such that for all k and all t*

$$\left| \frac{E \exp (z_o + it)\xi_{nk}}{E \exp z_o \xi_{nk}} \right| \leqslant \exp(-\beta \alpha_k)$$

where

$$\alpha_k = \frac{E\varepsilon_{nk} \exp z_o Y_{nk}}{E \exp z_o Y_{nk}}.$$

PROOF.

$E \exp (z_o + it)\xi_{nk}$

$$\begin{aligned} &= E \exp \left[(z_o + it)(Y_{nk} + \varepsilon_{nk} L_{nk}) \right] \\ &= E\left\{ \exp \left[(z_o + it) Y_{nk} \right] | \varepsilon_{nk} = 0 \right\} P(\varepsilon_{nk} = 0) \\ &\quad + E\left\{ \exp \left[(z_o + it) Y_{nk} \right] | \varepsilon_{nk} = 1 \right\} . E\left\{ \exp \left[(z_o + it) L_{nk} \right] \right\} . P(\varepsilon_{nk} = 1). \end{aligned}$$

Furthermore,

$$\begin{aligned} E \exp \left[(z_o + it) L_{nk} \right] &= \tfrac{1}{2} + \tfrac{1}{2} \exp (z_o + it) \\ &= \exp \left(\frac{z_o + it}{2} \right) \cosh \left(\frac{z_o + it}{2} \right). \end{aligned}$$

Hence,

$$E \exp (z_o + it)\xi_{nk} = E \exp \left[(z_o + it)\left(Y_{nk} + \tfrac{1}{2}\varepsilon_{nk} \right) \right] \left[\cosh \left(\frac{z_o + it}{2} \right) \right]^{\varepsilon_{nk}}.$$

Also,

$$E \exp z_o \xi_{nk} = E \exp \left[z_o\left(Y_{nk} + \tfrac{1}{2}\varepsilon_{nk} \right) \right] \left[\cosh z_o/2 \right]^{\varepsilon_{nk}}.$$

Hence,

$$\left|\frac{E\exp(z_o+it)\xi_{nk}}{E\exp z_o\xi_{nk}}\right| \leq \frac{E\exp\left[z_o\left(Y_{nk}+\frac{1}{2}\varepsilon_{nk}\right)\right]\left|\cosh\left(\frac{z_o+it}{2}\right)\right|^{\varepsilon_{nk}}}{E\exp\left[z_o\left(Y_{nk}+\frac{1}{2}\varepsilon_{nk}\right)\right]|\cosh z_o|^{\varepsilon_{nk}}}.$$

If $z = x + iy$ then $|\cosh z|^2 = \cosh^2 x - \sin^2 y$. Therefore

$$\left|\cosh\left(\frac{z_o+it}{2}\right)\right|^{\varepsilon_{nk}} = [\cosh z_o/2]^{\varepsilon_{nk}}\left(1-\frac{\sin^2 t/2}{\cosh^2 z_o/2}\right)\frac{\varepsilon_{nk}}{2}.$$

However for $|t| \in [\varepsilon, \pi]$ there exists an $0 \leq \alpha < 1$ such that

$$1-\frac{\sin^2 t/2}{\cosh^2 z_{o/2}} < \alpha^2.$$

Therefore

$$\left|\frac{E\exp(z_o+it)\xi_{nk}}{E\exp z_o\xi_{nk}}\right| \leq \frac{E\exp\left[z_o\left(Y_{nk}+\frac{1}{2}\varepsilon_{nk}\right)\right][\cosh z_{o/2}]^{\varepsilon_{nk}}\alpha^{\varepsilon_{nk}}}{E\exp\left[z_o\left(Y_{nk}+\frac{1}{2}\varepsilon_{nk}\right)\right][\cosh z_{o/2}]^{\varepsilon_{nk}}}.$$

Next,

$$\begin{aligned} E\exp\left[z_o\left(Y_{nk}+\tfrac{1}{2}\varepsilon_{nk}\right)\right][\cosh z_{o/2}]\alpha^{\varepsilon_{nk}} &= E\exp\left[z_o\left(Y_{nk}+\tfrac{1}{2}\varepsilon_{nk}\right)\right][\cosh z_{o/2}]^{\varepsilon_{nk}} \\ &\quad -(1-\alpha)e^{z_{o/2}}\cosh z_{o/2}E\varepsilon_{nk}\exp z_oY_{nk}. \end{aligned}$$

Hence,

$$\begin{aligned} \left|\frac{E\exp(z_o+it)\xi_{nk}}{E\exp z_o\xi_{nk}}\right| &\leq 1-(1-\alpha)e^{z_{o/2}}\cosh(z_{o/2})\frac{E\varepsilon_{nk}\exp z_oY_{nk}}{E\exp\left[z_o\left(Y_{nk}+\frac{1}{2}\varepsilon_{nk}\right)\right][\cosh z_{o/2}]^{\varepsilon_{nk}}} \\ &\leq 1-(1-\alpha)\frac{E\varepsilon_{nk}\exp z_oY_{nk}}{E\exp z_oY_{nk}} \\ &\leq e^{-(1-\alpha)\alpha_k} = e^{-\beta\alpha_k} \text{ where } \beta = 1-\alpha. \qquad \square \end{aligned}$$

DEFINITION 2. Let $q_k = \sum_{j=-\infty}^{\infty}\min\{P(\xi_{nk}=j), P(\xi_{nk}=j+1)\}$, and define $Q_n = \sum_{k=1}^{n} q_k$.

It is shown in [1] that ξ_{nk} may be written as $\xi_{nk} = Y_{nk} + \varepsilon_{nk}L_k$, where $\varepsilon_{nk}L_{nk}$ is a

Bernoulli part of ξ_{nk} and $P(\varepsilon_{nk} = 1) = q_k$. Hence a nontrivial Bernoulli part may be extracted.

For any random variable ξ the above decomposition simply implies the existence of a new probability space $\{\Omega, \mathcal{F}, P\}$ and random variables Y, ε and L defined on it such that

(a) $P(L = 0) = P(L = 1) = \frac{1}{2}$,

(b) $P(\varepsilon = 1) = q = \sum_{j=-\infty}^{\infty} \min\{P(\xi = j), P(\xi = j + 1)\}$, $P(\varepsilon = 0) = 1 - q$,

(c) L is independent of (Y, ε),

(d) $P(\xi = j) = P(Y + \varepsilon L = j)$.

Intuitively we interpret $Y + \varepsilon L$ as follows. We observe Y and then flip a coin (dependent on Y). If the coin is heads (corresponding to $\varepsilon = 1$) we add an independent Bernoulli value L to Y. If the coin is tails (corresponding to $\varepsilon = 0$) we add nothing. q is the probability the coin is heads and hence the probability the independent Bernoulli value L is added to Y. Hence q measures the amount of Bernoulli part in the distribution of ξ.

LEMMA 2. *If* (I) *holds then* (III′) *implies*

$\liminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}\alpha_k > 0$ where α_k is as defined in Lemma 1 with $z_o < \frac{a}{2}$.

PROOF.

$$\begin{aligned}
E \exp z_o|Y_{nk}| &\geqslant E[\varepsilon_{nk} \exp(-z_o Y_{nk})] \\
&= E\{(\exp z_o Y_{nk})^{-1}|\varepsilon_{nk} = 1\}P(\varepsilon_{nk} = 1) \\
&\geqslant \frac{P(\varepsilon_{nk} = 1)}{E\{\exp z_o Y_{nk}|\varepsilon_{nk} = 1\}} \quad \text{by Jensen's inequality,} \\
&= \frac{P^2(\varepsilon_{nk} = 1)}{E\varepsilon_{nk} \exp z_o Y_{nk}} \\
&= \frac{P^2(\varepsilon_{nk} = 1)}{\alpha_k E \exp z_o Y_{nk}} \\
&\geqslant \frac{P^2(\varepsilon_{nk} = 1)}{\alpha_k E \exp z_o |Y_{nk}|}.
\end{aligned}$$

Therefore $P(\varepsilon_{nk} = 1) \leqslant \alpha_k^{\frac{1}{2}} E \exp z_o|Y_{nk}|$. Hence,

$$\begin{aligned}
\left(\frac{1}{n}\sum_{k=1}^{n} P(\varepsilon_{nk} = 1)\right)^2 &\leqslant \left(\frac{1}{n}\sum_{k=1}^{n}\alpha_k^{\frac{1}{2}} E \exp z_o|Y_{nk}|\right)^2 \\
&\leqslant \left(\frac{1}{n}\sum_{k=1}^{n}\alpha_k\right)\left(\frac{1}{n}\sum_{k=1}^{n}(E \exp z_o|Y_{nk}|)^2\right) \\
&\leqslant \left(\frac{1}{n}\sum_{k=1}^{n}\alpha_k\right)\left(\frac{1}{n}\sum_{k=1}^{n}E \exp 2z_o|Y_{nk}|\right) \\
&\leqslant \left(\frac{1}{n}\sum_{k=1}^{n}\alpha_k\right)\left(\frac{1}{n}\sum_{k=1}^{n}E \exp 2z_o[|\xi_{nk}| + 1]\right).
\end{aligned}$$

Therefore,

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}\alpha_k \geq \frac{\left(\liminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} P(\varepsilon_{nk} = 1)\right)^2}{\left(\limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} E \exp 2z_o|\xi_{nk}|\right) \cdot \exp 2z_o}.$$

Since $2z_o < a$, the denominator of the above expression is finite by (I). By (III′) the numerator is positive. Hence $\liminf_{n\to\infty}\sum_{k=1}^{n}\alpha_k > 0$. □

PROOF OF THEOREM 1. The proof of the theorem in [2] depends upon bounding the integral in the expression in equation (8) in [2]. That is, for $\varepsilon > 0$ and $z_o > 0$ sufficiently small (we may take $z_o < a/2$) we must bound (1) here.

$$\left|\frac{M_n(z_o + it)}{M_n(z_o)}\right| \leq \prod_{k=1}^{n}\left|\frac{E \exp (z_o + it)\xi_{nk}}{E \exp z_o\xi_{nk}}\right|$$
$$\leq \exp\left(-\beta\sum_{k=1}^{n}\alpha_k\right)$$

where α_k and β are as in Lemma 1. Also, by Lemma 2, for n sufficiently large, $\exp(-\beta\sum_{k=1}^{n}\alpha_k) \leq \exp(-\beta\delta n)$ where $\liminf 1/n\sum_{k=1}^{n}\alpha_k \geq \delta > 0$. Hence

$$\left|\frac{M_n(z_o + it)}{M_n(z_o)}\right| \leq \exp(-\beta\delta n).$$

This estimate may now be used to complete the proof given in [2]. □
Consider the following independent random variables:

$$P(\xi_{nk} = 0) = \tfrac{1}{2} \quad \text{for all } k,$$
$$P(\xi_{nk} = 2) = \tfrac{1}{2} \quad \text{for } k \text{ odd},$$
$$P(\xi_{nk} = 3) = \tfrac{1}{2} \quad \text{for } k \text{ even}.$$

Clearly condition (III′) here and (III) in [2] are violated. Nevertheless, it is clear that the array

$$\xi_{n1} + \xi_{n2}, \xi_{n3} + \xi_{n4}, \cdots, \xi_{nn-1} + \xi_{nn} \qquad n = 1, 2, \cdots$$

(take $\xi_{n1} + \xi_{n2}, \cdots, \xi_{nn}$ if n is odd) satisfies (I), (II) and (III′). Hence

$$(\xi_{n1} + \xi_{n2}) + \cdots + (\xi_{nn-1} + \xi_{nn}) = \sum_{k=1}^{n}\xi_{nk}$$

satisfies Theorem 1. This "blocking" technique is used in [1].

NOTE. The Bernoulli part decomposition given gives other useful bounds. Suppose $S_n = \sum_{k=1}^{n}\xi_{nk}$ admits the decomposition given in [1] (note: we need not assume $\{\xi_{nk}\}_{k=1}^{n}$ independent):

$$S_n = Z_n + \sum_{k=1}^{N_n} L_k,$$

where N_n is a nonnegative, integer valued random variable and $\{L_k\}_{k=1}^{\infty}$ is a sequence of independent Bernoulli random variables such that $P(L_k = 0) = P(L_k$

$= 1) = \frac{1}{2}$ and $\{L_k\}_{k=1}^{\infty}$ is independent of (Z_n, N_n). If $f_n(s) = E \exp is S_n$ then

$$|f_n(s)| = |E \exp (is[Z_n + \Sigma_{k=1}^{N_n} L_k])|$$
$$= |\Sigma_{m=0}^{\infty} E \exp (is[Z_{n,m} + \Sigma_{k=1}^{m} L_k]) \cdot P(N_n = m)|$$

where $P(Z_{n,m} = z) = P(Z_n = z | N_n = m)$. Hence

$$|f_n(s)| \leqslant \Sigma_{m=0}^{\infty} |E(\exp (is \Sigma_{k=1}^{m} L_k))| \cdot P(N_n = m).$$

However

$$E \exp (is \Sigma_{k=1}^{m} L_k) = \Pi_{k=1}^{m} (\tfrac{1}{2} + \tfrac{1}{2} e^{is}).$$

Hence

$$|E \exp (is \Sigma_{k=1}^{m} L_k)| = (\cos s/2)^m.$$

Therefore

$$|f_n(s)| \leqslant \Sigma_{m=0}^{\infty} (\cos s/2)^m P(N_n = m)$$
$$= E(\cos s/2)^{N_n}.$$

If $\varepsilon > 0$ then for s such that $\varepsilon \leqslant |s| \leqslant \pi$ $\cos s/2 \leqslant \alpha < 1$ for some α. Hence

(2) $$|f_n(s)| \leqslant E\alpha^{N_n}$$

where

$$0 < \alpha < 1, \qquad \text{for } \varepsilon \leqslant s \leqslant \pi.$$

Clearly if $\xi_{n1}, \xi_{n2}, \cdots \xi_{nn}$ are independent and each has the decomposition given in Definition 1 then we may represent S_n as above:

$$S_n = Z_n + \Sigma_{k=1}^{N_n} L_k,$$

where $Z_n = \Sigma_{k=1}^{n} Y_{nk}$ and N_n has the same distribution as $\Sigma_{k=1}^{n} \varepsilon_{nk}$ (see [1]).

Therefore (2) gives:

$$|f_n(s)| \leqslant \Pi_{k=1}^{n} E\alpha^{\varepsilon_{nk}}$$
$$= \Pi_{k=1}^{n} (1 - (1 - \alpha) P(\varepsilon_{nk} = 1))$$
$$\leqslant \exp (-(1 - \alpha) \Sigma_{k=1}^{n} P(\varepsilon_{nk} = 1)),$$
$$\leqslant \exp (-(1 - \alpha) Q_n)$$

where $\alpha < 1$ and Q_n is as in Definition 2.

If $\xi_{n1}, \cdots, \xi_{nn}$ are not independent (2) may still be useful.

REFERENCES

[1] McDonald, D. (1979). Local limit theorems for non-identical integer valued random variables. Unpublished manuscript.

[2] Moskvin, D. A. (1972). A local limit theorem for large deviations. *Theor. Probability Appl.* **17** 678–684.

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