The Annals of Probability
1980, Vol. 8, No. 1, 1–67

SPECIAL INVITED PAPER

OCCUPATION DENSITIES[1]

BY DONALD GEMAN AND JOSEPH HOROWITZ
University of Massachusetts at Amherst

This is a survey article about occupation densities for both random and nonrandom vector fields $X : T \to \mathbb{R}^d$ where $T \subset \mathbb{R}^N$. For $N = d = 1$ this has previously been called the "local time" of X, and, in general, it is the Lebesgue density $\alpha(x)$ of the occupation measure $\mu(\Gamma) =$ Lebesgue measure $\{t \in T: X(t) \in \Gamma\}$. If we restrict X to a subset A of T we get a corresponding density $\alpha(x, A)$ and we will be interested in its behavior both in the space variable x and the set variable A. The first part of the paper deals entirely with nonrandom, nondifferentiable vector fields, focusing on the connection between the smoothness of the occupation density and the level sets and local growth of X. The other two parts are concerned, respectively, with Markov processes ($N = 1$) and Gaussian random fields. Here the emphasis is on the interplay between the probabilistic and real-variable aspects of the subject. Special attention is given to Markov local times (in the sense of Blumenthal and Getoor) as occupation densities, and to the role of local nondeterminism in the Gaussian case.

CONTENTS

Received March 27, 1978.

[1]This work was supported by National Science Foundation grant MCS 76-06599.

AMS 1970 *subject classifications.* 26A27, 60-02, 60G15, 60G17, 60J55.

Key words and phrases. Occupation density, local time, sample functions, Gaussian vector fields, local nondeterminism, path oscillations, Markov processes.

1. INTRODUCTION

0. Brownian occupation density. Let $X(t)$, $t \geqslant 0$, be a (nonrandom) real-valued Borel function. The *occupation measure of X up to time t* is $\mu_t(\Gamma) = \lambda\{s \leqslant t : X(s) \in \Gamma\}$, λ being Lebesgue measure and Γ a Borel set. Thus, $\mu_t(\Gamma)$ is the "amount of time spent by X in the set Γ during $[0, t]$." If $X(t, \omega)$ is the trajectory of a random process, then we have exactly the same definition, but now $\mu_t(\Gamma)$ will depend on the sample point ω as well.

For *Brownian* trajectories $X(t, \omega)$, Lévy (1965*, Section 50) showed that, for almost every trajectory, each $\mu_t(\Gamma)$ could be expressed as the "sum of times spent at each $x \in \Gamma$" in the following sense:

$$\lambda\{s \leqslant t : X(s) \in \Gamma\} = \int_\Gamma \alpha_t(x)\, dx \tag{0.1}$$

for all Borel sets Γ and $t \geqslant 0$, for some function $\alpha_t(x)$ for which $\alpha_0(x) = 0$ and $\alpha_t(x)$ is nondecreasing in t. Lévy called $\alpha_t(x)$ the "mesure du voisinage" and we regard it as the "amount of time spent by X *at* x during $[0, t]$"; it is the progenitor

*The first edition of the book was published in 1948.

of the *local times* for general Markov processes introduced by Blumenthal and Getoor (1964).

Now (0.1) is a purely real-variable statement, and a function X for which it holds, i.e., each μ_t is absolutely continuous, is said to *satisfy condition* (LT). We then call $\alpha_t(x)$ an occupation density to distinguish it from the above-mentioned local times which are sometimes, but not always, occupation densities. (The connection is spelled out in § 18.) It follows from (0.1) that for each $t \geqslant 0$,

$$(0.2) \qquad \alpha_t(x) = \lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\lambda\{s \leqslant t : x - \varepsilon < X(s) < x + \varepsilon\} \quad \text{for a.e. } x.$$

(Actually, this limit exists (finite) for a.e. x for any Borel function $X(t)$; it is a standard result on the differentiation of measures: $\alpha_t = d\mu_t/dx$.) As suggested by (0.2), the points of increase of the function $t \mapsto \alpha_t(x)$ are contained in the *level set* $M_x = \{t : X(t) = x\}$; all of this will be discussed in detail in § 6.

Having recognized the real-variable nature of occupation densities, one is led to study them apart from probability theory (see Part 2). We shall see that there are important connections between the behavior of the function $X(t)$ (such as the size of its level sets and the rate of its local growth) and the behavior of the function $\alpha_t(x)$.

In particular, the occupation density is ideally suited to the study of *nondifferentiable* functions (see, for instance, Example 1 below), and consequently provides a useful tool in the analysis of the sample paths of nondifferentiable random processes. (Occasionally, in fact, the applications simply amount to taking expectations in a real-variable equation.) Conversely, probabilistic methods are used to produce examples of functions with prescribed behavior which cannot be constructed directly. This type of interplay between real and random analysis is exemplified in the case of Fourier analysis by Kahane's survey paper (1971) and book (1968) which are very close in spirit to the present work. Here are four examples which further illustrate the point. (Perhaps the best example within the paper is Corollary (27.10), an explicit case of which is in (30.7).)

EXAMPLE 1. Everyone knows that the Brownian trajectory is nowhere differentiable (i.e., with probability 1). Following Berman (1970) we obtain a much stronger result from Trotter's (1958) theorem by a simple real variable argument. Trotter proved that *the Brownian trajectory has a jointly continuous occupation density* $\alpha_t(x)$.

Let $X(t)$ be a Borel function. If X had a finite derivative at t, there would be a double cone with vertex at $(t, X(t))$ and sufficiently wide aperture so that, for some $\delta > 0$, the entire graph of X lying over $[t - \delta, t + \delta]$ would fall inside the cone.

Suppose $X(t)$ has a jointly continuous occupation density, as does the Brownian trajectory. We will show that, for *every* cone with vertex at $(t, X(t))$, which we visualize as a pair of lines of slopes $\pm M$ going through $(t, X(t))$, t is a point of *zero density* of the set of times s at which $(s, X(s))$ is in the cone. Thus X not only fails to have a finite derivative but also an *approximate derivative*, so-called. The proof is trivial: the density in question is the limit, as $\delta \downarrow 0$, of

$$\frac{1}{2\delta}\lambda\{t - \delta < s < t + \delta : |X(s) - X(t)| < M|s - t|\}$$

$$\leqslant \frac{1}{2\delta}\lambda\{t - \delta < s < t + \delta : |X(s) - X(t)| < M\delta\}$$

$$= \frac{1}{2\delta}\int_{X(t)-M\delta}^{X(t)+M\delta}(\alpha_{t+\delta}(x) - \alpha_{t-\delta}(x))\,dx \to 0.$$

We will see later that the trajectory cannot even be contained in a "cone" with square-root shaped boundaries.

EXAMPLE 2. Here is a special case of a theorem of Meyer (1975): *let the process* $Y = (Y_t)$, *adapted to the natural* σ-*fields of a Brownian motion* $W = (W_t)$, *have trajectories of bounded variation*; *then the trajectories* $W_t + Y_t$ *are* (LT), i.e., *have an occupation density*.

It seems clear that a "tame" function, such as one of bounded variation, cannot dampen the Brownian oscillations enough to destroy (LT). There is a real variable "shadow" of this result (see § 12) which says that, *if a* (*nonrandom*) *function* $X(t)$ *has a sufficiently nice occupation density, then* $X(t) + Z(t)$ *is* (LT) *for any* $Z(t)$ *which is differentiable* a.e. Meyer's theorem is not a special case of this because, although jointly continuous, the Brownian occupation density is not "sufficiently nice". Typically such real variable results require stronger analytical hypotheses than their probabilistic counterparts.

There is a related result which should be mentioned here: *if* $f(t)$ *is an arbitrary* (*nonrandom*) *Borel function, then*, a.s., $W_t + f(t)$ *is* (LT). This is a probabilistic and not a real variable result in that the exceptional null set depends on $f(t)$.

EXAMPLE 3. It has been recently shown by Kaufman (1975) (see also Kahane (1976), page 153), by a rather difficult Fourier analytic argument, that *if* K *is a fixed set in* $[0, 1]$ *of Hausdorff dimension* $\dim K > \frac{1}{2}$, *then the Brownian image* $X(K)$ *has nonempty interior* a.s. Here is a simple proof, due essentially to Pitt (1978), based on occupation densities.

Let ψ be a finite measure carried by K and having a bounded potential with respect to the kernel $|s - t|^{-b}$, where $\frac{1}{2} < b < \dim K$:

$$\sup_{0 \leqslant s \leqslant 1}\int_0^1 |s - t|^{-b}\psi(dt) < \infty.$$

The existence of such a measure is standard. Now, although we shall not do so in this paper, the theory of occupation densities can be developed with other measures than λ on the time domain. Applying the results of §§ 25 and 26 relative to ψ instead of λ, we obtain the existence of an occupation density $\alpha(x, K)$, which is continuous in x, for the restriction of X to K. It follows (see Remark (b) at the end of § 6) that the set $\{x : \alpha(x, K) > 0\}$ is open, nonempty and contained in $X(K)$ a.s.

EXAMPLE 4. Let $X = (X_t)$ be a real-valued process with stationary, independent increments; assume that either a Gaussian component is present or that $M(\mathbb{R} \setminus \{0\}) = \infty$, where M is the Lévy measure. We give an easy proof of the following result of Getoor and Kesten (1972): *the event* $A = \{\omega : X_t(\omega)$ has a jointly continuous occupation density$\}$ *has probability* 0 *or* 1. The same argument will work for the event $\{\omega : X_t(\omega)$ has an occupation density$\}$.

Consider first a nonrandom function $X(t)$ and a *step function* $Z(t) = aI_{[0, t_0)}(t) + bI_{[t_0, 1]}(t)$. If $\alpha_t(x)$ is an occupation density for $X(t)$, then $X(t) + Z(t)$ will have the occupation density

$$\beta_t(x) = \alpha_{t \wedge t_0}(x - a) + \alpha_t(x - b) - \alpha_{t \wedge t_0}(x - b);$$

conversely, existence of $\beta_t(x)$ implies that of $\alpha_t(x)$, and α will be, e.g., jointly continuous iff $\beta_t(x)$ is likewise. Clearly the same will be true for any finite number of steps.

Turning to the *process* X_t, Getoor and Kesten, page 300, show easily that $P(A) = 1$ if the Gaussian component is present; thus, in the remainder of the proof, we may assume it is absent. Let X_t^n be the sum of the jumps during time $(0, t]$ of magnitude greater than $1/n$; this is a step function with a finite number of steps in $[0, 1]$. Thus A is equivalent to the corresponding event for the process $X_t - X_t^n$, for each n. Since A depends only on arbitrarily small jumps, we must have $P(A) = 0$ or 1, according to Theorem 14.30 of Breiman's book, *Probability*.

Finally, since much of our work is based on classical material in the theory of functions of a real variable, the treatise of Saks (1937) (unfortunately now out of print) has become invaluable to us. For this and other reasons we would like to dedicate this paper to the memory of Dr. Stanislaw Saks.

1. Agenda. This paper is a summary of occupation densities with emphasis on the application of real variable results to stochastic processes. The results are formulated for vector fields, and, by specialization, for functions (and processes) of one real variable. Except for Section 2, we shall deal exclusively with *nondifferentiable* functions.

The remainder of Part 1 (§§ 2–5) provides some orientation for the reader. Part 2 (§§ 6–14) takes up the study of occupation densities for nonrandom vector fields; this material has hitherto not been treated systematically. These results are applied in Part 3 (§§ 15–30) to random fields in a sometimes parallel development, e.g., §§ 10 and 11 and §§ 27 and 28. More specifically, §§ 15–20 deal with Markov processes (not fields), §§ 21 and 23 with general random fields, and §§ 22 and 24–30 largely with Gaussian random fields. Each general property described in Part 2 is possessed by a class of processes given in Part 3. The real variable results are summarized in a table in § 14; a corresponding table for processes is given in § 30, where the results of §§ 27 and 28 are illustrated.

Almost all the results have been reworked and are presented in more or less full generality. In addition, a large number of new results are included, among the most interesting of which are (we feel): (i) the results in §§ 9–11 on lack of (approximate) Hölder-type conditions and, in particular, the spiking behavior described in § 11; (ii) the complete Lebesgue decomposition of the occupation measure of a standard Markov process (§ 18); (iii) the generalization to Gaussian random fields of Berman's (1972) result on the uniform Hölder condition satisfied by certain occupation densities; this is applied in § 30 to obtain some new results for special classes of processes.

We give proofs only when they are simple and instructive, or in the case of new results. An exception to this policy is our retelling of the results of Berman (1972) and Pitt (1978) in §§ 24–26; here we have given rather complete proofs, and have added some details, either for clarity or because we felt they were insufficiently treated previously.

Many of the results rest on the classical but somewhat neglected concepts of metric density and approximate limit; an appendix on these points is placed after § 14. Included there is a proof of an often used theorem on approximate limits which we have not seen proven (correctly) hitherto; we hope this will be of general mathematical interest.

We have tried to mention all papers directly related to occupation densities. Any omissions are inadvertant and we hope no one will feel slighted on that account.

2. Smooth functions. Occupation measures appear rarely in real analysis, e.g., as "equi-measurable rearrangements", and in Saks (1937), page 291; occupation densities appear only in Duff (1970) and Sarkhel (1971), both concerned with differentiable functions.

A wide class of functions (including differentiable) for which $\alpha_t(x)$ is a pure jump function in t is treated by us (1976a); Cuzick (1977) treats smooth vector fields by a direct generalization of the same method. We will just describe the situation and point out an interesting connection with physics before leaving smooth functions altogether. The terminology for this discussion is in Federer (1969).

Let X be an *(N, d)-field*, i.e., X maps $T = [0, 1]^N$ into $\mathbb{R}^d$. The choice of T as domain is purely for convenience. Suppose that X is Lipschitzian and so differentiable a.e., and let $J_d(t)$ be the d-dimensional Jacobian at $t \in T$; assume $N \geqslant d$. The occupation measure of X is now defined as $\mu_B(\Gamma) = \lambda_N(B \cap X^{-1}(\Gamma))$, where λ_N is Lebesgue measure on $\mathbb{R}^N$, $B \subset T$, $\Gamma \subset \mathbb{R}^d$ Borel sets. Using the Hausdorff area and co-area theorems of Federer (1969), 3.2.5, 3.2.12, one finds the following decomposition of μ_B:

$$\mu_B(\Gamma) = \int_\Gamma \alpha(x, B)\, dx + \lambda_N\big(B \cap X^{-1}(\Gamma) \cap \{J_d = 0\}\big) \tag{2.1}$$

where

$$\alpha(x, B) = \int_{M_x \cap B \cap \{J_d \neq 0\}} (J_d(t))^{-1} H_{N-d}(dt). \tag{2.2}$$

Here, $M_x = \{t : X(t) = x\}$ and H_{N-d} is the $(N - d)$-dimensional Hausdorff measure on T. Obviously the first term on the right is λ_d-absolutely continuous; in order that (2.1) be the *Lebesgue* decomposition of μ_B we must impose (if $N > d$) higher smoothness on X. The exact statement is based on Sard's theorem (cf. Sternberg (1964), page 45ff.):

(2.3) THEOREM. (a) *If*

$$\lambda_N\{t : J_d(t) = 0\} = 0, \tag{2.4}$$

then X is (LT).

(b) *If $N = d$, then* (2.1) *is the Lebesgue decomposition of μ_B, and X is* (LT) iff (2.4) *obtains.*

(c) *If $N > d$ and X is of class $C^{(k)}$, $k \geq N - d + 1$, then the conclusion of* (b) *holds.*

If $N < d$, $\lambda_d(X(T)) = 0$, so μ_B is singular to λ_d. In this case it is natural to replace λ_d by an appropriate lower-dimensional Hausdorff measure living on $X(T)$. The theory is qualitatively similar to that above, but we shall not pursue it here. Observe that, in the (LT) cases of (2.3), $\alpha(x, B)$ plays the role played by $\alpha_t(x)$ in § 0.

When X is real-valued ($d = 1$), $J_d(t)$ is just the length of the gradient of X. If this is positive a.e. and X is $C^{(k)}$, $k \geq N$, we have (LT), $H_{N-1}(dt)$ in (2.2) becomes surface measure, and

$$\alpha(x, B) = \int_{M_x \cap B} \frac{H_{N-1}(dt)}{|\nabla X(t)|} \quad \text{for a.e. } x. \tag{2.5}$$

The function $\alpha(x) = \alpha(x, T)$ has appeared in the solid state (lattice dynamics) physics literature for $N = 2, 3$, where it is called the "density of states" and physical significance attaches to the so-called *van Hove singularities* of $\alpha(x)$ which appear for $N = 2$ when x is the image of a saddle point.

The function in (2.5) also arises in statistical mechanics (see Khinchin (1949)) under the name "structure function". If X is the Hamiltonian of a conservative dynamical system, then, under appropriate conditions, M_x is a surface of constant energy which is invariant under the natural motion in the phase space of the system. What is more important, *$\alpha(x, dt)$ is a measure on M_x which is preserved by the natural motion*, just as Lebesgue measure is preserved by the motion in the whole phase space (Liouville's theorem). (Here $t = (q_1, \cdots, q_s, p_1, \cdots, p_s)$ is given by the generalized position and momentum coordinates rather than as "time".) According to Khinchin (1949) page 37, the function $\alpha(x)$ "completely determines the most important features of the mechanical structure of the corresponding physical system".

3. Nondifferentiable functions. Here we turn to the main business of the paper: occupation densities for nondifferentiable functions and random processes. These never (to our knowledge) appear in the real analysis literature; indeed, the analytical properties of nondifferentiable functions are only rarely considered.

Most information in the past has been in the literature on Markov processes where the real variable content is either difficult to isolate or is missing altogether.

The subject was first treated somewhat systematically by Berman (1969) who studied the occupation measure μ_1 (§ 0) using Fourier analysis and then applied the results to trajectories of Gaussian processes. The theme running through his work is that "the smoothness of the local time [i.e., occupation density] of a Gaussian process implies the irregularity of its sample functions" (Berman (1972)). As has already been suggested by the examples in § 0, and as will be amply demonstrated in Part 2, this is a *real variable principle* which applies equally well to nonrandom functions.

The main application of occupation densities in the probabilistic setting has been to questions related to Hausdorff measure properties of the level sets (§§ 13 and 29). For our part, we find occupation densities more interesting as a tool for studying the local oscillations of nondifferentiable functions. They will be used, for instance, in § 10, to obtain some insight into a classical problem in real analysis, and in §§ 9–11 the method of Example 1 is extended to give strong results on the lack of (approximate) Hölder conditions. Sections 27 and 28 discuss Gaussian random fields whose trajectories have the analytical properties described in §§ 9–11. Most of these results show implications of properties of the occupation density for the behavior of the function (see especially Table 1 in § 14). One of the general open problems in this theory, specific instances of which are pointed out later, is to find implications in the other direction (i.e., to reverse the arrows in Table 1).

4. Markov processes. After Lévy and Trotter (§ 0), the structure of the Brownian occupation density *as a stochastic process with "time" parameter* x was determined by Knight, Ray and, later, Williams: for certain stopping times T, $\alpha_T(x)$ is a Markov process (in x) expressed in terms of Bessel processes. This allows a detailed analysis of $\alpha_t(x)$ which shows that it cannot be a very smooth function of x. The material in § 14 then suggests that *the Brownian trajectory cannot be "too irregular"* (cf. Kahane (1976)). An account of the work of Ray and Knight is given by Itô and McKean (1965) along with many applications of the Brownian occupation density; more recent material is summarized by McKean (1975), which we found to be tough going in some places. Some more recent real variable applications are indicated in § 16.

Occupation densities for more general Markov processes were obtained by Boylan (1964) and Griego (1967), the latter showing that the Blumenthal-Getoor local time (see below) is an occupation density under appropriate conditions. We remark that the term "local time" is generally used in the Markov literature instead of "occupation density". A potential-theoretic definition of the *local time* $L_t(x)$ at a regular point x for a standard Markov process X was given by Blumenthal and Getoor (1964): it is the (essentially) unique continuous additive functional whose points of increase comprise the level set M_x. As noted by Knight (1971) and

Williams (1969), $L_t(x)$ need *not* always behave like an occupation density. The best result on $L_t(x)$ as an occupation density is in Getoor and Kesten (1972), where one also finds results on joint continuity and the lack thereof; see also Getoor and Millar (1972) and Millar and Tran (1974). Although occupation measures appear in these papers, few implications are drawn concerning the trajectories of X. In § 18 we adapt the method of Getoor and Kesten to give, for the first time, the full Lebesgue decomposition of the occupation measure of X. This is applied in § 19 to a special case of Chung's problem and to the integral representation of additive functionals.

A voluminous literature on the application of Markov local times to the level sets M_x now exists and has been admirably summarized by Fristedt (1973) and Taylor (1973). A particularly remarkable result along these lines is that of Taylor and Wendel (1966) for stable processes, and Fristedt and Pruitt (1972) for the general case, which identifies $L_t(x)$ in terms of a *fixed* Hausdorff measure living on M_x. For processes with stationary, independent increments, the same measure works for almost all levels, which is very striking when construed in real variable terms, since no such behavior can be expected for a general measurable function.

The theory of Markov local times has also been carried far in the direction concerned with the *regenerative property* of the level sets and other "Markov random sets". Since this has little to do with occupation densities we will just refer the reader to the literature, e.g., Maisonneuve and Meyer (1974).

5. Gaussian processes. S. Berman initiated the study of Gaussian occupation densities (under the rubric "local times") in his papers (1969–1972) using Fourier analysis for the real variable part of the work. Another early contribution is Orey's (1970). Both authors study the Hausdorff dimension properties of the level sets. More recent variations on the same theme are given by Marcus (1976) and Hawkes (1977) using methods going back to Kahane (1968) who implicitly used occupation densities for Gaussian Fourier series. Our papers, Geman (1976, 1977, 1977a) and Geman and Horowitz (1976), are in the same spirit as Berman's but use more direct real variable techniques. A much finer result is that of Davies (1976, 1977) who gives the Gaussian analogue of the Wendel-Taylor theorem mentioned in § 4; part of the result is simplified by Kôno (1977).

Generalizations to Gaussian (and other) random *fields* are given by Davydov (1976, 1977), Cuzick (1977), Pitt (1978), Adler (1977, 1977a, 1977b), and Tran (1977); those of Cuzick and Pitt are closest in content and spirit to the real variable approach indicated above.

Section 21 begins with brief descriptions of the various methods used to obtain occupation densities for general random processes.

The main results on (Gaussian) random fields are then taken up in §§ 22–28. There is a short discussion in § 22 of when a Gaussian field is (LT) and we give our own proof of a new result of M. Lifschitz (communicated to us without proof by Davydov), which settles the question for Gaussian processes with orthogonal increments. An interesting consequence of the real variable approach becomes apparent at this point, namely that (LT) is really a property of the increments of the trajectories, hence of the *two-dimensional joint distributions* of the process; this affords an even greater simplification for Gaussian fields. The main result of §§ 25 and 26 is that certain Gaussian fields have an occupation density which is, in a precise sense, jointly continuous in space and "time", and Hölder continuous in the space variable. (The "time" set is $T = [0, 1]^N$ which now is multidimensional; "space" refers to $\mathbb{R}^d$, the state space of the random field.) An important concept (in § 24) is that of *local nondeterminism* which originated with Berman (1972) and was generalized in a slightly different form to random fields by Pitt (1978).

2. NONSTOCHASTIC OCCUPATION DENSITIES

6. Preliminaries. In this section we set down the basic properties of occupation densities for vector fields $X : T \to \mathbb{R}^d$, where, unless otherwise stated, $T = [0, 1]^N$. These are called *(N, d)-fields*.

We write $\mathfrak{B}_k$, λ_k for the Borel sets and Lebesgue measure in $\mathbb{R}^k$, $|x|$ for the usual Euclidean norm of $x \in \mathbb{R}^k$ and $B_k(x, r)$ for the open ball $\{y : |y - x| < r\}$. When no confusion is possible we write "a.e." and "dx" instead of "λ_k-a.e.", "$\lambda_k(dx)$". X is always assumed to be Borel measurable, and $\mathfrak{B}(T)$ denotes the Borel sets in T.

The *occupation measure of X* is

$$\mu_A(B) = \lambda_N(A \cap X^{-1}(B)), \qquad A \in \mathfrak{B}(T), \quad B \in \mathfrak{B}_d. \tag{6.1}$$

Interpreting T as a "time set" (as when $N = 1$), this is the "amount of time spent by X in B during the time period A". When $A = T$, we write simply $\mu(B)$. (Observe that, if X is regarded as a random variable on T, μ is just its distribution). If $\mu \ll \lambda_d$, we will also have $\mu_A \ll \lambda_d$ for every A; X is then said to *satisfy the condition* (LT), or just to "be (LT)" (LT: local time). The Radon-Nikodym derivative $d\mu_A/d\lambda_d$, denoted $\alpha(x, A)$, is then called the *occupation density over A*. Again, we write $\alpha(x)$ for $\alpha(x, T)$. Thus

$$\lambda_N(t \in A : X(t) \in B) = \int_B \alpha(x, A)\, dx, \qquad A \in \mathfrak{B}(T), \quad B \in \mathfrak{B}_d. \tag{6.2}$$

The density $\alpha(x, A)$ carries the interpretation: "the time spent at x during the time period A". This is further clarified as follows:

(6.3) THEOREM. *Suppose X is* (LT); *then a version of $\alpha(x, A)$ may be chosen which is a kernel, i.e.,*

(i) *$\alpha(\cdot, A)$ is $\mathfrak{B}_d$-measurable for each fixed A,*
(ii) *$\alpha(x, \cdot)$ is a finite measure on $\mathfrak{B}(T)$ for each x.*

Such a version will be called an *occupation kernel* and written $\alpha(x, dt)$.

The proof of (6.3) is standard: it is the same as finding regular conditional probabilities. The occupation kernel has, however, some more specialized properties. Define

$$M_x = \{t \in T : X(t) = x\},$$

the *level set at* x.

(6.4) THEOREM. *Let* α *be an occupation kernel; then*

(i) *for every Borel function* $f(t, x) \geqslant 0$ *on* $T \times \mathbb{R}^d$,

$$\int_T f(t, X(t))\, dt = \int_{\mathbb{R}^d} \int_T f(t, x)\alpha(x, dt)\, dx; \tag{6.5}$$

(ii) $\alpha(x, M_x^c) = 0$ a.e.

Equation (6.5) follows easily from (6.2) by standard approximations, and (ii) from (6.5) upon taking $f(t, x) = 1$ if $X(t) \neq x$, $= 0$ otherwise. By being more careful, we can get an even better occupation kernel. Write $\mathcal{H}$ for the family of all "rational boxes" in T, i.e., sets of the form $\Pi_1^N J_i$, where $J_i \subset [0, 1]$ is an interval with rational endpoints.

(6.6) THEOREM. *If* X *is* (LT), *there exists a version of* $\alpha(x, dt)$ *such that*

(i) $\alpha(x, J) = 0$ *if* $x \notin \overline{X(J)}$, $J \in \mathcal{H}$;

(ii) $\alpha(x, M_x^c) = 0$ *for every* x.

This result is due to Berman (1970) in case $N = d = 1$ and (for (ii)) X is continuous. Let K_n be an increasing sequence of compact sets in T such that $L = \cup K_n$ is full (i.e., $\lambda_N(T \setminus L) = 0$) and on each of which X is continuous. Berman's argument (with obvious modifications) shows the existence of kernels $\alpha_n(x, dt)$ such that $\alpha_n(x, dt)$ is carried by K_n, is an occupation kernel for $X|K_n$, and $\alpha_n(x, JK_n) = 0$ if $x \notin \overline{X(JK_n)}$, $J \in \mathcal{H}$.

Since $\alpha_{n+1}(x, dt)$ is an occupation kernel for $X|K_{n+1}$, hence also for $X|K_n$, the measures

$$\alpha_{n+1}(x, dt \cap K_n) \quad \text{and} \quad \alpha_n(x, dt)$$

must agree for a.e. x. Define $\tilde{\alpha}(x, dt) = 0$ on the exceptional x-set. We then have

$$\begin{aligned} \alpha_{n+1}(x, B) &= \alpha_{n+1}(x, BK_{n+1}) \\ &\geqslant \alpha_{n+1}(x, BK_n) \\ &= \alpha_n(x, BK_n) \\ &= \alpha_n(x, B) \end{aligned}$$

for every $B \in \mathcal{B}(T)$ and a.e. x, with equality if $B \subset K_n$. Now for the remaining x's, let

$$\tilde{\alpha}(x, B) = \lim_n \alpha_n(x, B).$$

The existence of the limit is obvious as is the fact that $\tilde{\alpha}$ is an occupation kernel. Finally, the proof of Lemma 1.5 of Berman (1970) shows that $\alpha_n(x, M_x^c) = 0$ for every x, so (ii) holds for $\tilde{\alpha}$. Similarly (i) is immediate.

We remark that, in general, $\tilde{\alpha}(x, T)$ can be zero even on a set of positive measure. On the other hand, for any occupation kernel,

(6.7) $\alpha(X(t), B_N(t, \varepsilon)) > 0$ for all $\varepsilon > 0$, for a.e. $t \in T$

as is seen by taking, in (6.5), $f(t, x) = 0$ if $\alpha(x, B_N(t, \varepsilon)) > 0$ for all $\varepsilon > 0$, and $= 1$ otherwise.

Here is another rather surprising application of (6.5) which relates the behavior of the measures $\alpha(x, dt)$ to that of $\alpha(X(t), ds)$.

(6.8) THEOREM. *The measure $\alpha(x, dt)$ will be atomless for* a.e. x iff $\alpha(X(t), \{t\}) = 0$ *for* a.e. t.

PROOF.

$$\int_T \alpha(X(t), \{t\})\, dt = \int_{\mathbf{R}^d} \alpha(x, \{t\})\alpha(x, dt)\, dx$$
$$= \int_{\mathbf{R}^d} \Sigma_{t \in T}(\alpha(x, \{t\}))^2\, dx.$$

REMARKS. (a) Occasionally it is useful to define the occupation measure relative to a finite measure on T other than Lebesgue measure; Example 3 (§ 0) is an instance of this.

(b) In connection with Example 3 we have the following real-variable result: *suppose X is continuous, K is compact in T, and $\alpha(x, K)$ is an occupation density* (not necessarily a kernel) *which is continuous in x; then $X(K)$ has nonempty interior.* Indeed, $\{x : \alpha(x, K) > 0\}$ is open and is nonempty by a remark similar to (6.7). Since

$$\int_{(X(K))^c} \alpha(y, K)\, dy = 0$$

we find $\alpha(y, K) = 0$ for a.e. $y \notin X(K)$, hence for all such y by continuity. This is especially interesting when $\alpha(x, K)$ is the occupation density relative to a singular measure carried by K, as in Example 3.

(c) Let Q_t be the "quadrant" in T with "upper right corner" at t. In Part 3 we will choose occupation densities with the property that $\alpha(x, Q_t)$ is jointly continuous in (t, x), and we wish to point out that we can retain the properties given in (6.6); specifically, the arguments in Berman (1970) can be easily modified to show *if $\alpha(x, Q_t)$ is jointly continuous, then $\alpha(x, Q_t)$ can be uniquely extended to an occupation kernel for which* (6.6i) *holds; if, in addition, X is continuous,* (6.6ii) *will hold as well.*

7. Existence of α.

We now give some sufficient conditions for an (N, d)-field X to be (LT). In Part 3 we will use these to give simple probabilistic conditions under which almost every trajectory of a *random* vector field will be (LT).

1°. *Via harmonic analysis.* This is Berman's approach. Let

$$\hat{\mu}(\theta) = \int_{\mathbf{R}^d} e^{i\theta \cdot x}\mu(dx), \qquad \theta \in \mathbb{R}^d$$

be the Fourier-Stieltjes transform (= characteristic function) of μ; here $\theta \cdot x$ is the ordinary dot product in $\mathbb{R}^d$. Standard results on characteristic functions tell us, e.g.

that X *is* (LT) *with* $\alpha(\cdot) \in L^2(dx)$ iff

(7.1) $$\int_{\mathbb{R}^d}\int_T\int_T e^{i\theta\cdot(X(s)-X(t))}\, ds\, dt\, d\theta < \infty,$$

since the inner integral is just $|\hat{\mu}(\theta)|^2$ by (6.5). Similar arguments involving integrability of $|\theta|^k|\,\hat{\mu}(\theta)|^2$ will be used later to deduce further regularity properties of $\alpha(x)$.

2°. *Via differentiation theory.* The following is almost classical: *if* ψ *is a finite measure on* $\mathbb{R}^d$, *then*

$$\psi'(x) \equiv \lim_{\varepsilon\downarrow 0}\frac{\psi(B_d(x, \varepsilon))}{\lambda_d(B_d(x, \varepsilon))}$$

exists ψ-a.e. (*finite or not*); *and* $\psi \ll \lambda_d$ iff $\psi'(x) < \infty$ ψ-a.e. Of course $\psi'(x)$ exists (finite) λ_d-a.e. also, but the ψ-a.e. conclusion is what is important here. This is applied to μ, recalling that $\mu(B) = 0$ iff $X(t) \notin B$ a.e., with the following result (Geman and Horowitz (1976) for $N = d = 1$).

(7.2) THEOREM. (a) *The limit*

$$V(t) \equiv \lim_{\varepsilon\downarrow 0}\frac{1}{c_d\varepsilon^d}\int_T I_{(0,\varepsilon)}(|X(s) - X(t)|)\, ds$$

exists ($\leqslant \infty$) *for* a.e. t (c_d *is given by* $\lambda_d(B_d(0, \varepsilon)) = c_d\varepsilon^d$).

(b) X *is* (LT) iff $V(t) < \infty$ *for* a.e. t.

(c) X *is* (LT) *with* $\alpha \in L^2(dx)$ iff

$$\liminf_{\varepsilon\downarrow 0}\varepsilon^{-d}\int_T\int_T I_{(0,\varepsilon)}(|X(s) - X(t)|)\, ds\, dt < \infty.$$

Moreover, under (LT), $V(t) = \alpha(X(t))$ a.e. The proof of (c) rests on

(7.3) $$\int_T \alpha(X(t))\, dt = \int_{\mathbb{R}^d} \alpha^2(x)\, dx$$

which comes from (6.5).

Although these results can be successfully applied to random functions and fields, it is difficult to apply them to particular nonrandom functions. For example an interesting open problem is *to determine which functions representable as Fourier series* (for instance) *are* (LT) *and to compute* α *in terms of the Fourier coefficients.* The same question applies to particular classical functions such as the Weierstrass nowhere differentiable function

$$X(t) = \sum_{k=0}^{\infty} b^k \cos(a^k\pi t), \qquad t \in \mathbb{R}$$

(a an odd positive integer, $0 < b < 1$, $ab \geqslant 1$).

8. Hilbert transform. Let $N = d = 1$. Since

$$\frac{1}{\varepsilon}\int_0^1 I_{(0,\varepsilon)}(|X(s) - X(t)|)\, ds \leqslant \int_0^1 |X(s) - X(t)|^{-1} I_{(0,\varepsilon)}(|X(s) - X(t)|)\, ds,$$

(7.2) leads us to conclude that

(8.1) $$\int_0^1 |X(s) - X(t)|^{-1}\, ds < \infty \text{ a.e.}$$

suffices for (LT). But (8.1) is never true! See the authors (1976) for a more comprehensive result. On the other hand, removing the absolute value marker, we have, formally,

$$\int_0^1 \frac{1}{X(s) - X(t)} = \int_{\mathbb{R}} \frac{\mu(dy)}{y - X(t)} \tag{8.2}$$

where μ is the occupation measure of X. These still will not converge, but it *is* known that

$$h(x) \equiv \lim_{\varepsilon \downarrow 0} \int_{|x-y|>\varepsilon} \frac{\mu(dy)}{y - x} \text{ exists (finite) for a.e. } x, \tag{8.3}$$

(for any finite measure on $\mathbb{R}$): $h(x)$ is the *Hilbert* (*-Stieltjes*) *transform of* μ. If X is (LT), then, for a.e. t, $X(t)$ is not in the exceptional set for (8.3), hence:

$$\lim_{\varepsilon \downarrow 0} \int_{\{s : |X(s) - X(t)| > \varepsilon\}} \frac{ds}{X(s) - X(t)} \text{ exists (finite) a.e.} \tag{8.4}$$

It is apparently not known whether (8.3) holds μ-a.e. as well; if such were the case, then (8.4) would hold even without (LT). If X is (LT) and $\alpha(x)$ satisfies other conditions, then (8.3) will hold for *every* x and so (8.4) for every t. A special case of this is given in Itô and McKean (1965), Exercise 1, page 72: *if* $W(t)$ *is Brownian motion,*

$$\lim_{\varepsilon \downarrow 0} \int_{\{s : |W(s)| > \varepsilon\}} \frac{ds}{W(s)} \text{ exists (finite) a.s.}$$

9. Hölder conditions and γ-variation. The (N, d)-field X satisfies an *approximate Hölder condition of order* γ *at* t_0 if t_0 is a density point (see appendix) for

$$B = \{t : |X(t) - X(t_0)| \leq c|t - t_0|^\gamma\}$$

for some $c > 0$.

(9.1) THEOREM. *Let X be* (LT) *with* $\alpha(x)$ *in* $L^2(dx)$ (resp. $\alpha(x)$ *essentially bounded*); *then. for every* $\gamma > 2N/d$ (resp. $\gamma > N/d$), *every* t_0, *and every* $c > 0$, *the density of* B *is zero at* t_0.

This implies no approximate or *ordinary* Hölder condition at t_0. The latter was proven for $N = d = 1$ by Berman (1969).

PROOF. Take $\psi = 1$, $g(x) = \alpha(x)$, $\phi(\varepsilon) = \varepsilon^\gamma$, and $t = t_0$ in the proof of (10.1) below. An entirely similar (but easier) calculation to the one there, using the Schwarz inequality in the L^2-case, gives the result.

A closely related subject, when $N = d = 1$, is that of γ-variation. Berman (1969a) gives some Fourier-analytic conditions on $\alpha(x, dt)$ which imply infinite γ-variation, but these are too complicated to reproduce, so the reader is referred to the original paper.

10. Smoothness in the set variable. The theme of this section is the relation between $\alpha(x, B)$ as a function of B and the approximate local growth of X. To illustrate, let $X(t)$, $0 \leqslant t \leqslant 1$, be real-valued (i.e., a (1, 1)-field) and call it a *Jarnik function* if it satisfies the condition

$$(J_1(t)) \qquad \text{ap} - \lim_{s \to t} \frac{|X(s) - X(t)|}{|s - t|} = \infty$$

at a.e. t. This behavior was described geometrically in Example 1, § 0, where it was shown that, *if $\alpha_t(x)$ is jointly continuous, then $(J_1(t))$ holds at every t*. It will appear as the special case $p = 0$ of (10.5) below that, *if $\alpha_t(x)$ is continuous in t for* a.e. x, *then X is a Jarnik function* ("is (J_1)" for short). V. Jarnik (1934) constructed a continuous Jarnik function and a function of Baire class 2 satisfying $(J_1(t))$ at every t. As discussed by Saks (1937), page 297, this was quite surprising because, *for an arbitrary function X*,

$$\lim_{s \to t} \frac{|X(s) - X(t)|}{|s - t|} = \infty$$

on at most a set of measure zero. Example 1 thus takes on the added significance that there exist *continuous* functions, viz. the Brownian trajectories, which satisfy $(J_1(t))$ at every t.

We now give a general result of this type for an (N, d)-field X having occupation kernel $\alpha(x, B)$. Assume that $\phi(r)$, $\psi(r)$, $r \geqslant 0$, are continuous, increasing, and vanish at $r = 0$.

(10.1) THEOREM. *Suppose $g(x) \geqslant 0$ is locally integrable and*

$$\alpha(x, B) \leqslant g(x)\psi(\lambda_N(B)) \text{ a.e.}$$

whenever B is a ball with rational center and radius. If

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-N/d} \phi(\varepsilon) \psi^{1/d}(c_N \varepsilon^N) = 0 \qquad (c_N \equiv \lambda_N(B_N(0, 1)))$$

then, for a.e. t (*and for every t if g is continuous*),

$$(10.2) \qquad \text{ap-}\lim_{s \to t} \frac{|X(s) - X(t)|}{\phi(|s - t|)} = \infty.$$

We need to show that for each $q > 0$,

$$(10.3) \qquad \lim_{\varepsilon \downarrow 0} \varepsilon^{-N} \lambda_N\{s \in B_N(t, \varepsilon) : |X(s) - X(t)| \leqslant q\phi(|s - t|)\} = 0$$

for a.e. t (or every t). Set $q = 1$ for simplicity, and let L be the Lebesgue set of $g(x)$. By (LT), $X(t) \in L$ a.e.; and if g is continuous, $L = \mathbb{R}^d$ so that $X(t) \in L$ for every t. Let $t \in X^{-1}(L)$ and let $\varepsilon_n \downarrow 0$. Choose a sequence $\{t_n\}$ of points in $\mathbb{R}^N$ with rational coordinates such that $|t_n - t| \leqslant \varepsilon_n$, and choose rational numbers $\delta_n \downarrow 0$ such that $2\varepsilon_n \leqslant \delta_n \leqslant 3\varepsilon_n$. Then $B_N(t, \varepsilon_n) \subset B_N(t_n, \delta_n)$ for each n and

$$\begin{aligned} &\varepsilon_n^{-N} \lambda_N\{s \in B_N(t, \varepsilon_n) : |X(s) - X(t)| \leqslant \phi(|s - t|)\} \\ &\qquad \leqslant \varepsilon_n^{-N} \lambda_N\{s \in B_N(t_n, \delta_n) : |X(s) - X(t)| \leqslant \phi(\varepsilon_n)\} \\ &\qquad = \varepsilon_n^{-N} \int_{B_d(X(t), \phi(\varepsilon_n))} \alpha(x, B_N(t_n, \delta_n))\, dx \\ &\qquad \leqslant \varepsilon_n^{-N} \psi(c_N \delta_n^N) \cdot \int_{B_d(X(t), \phi(\varepsilon_n))} g(x)\, dx. \end{aligned}$$

The integral term is $O((\phi(\varepsilon_n))^d)$ as $n \to \infty$. Since ϕ is increasing, the entire expression is bounded by a constant times $(\delta_n/3)^{-N}\psi(c_N\delta_n^N)(\phi(\delta_n))^d$, which converges to zero.

When (10.2) holds a.e. we call X a (J_ϕ)*-field* ((J_γ) if $\phi(r) = r^\gamma$). Geometrically, the straight lines bounding the double cone in Example 1, § 0, become curved in the shape of the curve $y = \phi(r)$, and the rest of the discussion there remains the same. It should be remarked, too, that (10.2) implies

$$\limsup_{s \to t} \frac{|X(s) - X(t)|}{\phi(|s - t|)} = \infty$$

which prohibits any kind of true "ϕ-Hölder condition".

The most interesting case of (10.1) is $\phi(r) = r^\gamma$ and $\psi(r) = r^\beta$: *under the hypothesis of* (10.1), *if* $\beta < 1$, *then* X *is* (J_γ) *for every* $\gamma > (1 - \beta)N/d$.

We now formulate a different type of condition on $\alpha(x, dt)$, the so-called [AC-p] *condition*, $0 \leqslant p < N$. For $p = 0$ this means that $\alpha(x, dt)$ *is atomless for* a.e. x. For $0 < p < N$, the meaning is that, *for* a.e. x, $\alpha(x, dt)$ *has a disintegration*

(10.4) $$\alpha(x, A_p \times A_{N-p}) = \int_{A_p} \alpha(s, x, A_{N-p})\lambda_p(ds)$$

for all $A_p \in \mathfrak{B}([0, 1]^p)$, $A_{N-p} \in \mathfrak{B}([0, 1]^{N-p})$, *where* $\alpha(s, x, A)$ *is a kernel on* $[0, 1]^p \times \mathbb{R}^d \times \mathfrak{B}([0, 1]^{N-p})$ *such that* $\alpha(s, x, du)$ *is atomless for* a.e. (s, x). This says that, for almost every $t^* = (t^{(1)}, \cdots, t^{(p)})$, the $(N - p)$-field $s \mapsto X(t^*, s)$ has an occupation kernel which is [AC-0]. The fixing of the first p coordinates here is arbitrary: any other set of p coordinates could have been chosen. We will leave aside the obvious formulation.

Another interpretation of [AC-p] is that the "conditional" measure obtained from $\alpha(x, dt)$ by fixing the last $N - p$ coordinates is absolutely continuous relative to λ_p. For $p = N$ this would be impossible since $\alpha(x, dt)$ lives on the level set M_x, which has Lebesgue measure zero. The following theorem is due to Geman (1977), page 244.

(10.5) THEOREM. *Let* $\alpha(x, dt)$ *be* [AC-p] *for some* p, $0 \leqslant p < N$; *then* X *is* (J_γ) *for* $\gamma = (N - p)/d$.

Thus, for example, if $N = d = 2$ and [AC-1] holds, then for a.e. $t \in [0, 1] \times [0, 1]$, the proportion of s's within ε units of t for which $X(s)$ is within $q\varepsilon^{\frac{1}{2}}$ units of $X(t)$ is asymptotically zero (as $\varepsilon \downarrow 0$) for every $q > 0$.

Obviously Jarnik functions, and the (J_ϕ) functions generally, are very wild. This leads naturally to the (*open*) *question of whether Jarnik functions are always* (LT).

Let us note, finally, the higher dimensional version of Example 1, § 0.

(10.6) THEOREM. *Suppose* $\alpha(x, dt)$ *is atomless for every* x *and* $\alpha(x, B)$ *is continuous in* x *for each rational ball* B; *then* $(J_{N/d}(t))$ *holds at every* t.

11. Smoothness in the space variable. This time we study the influence of $\alpha(x, B)$ as a function of x on the behavior of $X(t)$, first at a fixed $t_0 \in B$, then globally. We write $\xi = X(t_0)$ and assume (for now) B is open and $\alpha(\xi, B) = 0$. The last requirement will be explained below.

(11.1) THEOREM. *Suppose $\alpha(x, B)$ satisfies a Hölder condition of order β at ξ; then, for every $\gamma > N/(d + \beta)$, the condition $(J_\gamma(t_0))$ obtains.*

The requirement that $\alpha(\xi, B) = 0$ is easiest to visualize when $d = 1$. Suppose that $\alpha(x, B)$ is continuous in x, X is continuous, and that X takes on the extreme value $\xi = X(t_0)$ at, e.g., an interior point t_0 of B. If ξ is, say, a maximum, we shall have $\alpha(x, B) = 0$ for $x > \xi$, hence for $x = \xi$. Similarly, if $\alpha(x, B)$ is of class $C^{(k)}(\mathbb{R}^d)$, all partial derivatives of order $< k$ will vanish at ξ, and Taylor's theorem yields a Hölder condition of order k at ξ; thus we obtain (11.1) for $\gamma > N/(d + k)$.

Obviously it would be preferable to work with closed sets B in this context. If $N = 1$, it is shown by Freedman (1971), page 36 that, *if X is* a.e. *nondifferentiable on $[a, b]$ then X has an interior extremum.*

Referring to Example 1, § 0, and the discussion in § 10, we find that, at an interior extremum, one nappe of the cone is lost and X is (mostly) confined to a spike with curvilinear boundaries which points up or down according as ξ is a maximum or minimum. Finally, *if X is* (LT) *and α satisfies* [AC-0], then X is a.e. nondifferentiable and so *X has a dense set of spikes whose sharpness is governed by α via* (11.1). We will derive (11.1) as a special case of the following result, in which $\phi(r)$, $r \geq 0$, is a nonnegative, increasing function with inverse $\hat{\phi}(u)$ such that $\hat{\phi}(0) = 0$ and $\hat{\phi}(u) > 0$ for $u > 0$. Finally, put

$$\psi(u) = \text{ess. sup}_{x \in \bar{B}_d(\xi, u)} |\alpha(x, B) - \alpha(\xi, B)|.$$

(11.2) THEOREM. *Assume*

$$\int_0^\delta \psi(u)(\hat{\phi}(u))^{-N} u^{d-1}\, du < \infty \tag{11.3}$$

and

$$\liminf_{u \downarrow 0} \hat{\phi}(q^{-1}u)/\hat{\phi}(u) > 0 \textit{ for each } q > 0;$$

then (10.2) *holds at $t = t_0$,* i.e., *condition $(J_\phi(t_0))$.*

We must again prove (10.3), with $t = t_0$. We have

$$\varepsilon^{-N}\lambda_N\{s \in B_N(t_0, \varepsilon) : |X(s) - X(t_0)| \leq q\phi(|s - t_0|)\}$$

$$\leq \int_{B_N(t_0, \varepsilon)} |s - t_0|^{-N} I_{(0, q)}\left(\frac{|X(s) - X(t_0)|}{\phi(|s - t_0|)}\right) ds$$

$$\leq \int_{B_N(t_0, \varepsilon)} \left(\hat{\phi}(q^{-1}|X(s) - X(t_0)|)\right)^{-N} ds.$$

This will tend to zero as $\varepsilon \downarrow 0$ if

$$\int_B \left(\hat{\phi}(q^{-1}|X(s) - \xi|)\right)^{-N} ds < \infty. \tag{11.4}$$

But this is just

$$\int_{\mathbb{R}^d} (\hat{\phi}(q^{-1}|x - \xi|))^{-N} \alpha(x, B)\, dx$$

$$\leqslant \int_{B_d(\xi, \delta)} (\hat{\phi}(q^{-1}|x - \xi|))^{-N} \psi(|x - \xi|)\, dx + (\hat{\phi}(q^{-1}\delta))^{-N} \int_{(B_d(\xi, \delta))^c} \alpha(x, B)\, dx,$$

for $\delta > 0$. The second integral is at most $\lambda_N(B)$, hence finite, and a change to polar coordinates shows that the first integral is finite in view of (11.3). The theorem is proven.

We remark that the theorem remains true if B is merely Borel and t_0 is a point of dispersion for B^c.

Berman (1969) found that the *higher* smoothness of α could be used to obtain a probabilistic solution of a problem of Carathéodory, viz. *to construct a Borel function X such that $\lambda(D) > 0$ implies $\lambda(X^{-1}(D) \cap B) > 0$ for every interval B*. We call such an X a *Carathéodory function*.

(11.5) THEOREM. *Let $N = d = 1$; suppose X is* (LT) *and that $\alpha(x, B)$ is real-analytic on $\mathbb{R}$ for each rational interval B; then X is a Carathéodory function.*

Indeed, if $\lambda(X^{-1}(D) \cap B) = 0$, then $\alpha(x, B) = 0$ a.e. on D hence $\alpha(x, B) \equiv 0$ by analyticity; but this is impossible, e.g., by (6.2). Almost every trajectory of certain Gaussian processes satisfies the hypothesis of (11.5), hence the solution of Carathéodory's problem—see § 28.

We remark that a Carathéodory function cannot be continuous, but that *there exist continuous functions X such that $\alpha(x, B)$ is C^∞ for each rational interval B*, again given by Gaussian trajectories. A natural question is this: *does every Carathéodory function have an (analytic) occupation density*?

12. Perturbations. Let X, Z be (N, d)-fields. The *perturbation problem* is this: *if X has a nice occupation kernel $\alpha(x, dt)$ and $Z(t)$ is sufficiently smooth, is $X(t) + Z(t)$ (LT)*? In view of §§ 9–11, X should exhibit very wild behavior and gentle perturbation by Z should not be enough to dampen it. We discuss only $N = d = 1$. Proofs and generalizations will appear elsewhere.

Here are the main results. We write $\alpha_t(x)$ for $\alpha(x, [0, t])$ and $\alpha_t'(x)$ for $\partial\alpha_t(x)/\partial x$.

(12.1) THEOREM. *Assume $\alpha_t(x)$ is jointly continuous, $\alpha_t(\cdot)$ is absolutely continuous for* a.e. t, *and $\alpha_t'(x)$ is integrable on* $[0, 1] \times \mathbb{R}$; *then $X(t) + Z(t)$ is* (LT) *whenever Z is differentiable* a.e.

As mentioned in Example 2, the Brownian local time does not satisfy the hypotheses of (12.1), failing the absolute continuity requirement and *we do not know if mere joint continuity of $\alpha_t(x)$ suffices in* (12.1).

Now suppose Z is of bounded variation and write $Z(ds)$ for the corresponding signed measure. We now perform a little illegitimate calculation using the "δ-function": let $\gamma_t(x)$ be the occupation density of $X + Z$. We have

$$\begin{aligned}\gamma_t(x) &= \frac{d}{dx}\int_0^t I_{(-\infty, x]}(X(s) + Z(s))\, ds \\ &= \int_{-\infty}^{\infty}\int_0^t \frac{d}{dx} I_{[y+Z(s), \infty)}(x)\alpha(y, ds)\, dy \\ &= \int_{-\infty}^{\infty}\int_0^t \delta(x - y - Z(s))\alpha(y, ds)\, dy \\ &= \int_0^t d_s\alpha_s(x - Z(s)) + \int_0^t \alpha_s'(x - Z(s))Z(ds) \\ &= \alpha_t(x - Z(t)) + \int_0^t \alpha_s'(x - Z(s))Z(ds).\end{aligned}$$

Despite its origins, *the last line is correct for any continuous Z of bounded variation, if $\alpha_t'(x)$ exists and is (jointly) continuous for all (t, x), and then $\gamma_t(x)$ will itself be jointly continuous. Under the hypotheses of* (12.1) *alone, however, the formula for γ is not known.*

An interesting special case is obtained when X is continuous and $Z(t) = \max_{0 \leqslant s \leqslant t} X(s)$; Z is continuous, increasing. Let $\Delta = \{s : X(s) = Z(s)\}$ be the set of "progressive maxima". The measure $Z(ds)$ is carried by Δ as is $\gamma(0, ds)$, where $\gamma(x, ds)$ is the occupation kernel of $Z - X$; (instead of $X + Z$ as in (12.1)—obviously no problem in this). Since $Z - X \geqslant 0$, $\gamma(x, ds) \equiv 0$ for $x < 0$, hence also for $x = 0$ by joint continuity. This reflects the fact that Δ becomes smaller as X becomes more erratic.

It is interesting to contrast the above example with the trajectories of Brownian motion. Let $M_t = \max_{0 \leqslant s \leqslant t} W(s)$. Then $M - W$ is a copy of the reflecting Brownian motion $|W_t|$ and as such has a *nonzero* local time at the boundary $x = 0$, viz. $2\beta_t(0)$, where β is Brownian local time (see Itô and McKean (1965), Ch. 2). Since the occupation density of $M - W$ must be zero for $x < 0$, it must have a discontinuity *at* 0.

13. Level sets. Much of the literature on occupation densities is actually concerned with the size of the level sets $M_x = \{t : X(t) = x\}$, for a random vector field X, as measured by cardinality, capacity, or (Hausdorff) dimension. The basic idea here is that, for each x, $\alpha(x, dt)$ is a measure carried by M_x, and this has a direct bearing on each of the three types of "size". We restrict our attention to $N = d = 1$ here, though some of the results generalize to higher dimensions, and we assume X is (LT). Recall $\alpha(x) = \alpha(x, T)$ and $\alpha_t(x) = \alpha(x, [0, t])$.

Cardinality. A very simple result is

(13.1) THEOREM. *Suppose $\alpha(x, dt)$ is* [AC-0] (§ 10); *then, for* a.e. t,

$$L_t = M_{X(t)} = \{s : X(s) = X(t)\}$$

is uncountable.

This is because the support of $\alpha(x, dt)$ is either empty ($\alpha(x) = 0$) or uncountable ($\alpha(x) > 0$) for a.e. x, and is contained in M_x, and hence, for a.e. s, the support of

$\alpha(X(s), dt)$ is uncountable, and contained in L_s, whenever $\alpha(X(s)) > 0$. But (§ 6), $\alpha(X(t)) > 0$ a.e.

The following is due to Berman (1970), page 1265, when X is continuous:

(13.2) THEOREM. *If $\alpha_t(x)$ is jointly continuous, then $\{x : M_x$ is countable$\}$ is nowhere dense in the range of X.*

This follows immediately from

(13.3) LEMMA. *If $\alpha(x)$ is (lower semi-) continuous, then $\{x : \alpha(x) = 0\}$ is nowhere dense in the range of X.*

Berman (1969) proved (13.3) for X continuous and his proof, in conjunction with Lusin's theorem gives the general result.

Dimension. We refer to Kahane (1968) for the definition of Hausdorff dimension.

(13.4) THEOREM. (Berman (1972), pages 76, 78). *Suppose $\alpha_t(x)$ is jointly continuous and*

$$\sup_{x\in\mathbb{R};\,0\leqslant t\leqslant 1}\left[\alpha_{t+h}(x) - \alpha_t(x)\right] \leqslant D|h|^{\beta}, \qquad h \leqslant h_0, \tag{13.5}$$

for some β, D, $h_0 > 0$; then

(i) $\{x : \dim M_x < \beta\} \subset \{x : \alpha(x) = 0\}$;

(ii) *X does not satisfy a Hölder condition of order $\gamma > 1 - \beta$ at any t.*

In fact, from § 10 we see that X is $(J_\gamma(t))$ at every t, which is much stronger than (ii). This leads to

(13.6) THEOREM. *Let $0 < \gamma < 1$ and $\alpha_t(x)$ be as in* (13.4) *for each $\beta < \gamma$; assume that X satisfies a Lipschitz condition of each order less than $1 - \gamma$ at each t. Then*

$$\dim L_t = \gamma \qquad \textit{for a.e. } t. \tag{13.7}$$

For any $\varepsilon > 0$, $\dim M_x \leqslant \gamma + \varepsilon$ for a.e. x follows from Kahane (1968), page 142, and hence $\dim M_x \leqslant \gamma$ for a.e. x, and then $\dim L_t \leqslant \gamma$ because X is (LT). On the other hand, if $\beta < \gamma$, (13.4) tells us that $\dim L_t \geqslant \beta$ for a.e. t, since $\alpha(X(t)) > 0$ a.e.; this proves the theorem.

We note that almost every Brownian trajectory satisfies the hypotheses of (13.6) with $\gamma = \frac{1}{2}$.

Capacity. Results on capacity are, of course, intimately bound up with those on dimension. Here we present some theorems, due to Geman (1977a) which are rather different. These show the influence of certain energy integrals, such as arise in capacity theory, on the local behavior of X, which is now assumed to be a continuous (N, d)-field with occupation kernel $\alpha(x, dt)$.

Let

$$w(\varepsilon) = \sup_{|s-t|\leqslant\varepsilon}|X(s) - X(t)|$$

be the modulus of continuity of X, and let $\phi(r)$, $r \geqslant 0$, be a positive decreasing function, $\phi(0+) = \infty$, and such that $\phi(|t|)$, $t \in \mathbb{R}^N$, is integrable and has the representation $\phi(|t|) = \sup_k \phi_k(t)$, where $0 \leqslant \phi_k \leqslant \phi_{k+1} \in L^1(dt)$, each ϕ_k being the Fourier transform of a positive, integrable function. These functions $\phi(|t|)$ will serve as the potential kernels in our energy integrals; among them one finds all the usual things, such as the Riesz potentials $|t|^{-\beta}$, $0 < \beta < N$.

We now define

$$L(\varepsilon) = \left[\int_0^\varepsilon \phi(r) r^{N-1}\, dr\right]^{1/d},$$

finite since $\phi(|t|)$ is in $L^1(dt)$, and $\psi(r) = (r^d \phi(r))^{1/N}$. The *energy integral* corresponding to the level x is

$$I_x(\phi) = \int_T \int_T \phi(|s-t|) \alpha(x, ds) \alpha(x, dt),$$

and, finally, let $a = \lambda_N\{t \in T : I_{X(t)}(\phi) < \infty\}$.

(13.8) THEOREM. *Suppose $\alpha(x)$ is in L^2.*

(a) *If $a > 0$, then*

$$\lim_{\varepsilon \downarrow 0} \frac{w(\varepsilon)}{L(\varepsilon)} = \infty. \tag{13.9}$$

(b) *If $\psi(r)$ is increasing, and $a = 1$ then*

$$\text{ap} - \lim \sup_{s \to t} \frac{|X(s) - X(t)|}{\psi(|s-t|)} = \infty \text{ a.e.} \tag{13.10}$$

Thus, for instance, if $\phi(|t|) = |t|^{-\beta}$, then, in part (b), X cannot be Lipschitz of order $(d+\beta)/N$, whereas, in (a), X cannot be Lipschitz of order $(N-\beta)/d$.

As (13.8) shows, the finiteness of $I_x(\phi)$, which implies positive ϕ-capacity of M_x, for sufficiently many levels imposes a "lower bound" on the local (approximate) growth of X (note, however, that (13.10) is weaker than (J_ψ)). One may then ask *whether the conclusion of* (13.8) *remains valid if we replace* $I_{X(t)}(\phi) < \infty$ *by* $\mathrm{Cap}_\phi M_{X(t)} > 0$ *in the statement of the theorem.*

14. Summary. To get an overview of the real-variable part of the subject, we summarize in the following table the results of Part 2, restricting to the case $N = d = 1$ for simplicity, and as concerns the results of §§ 10 and 11 considering only Hölder conditions rather than the general "ϕ-conditions" appearing in the text. Reading down the table, we find progressively better behavior of $\alpha_t(x) = \alpha(x, [0, t])$ accompanied by correspondingly worse behavior of $X(t)$. Except for brief comments in Part 1, the entries concerning pure jump behavior of $\alpha_t(x)$ are not explained in this paper; see our (1976) article. The entry marked (*) shows the position of the Brownian motion trajectory with regard to the behavior induced by its occupation density. The Brownian trajectory is often held up as a badly behaved function from the point of view of classical analysis but appears tame by comparison with functions with smooth occupation densities. Indeed, a notable French mathematician, upon learning of such functions, is reported to have remarked "Je me détourne avec effroi et horreur de cette plaie lamentable de telles fonctions."

All the other entries carry a reference within the present paper. The table is largely self-explanatory; note that, in a phrase such as "Hölder condition $\beta < 1$", β is the order of the stated Hölder condition. Also, the implications in the fifth row of the table require some explanation: the conclusion that $J_\gamma(t)$ holds at every t follows from (10.1) by taking $g(x) =$ constant; in fact, the hypothesis "uniform Hölder condition $\beta < 1$" could be replaced by "uniform on finite x-intervals" since the version of (10.1) for g continuous remains valid for g bounded on finite intervals. The conclusion that $\dim L_t \geqslant \beta$ for a.e. t follows from the proof of (13.6) together with the observation that (13.4) (i) is true for *almost every* x without assuming that $\alpha_t(x)$ is jointly continuous (see the proof of Lemma 6.2 of Berman (1972)).

TABLE 1

$\alpha_.(x)$	$\alpha_t(\cdot)$		$X(t)$
pure jump	ess. unbdd.	$\Leftarrow$	$C^{(1)}$, $X'(t) = 0$ for some t
pure jump		$\Leftarrow$	M_x countable for a.e. x
	ess. bdd.	$\Rightarrow$	No local Hölder conditions of order $\gamma > 1$. (§ 9)
continuous (for a.e. x)		$\Rightarrow$	(J_1) (§ 10), $M_{X(t)}$ uncountable for a.e. t (§ 13)
uniform Hölder condition $\beta < 1$		$\Rightarrow$	$J_\gamma(t)$ for every t, $\gamma > 1 - \beta$ (§ 10); $\dim L_t \geqslant \beta$ for a.e. t (§ 13)
jointly continuous		$\Rightarrow$	$J_1(t)$ for every t (§ 10)
(*) uniform Hölder condition $\beta < \frac{1}{2}$	uniform Hölder condition $\beta < \frac{1}{2}$	$\Leftarrow$	Brownian motion
continuous	abs. continuous for a.e. t	$\Rightarrow$	$X + Z$ is (LT), Z diff. a.e. (§ 12)
	$C^{(k)}$	$\Rightarrow$	$J_{1/k}(t)$ at extreme points (§ 11)
	analytic (for all rational t)	$\Rightarrow$	Carathéodory

APPENDIX

Metric density and approximate limits. The *upper* (*lower*) *density of* $A \in \mathfrak{B}_k$ *at* $x \in \mathbb{R}^k$ is the limit superior (resp. inferior) of

$$g(\varepsilon) = \lambda_k(B_k(x, \varepsilon) \cap A)/c_k\varepsilon^k$$

where $c_k = \lambda_k(B_k(0, 1))$. A *point of density* (resp. *dispersion*) *of* A is one at which $\lim_\varepsilon g(\varepsilon)$ exists and equals 1 (resp. 0). Almost every $x \in A$ is a point of density for A and a point of dispersion for A^c. These hold as stated for Lebesgue measurable sets, and, with modifications, for arbitrary sets; see Saks (1937) for this and the following definition.

Let f be a real-valued Borel function on $E \in \mathfrak{B}_k$ and let t be a density point of E. The *approximate limit superior* (or *upper limit*), denoted

$$\text{ap} - \limsup_{s \to t} f(s)$$

is the infimum of those numbers a such that t is a point of dispersion for $\{s \in E : f(s) > a\}$. Similarly,

$$\text{ap} - \liminf_{s \to t} f(s)$$

is the supremum of those a for which t is a point of dispersion for $\{s \in E : f(s) < a\}$. (Obvious conventions apply when no such a's exist.) When these two quantities are equal, say, to L, we say that f has an *approximate limit* at t, denoted $\text{ap} - \lim_{s \to t} f(s)$, the value of which is the common value L. We note that

$$- \infty \leqslant \liminf_{s \to t} f(s) \leqslant \text{ap} - \liminf_{s \to t} f(s)$$
$$\leqslant \text{ap} - \limsup_{s \to t} f(s) \leqslant \limsup_{s \to t} f(s) \leqslant \infty,$$

so that, e.g., the existence of a true limit implies that of an approximate limit, but not conversely. It is easy to see that $\text{ap} - \lim_{s \to t} f(s) = L$ iff, *for each* $\varepsilon > 0$, *t is a density point for* $\{s \in E : |f(s) - L| < \varepsilon\}$; here L is assumed to be finite.

The following result, which makes clearer the intuitive meaning of approximate limit, is part of the folklore. It is alluded to by Saks (1937), and Hobson (1927), page 312, has a proof, for $k = 1$, which appears to us to be incorrect; we could not find a proof anywhere else.

THEOREM. *For* $\text{ap} - \lim_{s \to t} f(s) = L$ *it is necessary and sufficient that there exist a Borel set* $G \subset E$ *of which t is a density point such that*

$$(\dagger) \qquad \lim_{s \to t;\, s \in G} f(s) = L.$$

The sufficiency is trivial. For simplicity, we take $t = 0$ and L finite in proving necessity. Define

$$A_n = \{s \in E : L - 1/n < f(s) < L + 1/n\}, \qquad n \geqslant 1.$$

Then $A_n \in \mathfrak{B}_k$ and 0 is a point of density of A_n. Let $\alpha_n \uparrow c_k$ and $\delta_n \downarrow 0$ $(n \to \infty)$, choose $\varepsilon_n \downarrow 0$ so that

$$\lambda_k(B_k(0, \varepsilon) \cap A_n) \geqslant \alpha_n \varepsilon^k \qquad \text{for all} \quad \varepsilon \leqslant \varepsilon_n,$$
$$\varepsilon_{n+1}^k \leqslant \alpha_n \delta_n \varepsilon_n^k,$$

and let

$$G = \cup_{n=1}^{\infty} A_n \cap (B_k(0, \varepsilon_n) \setminus B_k(0, \varepsilon_{n+1})).$$

That $(\dagger)$ holds is trivial; we need only prove that 0 is a density point of G. Let

$\varepsilon_n < \varepsilon < \varepsilon_{n-1}$; then

$$\begin{aligned}
\varepsilon^{-k}\lambda_k(B_k(0,\varepsilon)\cap G) &= \varepsilon^{-k}\lambda_k(B_k(0,\varepsilon_n)\cap G) + \varepsilon^{-k}\lambda_k\big[(B_k(0,\varepsilon)\setminus B_k(0,\varepsilon_n))\cap G\big] \\
&\geqslant \varepsilon^{-k}\big[\lambda_k(B_k(0,\varepsilon_n)\cap A_n) - \lambda_k(B_k(0,\varepsilon_{n+1})\cap A_n) \\
&\qquad + \lambda_k(B_k(0,\varepsilon)\cap A_{n-1}) - \lambda_k(B_k(0,\varepsilon_n)\cap A_{n-1})\big] \\
&\geqslant \varepsilon^{-k}\big[\alpha_n\varepsilon_n^k - c_k\varepsilon_{n+1}^k + \alpha_{n-1}\varepsilon^k - c_k\varepsilon_n^k\big] \\
&\geqslant \varepsilon^{-k}\big[\alpha_n\varepsilon_n^k - c_k\alpha_n\delta_n\varepsilon_n^k + \alpha_{n-1}\varepsilon^k - c_k\varepsilon_n^k\big] \\
&= \alpha_{n-1} - (\varepsilon_n/\varepsilon)^k(c_k + c_k\alpha_n\delta_n - \alpha_n) \\
&\geqslant \alpha_{n-1} - (c_k + \alpha_n(c_k\delta_n - 1)) \to c_k \qquad (n\to\infty).
\end{aligned}$$

3. STOCHASTIC OCCUPATION DENSITIES

15. Markov local time L. Let $X = (X_t)$, $t \in T = [0, \infty)$, be a standard Markov process as in Blumenthal and Getoor (1968) whose terminology we use freely. The σ-fields $\mathcal{F}_t$ on the sample space Ω are the usual completions of $\sigma\{X_s : s \leqslant t\}$ and $\mathcal{F} = \vee_t \mathcal{F}_t$. The state space is $(E, \mathcal{E})$ with $\mathcal{E}$ separable and $\{x\} \in \mathcal{E}$ for each $x \in E$. For each $\omega \in \Omega$, the *trajectory* $t \mapsto X_t(\omega)$ is a Borel function and so has an occupation measure $\mu_B(\Gamma) = \lambda\{s \in B : X_s \in \Gamma\}$, $B \in \mathcal{B}(T)$, $\Gamma \in \mathcal{E}$, where we have suppressed the symbol ω from our writing but not from our thoughts. We say that X *is* (LT) *relative to* π, a σ-finite measure on $\mathcal{E}$, if $\mu = \mu_T \ll \pi$ a.s., i.e., almost every trajectory is (LT). Recall that "a.s." here means on a set which is of $\mathbb{P}^x$-probability 1 for every x. In the (LT) case, the occupation kernel $\alpha(x, B)$ can be chosen jointly (x, ω)-measurable under minimal hypotheses on $\mathcal{E}$ and $\mathcal{F}$. Since this is rather standard, we will accept the conclusion and not dwell on the point any longer. When $B = [0, t]$ we write $\mu_t(\Gamma)$ and $\alpha_t(x)$ instead of $\mu_B(\Gamma)$, $\alpha(x, B)$. It is not difficult to show that, for every Γ, $\mu_t(\Gamma)$ is an additive functional, and it will emerge below that the same is true of $\alpha_t(x)$. In the course of this work, we will also fully explain the connection between $\alpha_t(x)$ and the Blumenthal-Getoor local time $L_t(x)$ which we now define.

Let T_x be the hitting time of $\{x\}$, $x \in E$, and let $E_r \subset E$ be the set of *regular* points: $\mathbb{P}^x(T_x = 0) = 1$. (By Blumenthal's 0-1 law, $\mathbb{P}^x(T_x = 0) = 0$ for x irregular.) Following Getoor and Kesten (1972) *we assume once and for all that*

$$(15.1) \qquad \psi^1(x, y) \equiv \mathbb{E}^x(e^{-T_y}) \text{ is } \mathcal{E} \otimes \mathcal{E}\text{-measurable.}$$

This implies that $E_r = \{x : \psi^1(x, x) = 1\}$ is in $\mathcal{E}$.

Citations such as [V, (3.8)] will refer to Blumenthal and Getoor (1968); for simplicity, we always take the multiplicative functional there to be $\mu_t = 1_{\{t<\zeta\}}$, where ζ is the lifetime of the process.

For $y \in E_r$, the *local time at y* is the unique, continuous additive functional $L(y) = (L_t(y))$ such that

(15.2) $$\psi^1(x, y) = \mathbb{E}^x \int_0^\infty e^{-t}\, dL_t(y), \qquad x \in E.$$

The existence and uniqueness of $L(x)$ are guaranteed by probabilistic potential theory. One of the basic results on local times, from which the name derives and which connects $L(x)$ with the behavior of X, is this [V, (3.8)]: for every $x \in E_r$,

(15.3) $$J_x^+ \subseteq M_x \subseteq J_x \text{ a.s.},$$

where $J_x = \{t : L_{t+\varepsilon}(x) - L_{t-\varepsilon}(x) > 0 \text{ for all } \varepsilon > 0\}$ and $J_x^+ = \{t : L_{t+\varepsilon}(x) - L_t(x) > 0 \text{ for all } \varepsilon > 0\}$. Thus the local time may be construed as a continuous measure $L(x, dt)$ carried by M_x; indeed, the closed support of $L(x, dt)$ is J_x, which differs from J_x^+ (hence from M_x) by a countable set. We then have, for $x \in E_r$,

(15.4) $$L(x, M_x^c) = 0.$$

This condition essentially determines $L(x)$ up to a multiplicative constant.

Other constructions of $L(x)$, depending more or less on the "regenerative property" of the set M_x, were given by Horowitz (1968, 1972), Maisonneuve and Morando (1970), and Maisonneuve (1971).

Still another approach is due to Kingman (1973). As we shall see in §§ 17 and 18, under very general conditions, $L_t(x)$ can be realized as an occupation density, and so as a limit of ratios involving $\mu_t(\Gamma)$ for sets $\Gamma \downarrow \{x\}$. In this sense, the definition of $L_t(x)$ requires knowledge of the behavior of the process in a *neighborhood* of x, as well as *at* x; Kingman refers to this as an *extrinsic* specification of the local time. Similarly, the definition at (15.2) is extrinsic. On the other hand, Kingman's construction of $L(x)$ depends on M_x alone.

Finally, various attempts have been made to define "local time on Γ" for a set $\Gamma \subset E$; for example, see Sato and Tanaka (1962) who discuss the so-called "diffusion process on the boundary". A discussion of some of the difficulties which arise is given by Blumenthal and Getoor (1968); see also Maisonneuve (1972).

16. Applications of L. The best-known applications of Markov local time are generalizations of those in Itô and McKean (1965) for *Brownian local time* $\beta_t(x)$. We remark that β coincides with L in this case *and* is an occupation density relative to Lebesgue measure. A full, recent account of β is found in the survey paper of McKean (1975) where one finds Levy's results concerning the component intervals of $[0, t] \setminus M_x$, Tanaka's representation of β as a stochastic integral and the proof of its joint continuity, and the results of Ray, Knight and Williams on the *Markovian* nature of the process $\beta_T(x)$, $0 \leqslant x \leqslant 1$, for certain stopping times T. The results of Pittenger and Shih (1972) and Meyer, Smythe and Walsh (1971) suggest that the Markov property of $\beta_T(x)$ may persist in more general cases. We would like to remark that this limits the smoothness of $x \mapsto \beta_T(x)$, and, in turn, the irregularity of the trajectories, in contrast to what is possible, e.g., for Gaussian processes.

Further results on $[0, t] \setminus M_x$ (for general Markov processes) are given by Blumenthal and Getoor (1964) and Getoor and Millar (1972). The Hausdorff dimension of M_x (x fixed) was found by Taylor (1955) in the Brownian case; generalizations and related material are given by Blumenthal and Getoor (1962), (1968), and Stone (1963).

These results culminate in a result of Fristedt and Pruitt (1971), which, for our purposes, can be stated as follows: *for each* $x \in E_r$,

$$L_t(x) = H_x([0, t] \cap M_x) \qquad \text{for all} \quad t \geq 0 \text{ a.s.} \tag{16.1}$$

where H_x *is the Hausdorff measure generated by a certain* nonrandom *function* $h_x(t)$. (Of course, h_x is then the "exact measure function" of M_x.) This was first done for stable processes by Taylor and Wendel (1966): if the index is $\alpha > 1$, then $h_0(t) = t^{\beta}(\log \log 1/t)^{1-\beta}$, where $\beta = 1 - (1/\alpha)$. Obviously, the same h works for every level x in the case of stationary, independent increments. For a full account, see Fristedt (1973) and Taylor (1973). For other analytical properties of $L_t(x)$ as a function of t ($x \in E_r$ fixed), see, e.g., Hawkes (1974) and Millar (1972).

A recent application of Brownian local time concerns the set of points $A(\omega) \subset [0, \infty)$ at which the law of the iterated logarithm fails, i.e.,

$$\limsup_{h \to 0^+} \frac{|W_{t+h}(\omega) - W_t(\omega)|}{(2h \log \log 1/h)^{\frac{1}{2}}} < 1. \tag{16.2}$$

Knight (1974) showed that $A(\omega) \cap M_0(\omega) \neq \varnothing$ a.s. and Kahane (1976) improved this by replacing $A(\omega)$ with the set where the "lim sup" in (16.2) is zero. A related result is given by Bruneau (1974): let H_0 be as in (16.1) (for Brownian motion); then, a.s., $H_0(A(\omega) \cap M_0(\omega)) = 0$.

In the same paper, Knight described the spiking of the Brownian path as follows: "the path exhibits a dense set of spine-like projections of sharpness exceeding $|h|^{\frac{1}{2}}(\log(1/|h|))^{-(1+\varepsilon)}$ for every $\varepsilon > 0$"; i.e., *with* $\mathbb{P}^0$*-probability* 1, *there is a dense, random set* $D \subset \mathbb{R}_+$ *such that, for each* $t \in D$ *and each* $\varepsilon > 0$,

$$|W_s - W_t| \geq C(|s - t|)^{\frac{1}{2}}\left[\log \frac{1}{|s - t|}\right]^{-(1+\varepsilon)} \tag{16.3}$$

for some constant C and all sufficiently small $|s - t|$ *(depending on* ε*)*. The real-variable aspect of this result is revealed by passing to an "approximate double-spike" (a "spike" at extrema), in which case the factor $(\log(1/|h|))^{-(1+\varepsilon)}$ can be improved somewhat and the result becomes valid for every t. Indeed, according to Itô and McKean (1965, page 71)

$$\limsup_{\delta \downarrow 0} \sup_{x \in \mathbb{R}} \sup_{t \leq 1} \frac{\beta_{t+\delta}(x) - \beta_t(x)}{\delta^{\frac{1}{2}} \log 1/\delta} \leq 8, \ \mathbb{P}^0\text{-a.s.},$$

from which it follows from (10.1) that, *with* $\mathbb{P}^0$*-probability* 1: *for every* $C > 0$, *every* *t is a point of dispersion for the set*

$$\left\{ s : |W_s - W_t| \leqslant C|s - t|^{\frac{1}{2}} \left| \frac{1}{\log|s - t|} \right| p(|s - t|) \right\},$$

for any $p(h) \to 0$ as $h \downarrow 0$.

Let us mention, in passing, a few other applications of the Blumenthal-Getoor local time. Çinlar (1975) studies the "dry periods" in a model for a storage process or dam by constructing the local time at 0 for the process X_t which represents the content at time t. Greenwood (1975) uses the local time as a continuous time analogue of "ladder epochs" in random walk theory. Similar applications are given by Bingham (1975) for processes with stationary, independent increments and for various models of queues and dams.

17. ***L* as occupation density.** In view of (15.4), the question naturally arises whether $L_t(x)$ is itself an occupation density, assuming all points regular. As indicated in § 4, various sufficient conditions have been given in which π is a reference measure and there is a potential "kernel" $u^1(x, y)$ satisfying suitable continuity hypotheses. The best result involving only the existence of an occupation density is due to Getoor and Kesten (1972):

(17.1) THEOREM. *Assume* $E = E_r$ *and* (15.1) *holds*; *let* π *be a reference measure for* X; *then a version of* $(L_t(x))$ *can be chosen such that*

(a) $(s, x, \omega) \mapsto L_s(x, \omega)$ *is a* $\mathfrak{B}([0, t]) \otimes \mathcal{E} \otimes \mathcal{F}_t$*-measurable mapping* $(s \leqslant t)$;

(b) $t \mapsto L_t(x, \omega)$ *is left continuous and nondecreasing for each* (x, ω);

(c) $L(x)$ *is a continuous additive functional for each* x;

(d) *there is a finite, positive,* $\mathcal{E}$*-measurable function* $g(x)$ *such that* $\alpha_t(x) = g(x)L_t(x)$ *is an occupation density.*

It should be emphasized that, whereas (17.1) implies a.s.

$$(*) \qquad \alpha_t(x) = \lim\nolimits_{\Gamma \downarrow \{x\}} \frac{\mu_t(\Gamma)}{\pi(\Gamma)} \qquad \pi\text{-a.e. } x,$$

it is not true that this relation holds a.s. at every x. Knight (1971) shows that, for some constant C_0,

$$L_t(0) = C_0 \lim\nolimits_{\delta \downarrow 0} \delta^{-\alpha/2} \mu_t([0, \delta]) \text{ a.s.}$$

for the "reflected" (in a suitable sense) symmetric stable process of index $1 < \alpha \leqslant 2$; these processes have all points regular with λ as a reference measure, so that (*) holds and the property at $x = 0$ appears exceptional. Williams (1969) exhibits a Markov process with state space $\{0, 1, \cdots\}$ with 0 as the unique instantaneous state and such that *no* formula of the type

$$L_t(0) = \lim\nolimits_{\delta \downarrow 0} \mu_t(B(\delta))/h(\delta)$$

is valid, where $B(\delta)$ is the ball (in a suitable metric) of radius $\delta > 0$, center 0; (*) holds in a trivial way for all $x \neq 0$ in this case. Of course, from the point of view of occupation densities, such aberrant behavior at a single state is irrelevant, and the examples only serve to show that there is no *a priori* connection between the Blumenthal-Getoor local time and the occupation density. As mentioned in § 4, Griego (1965) gives conditions under which (*) holds for each x a.s.

18. Lebesgue decomposition. We are going to extend (17.1) by giving the complete Lebesgue decomposition of the occupation measure μ_t, and, consequently, a necessary and sufficient condition for a Markov process to be (LT). Let $E_p = \{y : \psi^1(x, y) = 0 \text{ for all } x \in E\}$ (the "polar" points), $E_i = E \setminus (E_r \cup E_p)$ (irregular but not polar), and $\tilde{E} = E_r \cup E_i$ (nonpolar); all these are in $\mathcal{E}$. The definition of $L(x)$ is now extended to all $x \in E$ as follows: if $x \in E_r$, $L(x)$ is already defined as in § 15; if $x \in E_p$, $L_t(x) \equiv 0$; if $x \in E_i$,

$$L_t(x) = \#\{0 < s \leqslant t : X_s = x\}, \tag{18.1}$$

where # denotes the cardinality of the indicated set. Clearly, $L(x)$ is again an additive functional, though not continuous (or even natural) in general, and (15.4) still holds. According to [V, (3.40)], $L_t(x) < \infty$ for all t a.s. for $x \in E_i$.

It can be shown that, for each t, the mapping $(s, x, \omega) \mapsto L_s(x, \omega)$ on $[0, t] \times E \times \Omega$ is $(\mathcal{B}([0, t]) \otimes \mathcal{E} \otimes \mathcal{F}_t)^*$-measurable, the * denoting universal completion. This is accomplished by looking separately at E_p, E_r and E_i: on E_p the situation is trivial, on E_r the method of Getoor and Kesten (1972) applies, and on E_i one uses the approach in our paper (1976a). We omit the details.

Let us denote (temporarily) by $\gamma_t(x)$ the density of the absolutely continuous part of the occupation measure:

$$\mu_t(\Gamma) = \int_\Gamma \gamma_t(x)\pi(dx) + \bar{\mu}_t(\Gamma), \qquad \bar{\mu}_t \perp \pi, \Gamma \in \mathcal{E}. \tag{18.2}$$

A priori, $\gamma_t(x)$ cannot be very different from $L_t(x)$: one can show that $\gamma(x)$ can be chosen as an additive functional carried by M_x, and this already determines $\gamma(x)$ up to a constant.

Let $R(\omega)$ denote the range of $X_s(\omega)$, $s \geqslant 0$. For any $x \in E$

$$\mathbb{E}^x\big(\pi(R(\omega) \cap E_p)\big) = \mathbb{E}^x \int_{E_p} \sup_{t>0} I_{\{y\}}(X_t)\pi(dy) = 0.$$

Hence:

$$\pi\big(R(\omega) \cap E_p\big) = 0 \text{ a.s.} \tag{18.3}$$

Thus the polar points cannot contribute to the absolutely continuous component of μ_t. As we shall see, E_p carries all of the singular component if π is a reference measure.

Let us recall a bit of notation. The 1-potential operator of an additive functional A is defined by

$$U_A^1 f(x) = \mathbb{E}^x \int_0^\infty e^{-t} f(X_t)\, dA_t;$$

when $f \equiv 1$ we write simply $u_A^1(x)$. When $A_t = t$, $U_A^1 f$ becomes the 1-potential $U^1 f$ (or resolvent) of the process, and, finally, taking $f = I_\Gamma$ we retrieve the 1-potential kernel $U^1(x, \Gamma)$.

We now show that, under minimal assumptions, if X is (LT) relative to π, then π must be a reference measure. Indeed, if X is (LT), then (18.2) holds for all $t \geqslant 0$ with $\bar{\mu}_t = 0$ and then

$$U^1(x, \Gamma) = \int_\Gamma u^1_{\gamma(y)}(x)\pi(dy).$$

A routine calculation shows $u^1_{\gamma(y)}(x) = \psi^1(x,y)u^1_{\gamma(y)}(y)$. Thus a π-null set always has potential zero, and the converse holds if $u^1_{\gamma(y)}(x) > 0$ for every pair $x, y \in E$. *We assume, from now on, that π is a reference measure and, with no harm done, is finite.*

(18.4) THEOREM. *The Lebesgue decomposition of μ_t relative to π is*

$$\mu_t(\Gamma) = \mu_t(\Gamma \cap \tilde{E}) + \mu_t(\Gamma \cap E_p);$$

the term $\mu_t(\Gamma \cap E_p)$ is singular (by (18.3)) whereas

$$\mu_t(\Gamma \cap \tilde{E}) = \int_\Gamma g(y)L_t(y)\pi(dy) \tag{18.5}$$

for a positive, $\mathcal{E}$-measurable function g.

For later use we note that, if $\pi(\{y\}) > 0$, then $y \in E_r : \pi(\{y\}) > 0$ implies $U^1(x, \{y\}) > 0$ for some x, and the strong Markov property at T_y shows that M_y has positive Lebesgue measure $\mathbb{P}^y$-a.s., which means $y \in E_r$.

We now write down the 1-potential operators of the additive functionals $L(y)$, writing U^1_y instead of $U^1_{L(y)}$:

$$U^1_y f(x) = \psi^1(x,y)f(y) \qquad \text{if} \quad y \in E_r$$
$$= \frac{\psi^1(x,y)}{1 - \psi^1(y,y)} f(y) \qquad \text{if} \quad y \in E_i.$$

These are given in Chapter V.3 of Blumenthal-Getoor (1968). Define a function

$$\phi(y) = 1 \qquad \text{if} \quad y \in E_r \cup E_p$$
$$= (1 - \psi^1(y,y)) \qquad \text{if} \quad y \in E_i,$$

and an additive functional

$$A_t = \int_E \phi(y)L_t(y)\pi(dy).$$

We have

$$u^1_A(x) = \int_E \phi(y)u^1_y(x)\pi(dy) \leqslant \pi(E) < \infty$$

so that A has a bounded 1-potential.

We now observe that A is a *continuous* additive functional: it suffices to prove continuity at $T(\omega) < \infty$, where T is a stopping time. Let $\xi = X_T(\omega)$. If $\pi(\{\xi\}) > 0$ we have $\xi \in E_r$, so $L(\xi)$ is continuous, and so cannot contribute to a discontinuity of A. If $y \neq \xi$, then, for $y \in E_r \cup E_p$, $L(y)$ is continuous, and if $y \in E_i$, then $L(y)$ is continuous at $T(\omega)$, since it jumps only at times t when $X_t(\omega) = y$.

Let f be nonnegative and $\mathcal{E}$-measurable; then

$$U_A^1 f(x) = \int_E \phi(y) f(y) u_y^1(x) \pi(dy)$$
$$= \int_E f(y) \psi^1(x, y) \pi(dy).$$

Let

$$B_t = \int_0^t I_{\tilde{E}}(X_s)\, ds.$$

This is obviously a continuous additive functional with a bounded 1-potential; the 1-potential *operator* is

$$U_B^1 f(x) = \mathbb{E}^x \int_0^\infty e^{-t} I_{\tilde{E}}(X_t) f(X_t)\, dt.$$

Suppose $0 \leqslant f \leqslant 1$ and $U_A^1 f = 0$; then, for each x,

$$f(y)\psi^1(x, y) = 0 \qquad \text{for} \quad \pi\text{-a.e. } y,$$

and, by Fubini, for π-a.e. y, $f(y)\psi^1(x, y) = 0$ for π-a.e. x. Now, for $y \in E_p$, $\psi^1(x, y) = 0$ for all x. Suppose $\pi(\{f > 0\} \cap \tilde{E}) > 0$; then, for some $y \in \{f > 0\} \cap \tilde{E}$, $f(y)\psi^1(x, y) = 0$ for a.e. x. Since $x \mapsto \psi^1(x, y)$ is 1-excessive, it would follow that $\psi^1(x, y) \equiv 0$ which is impossible because y is not polar. It follows that $\pi(\{f > 0\} \cap \tilde{E}) = 0$, and thus

$$U_B^1 f(x) \leqslant U^1(x, \tilde{E} \cap \{f > 0\}) = 0,$$

since π is a reference measure. Motoo's theorem [V, (2.8)] now gives a function $h(x) \geqslant 0$ such that

$$\int_0^t I_{\tilde{E}}(X_s)\, ds = \int_0^t h(X_s)\, dA_s \qquad \text{for all} \quad t \geqslant 0 \text{ a.s.}$$

From the definition of A we derive easily

$$\mu_t(\tilde{E} \cap \Gamma) = \int_E \phi(y) \int_0^t h(X_s) I_\Gamma(X_s)\, dL_s(y) \pi(dy)$$
$$= \int_{\tilde{E}\Gamma} \phi(y) h(y) L_t(y) \pi(dy),$$

which is (18.5) with $g = \phi \cdot h$, and (18.4) is proven.

(18.6) COROLLARY. *Suppose* $\pi(R(\omega)) = 0$ a.s., *then* π-a.e. *point is polar.*

Since $\pi(R(\omega)) = 0$ a.s., μ_t has only a singular component and so $\mu_t(\tilde{E}) = 0$; but then $\tilde{E}$ is of potential zero, hence $\pi(\tilde{E}) = 0$.

(18.7) COROLLARY. *Suppose* μ_t *is purely singular* a.s.; *then* $\pi(R(\omega)) = 0$ a.s. *and* π-a.e. *point is polar.*

We have already seen that $\pi(R(\omega) \cap E_p) = 0$ a.s. (18.3); since μ_t is singular, we have $\mu_t(\tilde{E}) = 0$ a.s. by (18.4) and thus $\tilde{E}$ has potential 0. Thus $\pi(\tilde{E}) = 0$ and the result follows.

Summarizing, we have: if π is a reference measure, then the following are equivalent:

(i) $\pi(R(\omega)) = 0$ a.s.
(ii) $\pi(\tilde{E}) = 0$
(iii) μ_t purely singular a.s.

We should point out that a similar decomposition for the occupation measure of an *inhomogeneous* Markov process is apparently not valid: there are strictly increasing, continuous functions $F(t)$ such that $X_{F(t)}$ is a.s. not (LT), where X denotes Brownian motion (see § 22). Such processes have no polar points (under any reasonable definition of "polar") and yet μ_t is purely singular.

19. Applications. We will give two applications of the results in § 18, viz. to the "Chung problem" and to the representation of continuous additive functionals as "local time integrals".

(a) Suppose X is a subordinator having exponent $g(\lambda) = a\lambda + \int_0^\infty(1 - e^{-\lambda y})\nu(dy)$ where ν is its Lévy measure (terminology: Blumenthal and Getoor (1968), Fristedt (1973)). Consider the case where $a = 0$ and $\nu((0, 1)) = \infty$. It is well known (e.g. Itô and McKean (1965) pages 31–33) that the trajectories are almost all monotone increasing saltus functions. Let $F(x)$ be a continuous distribution function. If $F(X_t)$ has derivative 0 a.e. (a.s.), then it follows from Geman and Horowitz (1976a) that a.s. X is not (LT) relative to the measure π having distribution function F; in fact, μ_t is purely π-singular. Clearly $F(x) = x$ has this property and we conclude *if* λ *is a reference measure for* X, *then* λ-a.e. *point is polar, and so every point is polar*. This is a partial solution to a famous problem posed by Chung and solved by him in the case that the measure $U(\Gamma) = \mathbb{E}^0\int_0^\infty I_\Gamma(X_t)\,dt$ is $\ll \lambda$, viz. *is every point polar for subordinators of the given description*? Complete solutions (in the affirmative) were given later by Kesten (1969) (see also Bretagnolle (1971) and L. Carleson (unpublished)), though these are fairly difficult.

(b) Various authors (Blumenthal and Getoor (1968, page 294), Griego (1967), Geman and Horowitz (1973)) have given conditions for the validity of the following statement: *let* $A = (A_t)$ *be a continuous additive functional of* X; *then there exists a* (*unique*) *measure* ν *on* $\mathcal{E}$ *such that*

$$(19.1) \qquad A_t = \int_E L_t(x)\nu(dx) \qquad \textit{for all} \quad t \geq 0, \text{ a.s.}$$

Obviously $L(x)$ must be defined at every $x \in E$ for (19.1) to make sense, and hitherto this has meant $E = E_r$.

Let π be a σ-finite, excessive reference measure, and suppose *the duality hypothesis of* [VI, (1.2)] *holds*. Following Revuz (1970), we say that an additive functional A is *σ-integrable* if

$$(19.2) \qquad \nu_A(\Gamma) = \lim_{\lambda\to\infty}\lambda\int U_A^\lambda I_\Gamma(x)\pi(dx)$$

exists (finite) for a sequence $\Gamma_n\uparrow E$. In this case, the limit in (19.2) exists for all $\Gamma \in \mathcal{E}$ and defines a measure. Moreover, ν_A charges no polar (resp. semipolar) set if A is σ-integrable (resp. continuous). For us the main result of Revuz is this: *if* A *is σ-integrable, natural, and has* $u_A^1 < \infty$ π-a.e., *then*

$$(19.3) \qquad u_A^1(x) = \int_E u^1(x, y)\nu_A(dy).$$

An equivalent formulation of (19.3) is

(19.4) $$U_A^1 f(x) = \int_E u^1(x,y) f(y) \nu_A(dy), \qquad f \geqslant 0, \mathcal{E}\text{-measurable}.$$

Consider now the decomposition $E = \tilde{E} + E_p$; we obtain corresponding decompositions $A = \tilde{A} + A_p$, $\nu_A = \nu_{\tilde{A}} + \nu_{A_p}$ where $A_p = A - \tilde{A}$ and

$$\tilde{A}_t = \int_0^t I_{\tilde{E}}(X_s)\, dA_s.$$

If A satisfies the hypothesis for (19.3) the same will be true of $\tilde{A}$, and (19.4) will be valid with E, A replaced by $\tilde{E}, \tilde{A}$. We now identify $u^1(x,y)$ in terms of the local times $L_t(x)$:

$$\begin{aligned} U^1 f(x) &= \mathbb{E}^x \int_0^\infty e^{-t} f(X_t)\, dt \\ &= \mathbb{E}^x \int_{\tilde{E}} g(y) f(y) \int_0^\infty e^{-t}\, dL_t(y) \pi(dy) + \mathbb{E}^x \int_0^\infty e^{-t} f(X_t) I_{E_p}(X_t)\, dt \\ &= \int_{\tilde{E}} g(y) f(y) u_y^1(x) \pi(dy) + \int_{E_p} f(y) u^1(x,y) \pi(dy), \end{aligned}$$

where $g(y)$ is given in (18.4), and $u_y^1(x) = u_{L(y)}^1(x)$. This suggests

(19.5) $$u^1(x,y) = g(y) u_y^1(x), \qquad y \in \tilde{E}.$$

Clearly the uniqueness theorem for additive functionals tells us that:

(19.6) THEOREM. *If* (19.5) *holds, then*

(19.7) $$A_t = \int_{\tilde{E}} g(y) L_t(y) \nu_A(dy) + A_p(t) \text{ a.s.}$$

We prove that a version of (19.7) holds under the supplementary hypothesis

(19.8) $$u^1(y,y) > 0 \quad \text{for} \quad y \in \tilde{E}.$$

This is relatively harmless, even though it fails for some common processes; the result is probably true without (19.8) if more care is used.

Let $A \in \mathcal{E}$; then [VI, (1.16)]

(19.9) $$\mathbb{E}^x\left[e^{-T_A} u^1(X_{T_A}, y)\right] = \hat{\mathbb{E}}^y\left[e^{-\hat{T}_A} u^1(x, \hat{X}_{\hat{T}_A})\right]$$

where T_A, $\hat{T}_A$ are the hitting times of A by the processes X, $\hat{X}$ which are assumed to be in duality. Setting $x = y$ and $A = \{y\}$, we obtain, with a self-explanatory notation,

$$\psi^1(y,y) u^1(y,y) = \hat{\psi}^1(y,y) u^1(y,y),$$

whence $\psi^1(y,y) = \hat{\psi}^1(y,y)$ if $y \in \tilde{E}$ by (19.8) (and automatically if $y \in E_p$).

Next, take $A = \{y\}$ in (19.9) to get

(19.10) $$\psi^1(x,y) u^1(y,y) = \psi^1(y,y) u^1(x,y)$$

for every $x, y \in E$, in view of the preceding sentence.

If $y \in \tilde{E}$, there is an $x_0 \in E$ such that $\psi^1(x_0, y) > 0$; (19.10) then shows that $\psi^1(y,y) > 0$, that $u^1(x,y)$ and $\psi^1(x,y)$ are positive or vanish simultaneously, and that the ratio $u^1(x,y)/\psi^1(x,y)$ is independent of x, at least when the denominator is positive. Define $\tilde{g}(y) = u^1(y,y)/u_y^1(y)$, $y \in \tilde{E}$. Using the material in § 18, simple calculations show that $g(y) = \tilde{g}(y)$ π-a.e. on $\tilde{E}$ and that (19.5) holds for all $x \in E$, $y \in \tilde{E}$, if g is replaced by $\tilde{g}$, and so also (19.7) with $\tilde{g}$ instead of g. We note that the use of $\tilde{g}$ does not hurt the conclusion of (18.4).

20. Joint continuity of *L*. The definitive result on joint continuity is due to Getoor and Kesten (1972). Here, $E = E_r = \mathbb{R}$.

(20.1) THEOREM. *Assume that*

$$\int_0^1 u^{-1} p(u)\, du < \infty \tag{20.2}$$

where $p(u) = \sup_{|x-y| \leq u}(1 - \psi^1(x, y)\psi^1(y, x))^{\frac{1}{2}}$. *Then a version of* $L_t(x)$ *may be chosen such that* a.s. *the mapping* $(t, x) \mapsto L_t(x)$ *is continuous.*

Condition (20.2) is equivalent to $\Sigma p(2^{-n}) < \infty$ which improves the condition $\Sigma np(2^{-n}) < \infty$ in Boylan's original (1964) generalization of Trotter's theorem. An explicit Hölder-type condition (in x) is given by Getoor and Kesten, and obviously all of these results can be used in conjunction with the material of Part 2 to describe the behavior of the trajectories. We give only one simple example: *suppose* λ *is a reference measure*; *then, under the conditions of* (20.1), a.s. *the trajectories are nowhere approximately differentiable.*

Getoor and Kesten also describe conditions under which $L_t(x)$ a.s. fails to be jointly continuous. Their result was soon improved by Millar and Tran (1974) who proved

$$\sup_{x \in D} L_t(x) = \infty \quad \text{for each } t > 0, \text{ a.s.,} \tag{20.3}$$

where D denotes the dyadic rationals, under a variety of conditions on the functions $\psi^1(x, y)$ and $(\alpha, x) \mapsto \mathbb{E}^x \int_0^\infty e^{-\alpha t}\, dL_t(x)$.

A somewhat different approach to joint continuity for the occupation densities of certain diffusions has recently been given by Yor (1978) and Jeulin and Yor (1977). These diffusions are solutions of a class of stochastic differential equations and fall under Meyer's (1975) theory of occupation densities for semimartingales. Meyer gives a generalization of Tanaka's formula for the Brownian occupation density (for which see McKean (1975)) which gives an explicit representation for $L_t(x)$ in terms of stochastic integrals, where $L_t(x)$ is an occupation density for the semimartingale X relative to the measure $d\langle X^c, X^c\rangle_s$ on the time domain; here X^c is the continuous (local) martingale part of X. For the diffusion processes in question, the random measure $d\langle X^c, X^c\rangle_s$ becomes just ds, and the explicitness of the formula for $L_t(x)$ allows Jeulin and Yor to prove joint continuity by a Kolmogorov argument, thus retrieving Trotter's (1958) theorem in the Brownian case (see Example 1, § 0), as well as the Ray-Knight theorem mentioned in § 4. Of course, the semimartingale occupation density reduces to the Blumenthal-Getoor local time in the Markovian case.

For the sake of completeness we give a quick derivation of the Meyer-Tanaka formula in the case of a *continuous* semimartingale, which simplifies matters a bit, denoting by A_t the increasing process $\langle X, X\rangle_t$. The formula states

(20.4) $$\int_0^t f(X_s)\,dA_s = \int_{-\infty}^{\infty} f(x)L_t(x)\,dx,$$

for every measurable function $f \geqslant 0$, where

(20.5) $$\tfrac{1}{2}L_t(x) = (X_t - x)^+ - (X_0 - x)^+ - \int_0^t I_{[x,\infty)}(X_s)\,dX_s.$$

Let g be a C^∞ function such that g'' has compact support and $g'(x) = \int_{-\infty}^x g''(u)\,du$. According to Itô's formula (Meyer (1975), IV.21)

$$\int_0^t g''(X_s)\,dA_s = 2(g(X_t) - g(X_0)) - 2\int_0^t g'(X_s)\,dX_s.$$

A simple integration by parts shows that the first term equals

$$2\int_{-\infty}^{\infty} g''(x)\big((X_t - x)^+ - (X_0 - x)^+\big)\,dx,$$

and an easily justified Fubini-type argument gives

$$\int_0^t g'(X_s)\,dX_s = \int_{-\infty}^{\infty} g''(x)\int_0^t I_{[x,\infty)}(X_s)\,dX_s\,dx.$$

We conclude that (20.4) holds for $f = g''$, which suffices to give the full result.

21. General processes: existence results. Let $X = (X_t)$, $t \in T$, be a stochastic process with (measurable) state space $(E, \mathcal{E})$. The terminology is the same as that introduced in § 15 except that now T will usually be $[0, 1]^N$, and we must assume that the mapping $(t, \omega) \mapsto X_t(\omega)$ is measurable relative to $\mathcal{B}(T) \otimes \mathcal{F}$ and $\mathcal{E}$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is now the underlying probability space. Each trajectory $t \mapsto X_t(\omega)$ is a Borel function and we have the occupation measure $\mu_B(\Gamma) = \lambda_N\{s \in B : X_s \in \Gamma\}$, $B \in \mathcal{B}(T)$, $\Gamma \in \mathcal{E}$, defined for each trajectory. As in § 15, we say X *is* (LT) *relative to* π if $\mu = \mu_T \ll \pi$ a.s. (i.e., $\mathbb{P}$-a.s.), and we make the same assumption on the measurability of the occupation kernel $\alpha(x, B)$. Finally, if $E = \mathbb{R}^d$ and $\pi = \lambda_d$, we omit the phrase "relative to π".

Results on the existence of occupation densities have been obtained by four methods which we now summarize: 1° martingales on the state space; 2° martingales on Ω; 3° harmonic analysis; and 4° differentiation of measures.

1°. This is the approach of Orey (1970), Davydov (1976, 1977), and Pitt (1978). We assume $\mathcal{E}$ separable and replace π by an equivalent probability measure, again denoted π. Let $G_n = \{A_{n_1} \cdots A_{n,k_n}\}$ be a sequence of measurable partitions of E, linearly ordered by refinement, and such that $\cup_n G_n$ generates $\mathcal{E}$. Now, for a fixed trajectory, it is well known that

(21.1) $$p_n(x) = \sum_{i=1}^{k_n} \frac{\mu(A_{ni})}{\pi(A_{ni})} I_{A_{ni}}(x), \qquad x \in E,$$

defines a nonnegative supermartingale (a martingale if $\pi(A_{ni}) > 0$ for each n, i) relative to (E, π) and $\sigma(G_n)$, $n \geqslant 1$. Thus, $p_n(x)$ converges π-a.e. to a limit $p_\infty(x)$ which is a version of the Radon-Nikodym derivative $(d\mu/d\pi)(x)$. Moreover $\mu \ll \pi$ if and only if (p_n) is π-uniformly integrable. (It is also known that $\mu \ll \pi$ iff $\mu\{p_\infty = \infty\} = 0$; this is the martingale analogue of (7.2).)

A sufficient condition for uniform integrability is

(21.2) $$\sup_n \int_E p_n^2(x)\pi(dx) < \infty.$$

Hence, a sufficient condition for (LT) is

(21.3) $$\sup_n \int_E \mathbb{E}p_n^2(x, \omega)\pi(dx) < \infty,$$

and this integral is often easily expressed in terms of the two-dimensional distributions of X. For example, for $T = [0, 1]$, $E = \mathbb{R}$, and X Gaussian (with means 0), (21.3) is equivalent to

(21.4) $$\int_0^1\int_0^1\left(\mathbb{E}X_s^2\mathbb{E}X_t^2 - (\mathbb{E}X_sX_t)^2\right)^{-\frac{1}{2}} ds\, dt < \infty;$$

this is Orey's (1970) result. The conclusions of Davydov and Pitt about Gaussian processes are retrieved by different methods in § 22 (see § 30 for specific examples). Method 4° below is almost the same as this one, but is closer in spirit to the paper, so we will follow it in the sequel.

2°. This is a direct generalization of the methods used for Markov processes. Here $T = [0, \infty)$ and there is an increasing family of σ-fields $(\mathcal{F}_t)$ with respect to which X_t is progressively measurable. For details of the following material, see our paper (1973).

The basic assumption is that for each $t > 0$, there is a function $Z_t(x, \omega)$, $x \in E$, $\omega \in \Omega$, such that

(21.5) $$\mathbb{E}\left(\int_t^\infty e^{-s}I_\Gamma(X_s)\, ds|\mathcal{F}_t\right) = \int_\Gamma Z_t(x)\pi(dx), \qquad \Gamma \in \mathcal{E}, \text{ a.s.}$$

One then shows that, for π-a.e. x, $(Z_t)_{t\geq 0}$ is a (nonnegative) supermartingale relative to $(\mathcal{F}_t)$ with $\lim_{t\to\infty}\mathbb{E}Z_t(x) = 0$.

(21.6) THEOREM. *X is* (LT) *if and only if $(Z_t(x))_{t\geq 0}$ is of class (D) for π-a.e. x.*

("Class (D)" means that $\{Z_T(x)\}$ is a uniformly integrable family of random variables as T ranges over all finite stopping times.)

A sufficient condition for (21.5) is that for each $0 < t < s < \infty$, $\mathbb{P}(X_s \in dx|\mathcal{F}_t)(\omega)$ is a.s. π-absolutely continuous, in which case, for π-a.e. x,

(21.7) $$Z_t(x, \omega) = \int_t^\infty \psi_t(s, x, \omega)\, ds, \qquad t \geq 0,$$

where $\mathbb{P}(X_s \in dx|\mathcal{F}_t) = \psi_t(s, x)\pi(dx)$ a.s. (The equality in (21.7) is for the right-continuous versions of the two sides.) For example, let (X_t), $t \in \mathbb{R}$, be a real, mean 0, nondeterministic Gaussian process with $X_{st} = \mathbb{E}(X_s|\mathcal{F}_t)$, $V_{st}^2 = \mathbb{E}((X_s - X_{st})^2|\mathcal{F}_t)$, and $\mathcal{F}_t = \sigma\{X_s;\ -\infty < s \leq t\}$. Then

(21.8) $$\psi_t(s, x) = (2\pi)^{-\frac{1}{2}}V_{st}^{-1}\exp\left[-(x - X_{st})^2/2V_{st}^2\right], \qquad s > t.$$

(In particular, for standard Brownian motion, one gets

$$Z_t(x) = 2^{-\frac{1}{2}}e^{-t}e^{-\sqrt{2}|x - W_t|}.)$$

Unfortunately, however, the $Z_t(x)$'s have not been amenable to calculation outside the Markov case.

3°. Suppose here and in 4° that for each ω, $X.(\omega)$ is an (N, d)-field as described in § 6. Taking the expected value of the expression in (7.1) and interchanging integrals yields:

(21.9) THEOREM. *X is* (LT) *with* $\alpha = \alpha(x) \in L^2(\lambda_d \times \mathbb{P})$ *if and only if*

$$\int_{\mathbb{R}^d}\int_T\int_T \mathbb{E}e^{i\theta\cdot(X_s - X_t)}dsdtd\theta < \infty. \tag{21.10}$$

In particular, if the characteristic function of $X_s - X_t$ is nonnegative and integrable ($d\theta$) for a.e. $(s, t) \in T \times T$, *then for such (s, t), $X_s - X_t$ has a (continuous) density $P_{s,t}(x)$ with $P_{s,t}(x) \leqslant P_{s,t}(0)$ for all x, and* (21.10) *becomes*

$$\int_T\int_T P_{s,t}(0)\, ds\, dt < \infty. \tag{21.11}$$

The criteria (21.10) and (21.11) are due to Berman (1969).

4°. Taking the expected value of the expression in (7.2) (a), and using Fatou's lemma, gives:

(21.12) THEOREM. *If*

$$\liminf_{\varepsilon\downarrow 0}\varepsilon^{-d}\int_T \mathbb{P}\{|X_s - X_t| \leqslant \varepsilon\}\, ds < \infty \quad \textit{for} \quad \text{a.e. } t \in T, \tag{21.13}$$

then X is (LT).

Actually, the following "local" condition suffices for (LT): *for each* $n = 1, 2, \cdots$

$$\liminf_{\varepsilon\downarrow 0}\varepsilon^{-d}\int_{T_n}\mathbb{P}\{|X_s - X_t| \leqslant \varepsilon, X_s \in \Gamma_n, X_t \in \Gamma_n\}\, ds < \infty, \quad \text{a.e. } t \in T_n, \tag{21.14}$$

where $T_n \uparrow T$ a.e. *and* $\Gamma_n \uparrow \mathbb{R}^d$ a.e. ($T_n \uparrow T$ a.e. *means that* $T_1 \subseteq T_2 \subseteq \cdots$ and $T \setminus \cup_n T_n$ is null.)

The analogue of (21.9), via differentiation methods, is

(21.15) THEOREM. *A necessary and sufficient condition for* (LT) *with* $\alpha \in L^2(\lambda_d \times \mathbb{P})$ *is*

$$\liminf_{\varepsilon\downarrow 0}\varepsilon^{-d}\int_T\int_T \mathbb{P}\{|X_s - X_t| \leqslant \varepsilon\}\, dsdt < \infty. \tag{21.16}$$

The "sufficiency" part is immediate from (7.2) (c); the "necessity" part hinges on the Hardy-Littlewood maximal theorem, and here, briefly, is the idea. Assuming $\alpha \in L^2(\lambda_d \times \mathbb{P})$, it follows, for almost every $\omega \in \Omega$ that $\alpha(x, \omega)$ is in $L^2(\mathbb{R}^d)$, that the "maximal function"

$$\tilde{\alpha}(x, \omega) = \sup_{\varepsilon>0}\varepsilon^{-d}\int_{B_d(x,\varepsilon)}\alpha(y, \omega)\, dy$$

is in $L^2(\mathbb{R}^d)$, and that there is a constant C, *depending only on the dimension d*, such that

$$\int\tilde{\alpha}^2(x)\, dx \leqslant C\int\alpha^2(x)\, dx.$$

Thus,

$$\liminf_{\varepsilon\downarrow 0}\varepsilon^{-d}\int_T\int_T \mathbb{P}\{|X_s - X_t| \leqslant \varepsilon\}\, dsdt$$
$$\leqslant \mathbb{E}\int_T \sup_{\varepsilon>0}\varepsilon^{-d}\int_T I_{[0,\varepsilon]}(|X_s - X_t|)\, ds\, dt$$
$$= \mathbb{E}\int_T \tilde{\alpha}(X_t)\, dt = \mathbb{E}\int_{\mathbf{R}^d}\tilde{\alpha}(x)\alpha(x)\, dx \qquad \text{(by (6.5))}$$
$$\leqslant C^{\frac{1}{2}}\mathbb{E}\int\alpha^2(x)\, dx < \infty.$$

Finally, to state the results of L. Pitt (1978) on the existence and higher moments of α, we introduce his *condition* (A_k) *(on H)*: fix an $H \in \mathfrak{B}_d$ with $\lambda_d(H) < \infty$ and suppose for each $(t_1, \cdots, t_k) \in T^k$ with distinct t_i's, the vector $(X_{t_1}, \cdots, X_{t_k})$ has a density $p_k(t_1, \cdots, t_k; x_1, \cdots, x_k)$ such that

$$p_k(t_1, \cdots, t_k; x_1, \cdots, x_k) \leqslant g_k^H(t_1, \cdots, t_k) \text{ for a.e. } (x_1, \cdots, x_k) \in H^k$$

for some g_k^H such that

$$\int_{T^k} g_k^H(t_1, \cdots, t_k)\, dt_1 \cdots dt_k < \infty.$$

(21.17) THEOREM. *Suppose (A_k) holds on H for some $k \geqslant 2$. Then X is* (LT) on *$\{X \in H\}$ and $\alpha(x, \{X \in H\}) \in L^k(dx)$.*

Pitt obtained this by martingale methods; here is a proof based upon a k-dimensional version of (21.14). Fix an ω. Arguing exactly as in § 7, the limit

$$V_H(t) = \lim_{\varepsilon\downarrow 0}(\varepsilon^d c_d)^{-1}\int_T I_{(0,\varepsilon)}(|X_s - X_t|)I_H(X_s)\, ds$$

exists ($\leqslant \infty$) for a.e. t such that $X_t \in H$ and X is (LT) *on* $\{X \in H\}$ iff $V_H(t) < \infty$ a.e. on $\{X \in H\}$. Let $C = c_d^{-(k-1)}$ and $m = k - 1$.

$$\mathbb{E}\int_T V_H^{2m}(t)I_H(X_t)\, dt$$
$$\leqslant C\liminf_{\varepsilon\downarrow 0}\varepsilon^{-md}\int_{T^k}\mathbb{E}\prod_{i=1}^m I_{(0,\varepsilon)}(|X_{s_i} - X_t|)I_H(X_{s_i})I_H(X_t)\, ds_1 \cdots ds_m\, dt$$
$$\leqslant C\liminf_{\varepsilon\downarrow 0}\varepsilon^{-md}\int_{T^k}\int_{H^k}\prod_{i=1}^m I_{(0,\varepsilon)}(|x_i - y|)p_k(s_1, \cdots, s_m, t; x_1, \cdots, x_m, y)$$
$$\times\prod_1^m dx_i\prod_1^m ds_i\, dy\, dt$$
$$\leqslant \text{const.}\times\int_{T^k} g_k^H(s_1, \cdots, s_m, t)\, ds_1 \cdots ds_m\, dt < \infty.$$

This shows that α exists on $\{X \in H\}$ a.s. and the rest follows because, using (6.5),

$$\int_H \alpha^k(x, \{X \in H\})\, dx = \int_T V_H^m(t)I_H(X_t)\, dt.$$

22. The Gaussian case. *Throughout this section,* $X = (X_t) = (X_t^{(1)}, \cdots, X_t^{(d)})$, $t \in T = [0, 1]^N$, *will denote an* (N, d)-Gaussian field, with $EX_t^{(k)} \equiv 0$, each $X^{(k)}$ continuous in probability, and such that the determinant $\Delta(s, t)$ of the covariance matrix of $X_s - X_t$ is positive for a.e. $(s, t) \in T \times T$.

Evaluating the expressions in (21.9) (or (21.14)) and (21.11) yields:

(22.1) THEOREM. *X is* (LT) *if*

$$\int_T(\Delta(s, t))^{-\frac{1}{2}}\, ds < \infty \qquad \textit{for} \quad \text{a.e. } t \in T; \tag{22.2}$$

X is (LT) *with* $\alpha \in L^2(\lambda \times \mathbb{P})$ *if and only if*

(22.3) $$\int_T \int_T (\Delta(s, t))^{-\frac{1}{2}} \, ds \, dt < \infty.$$

Until the end of this section, we take $N = d = 1$ in which case

(22.4) $$\Delta(s, t) = E(X_s - X_t)^2 = R(s, s) + R(t, t) - 2R(s, t),$$

R being the covariance function of X. (The condition (21.4) implies (22.2), but not conversely.) In general, (22.2) is not necessary for (LT) because it is never satisfied when X has differentiable trajectories. Moreover, *there exist examples of nondifferentiable Gaussian processes which are not* (LT). Two such examples were given by the authors (1976). Let $W = (W_t)$ be a standard Brownian motion. If $F(t)$, $0 \leq t \leq 1$, is a strictly increasing, continuous distribution function, then $Y_t(\omega) = W_{F(t)}(\omega)$ is Gaussian with orthogonal increments, and all such processes arise in essentially this manner. For $0 < p < 1$, let $F_p(t)$ be the distribution on [0, 1] corresponding in the usual way (Billingsley (1965), page 35) to an infinite sequence of independent tosses of a coin having probability p of "heads"; then F_p satisfies the above conditions, and so has an inverse function $\hat{F}_p$ which also satisfies the conditions. Let $Y_t^p = W_{F_p(t)}$ $\hat{Y}_t^p = W_{\hat{F}_p(t)}$; then

(i) Y^p *is not* (LT) *if* $pq \leq \frac{1}{16}$ $(q = 1 - p)$;

(ii) $\hat{Y}^p$ *is not* (LT) *if* $p^{2p}q^{2q} > \frac{1}{2}$.

The authors have learned from Yu. Davydov that the orthogonal increments case has been completely resolved by M. Lifschitz:

(22.5) THEOREM. *The condition* (22.2) *is necessary and sufficient in order that* $Y_t = W_{F(t)}$ *be* (LT).

Here (22.2) becomes

(22.6) $$\int_0^1 |F(s) - F(t)|^{-\frac{1}{2}} \, ds < \infty \qquad \text{for a.e. } t \in [0, 1].$$

We have not seen Lifschitz' proof; however, here is a proof based on a recent result of Hawkes (1977). We need only show that (LT) implies (22.6). Define

$$\zeta(r) = \int_0^1 |F(s) - r|^{-\frac{1}{2}} \, ds = \int_0^1 |s - r|^{-\frac{1}{2}} \, d\hat{F}(r),$$

where $\hat{F}$ is the inverse function of F. Since $\int_0^1 \zeta(r) \, dr < \infty$, the set $L = \{\zeta = \infty\}$ has $\lambda(L) = 0$. It also has $\frac{1}{2}$-capacity zero: if not, there would be a probability measure ν concentrated on L such that $\int_0^1 |s - r|^{-\frac{1}{2}} \nu(ds) \leq k$ for all $r \in [0, 1]$, for some $k < \infty$. Integration against $d\hat{F}(r)$ gives the contradiction. Now Hawkes (1977) tells us that, *with positive probability the image* $W(L)$ *of* L *by a Brownian trajectory* W_t *has measure zero*. Fix $\omega \in \Omega$ such that $\lambda(W(L)) = 0$ *and* $Y_t(\omega)$ is (LT); then

$$0 = \lambda\{t : Y_t \in W(L)\} \geq \lambda\{t : F(t) \in L\}.$$

Thus $\zeta(F(t)) < \infty$ a.e., and the proof is concluded.

It is an open question whether (22.2) is necessary for other Gaussian processes to be (LT), e.g., for X stationary with infinite second spectral moment, i.e., $-r''(0) = \infty$, where $r(|s - t|) = R(s, t)$. Here (22.2) becomes

$$\int_0^1 [r(0) - r(s)]^{-\frac{1}{2}}\, ds < \infty. \tag{22.7}$$

There are other unresolved issues when (22.7) fails and $-r''(0) = \infty$; for instance, *is M_x infinite with positive probability for every x?* Klein proved this is so *if* (22.7) *holds* (see § 29).

Finally, the general effect of imposing conditions such as (22.3) is to cause the trajectories to oscillate wildly. This will be illustrated in later sections, but, as a simple example, notice that (22.3) implies that, a.s., $\alpha(x) \in L^2(dx)$, and hence (by (9.1)) *the trajectories cannot satisfy at any point a Hölder condition of any order* $> 2N/d$. For $N = d = 1$, Berman (1969a) proves the following more interesting result by a real variable argument using Fourier analysis:

(22.8) THEOREM. *Let $m \geq 0$ be an integer, $0 < \varepsilon \leq 1$, and put $p = 2m + \varepsilon$. Suppose*

$$\int_0^1\int_0^1 (\Delta(s, t))^{-((p+1)/2)}\, ds\, dt < \infty; \tag{22.9}$$

then a.s.,

(a) *the γ-variation of the trajectory is infinite for $\gamma = p + 1$;*

(b) *the trajectory does not satisfy a Hölder condition of order $2/(p + 1)$ at any point t.*

Of course, (22.9) (even with $p = 0$) already implies the existence of α; when $p = 1$, it follows from (b) that the trajectories are *nowhere differentiable.*

23. The [AC-*p*] conditions. These conditions are discussed in § 10, in particular the consequences for the local growth of X. Here is a sufficient condition for a (general) (N, d)-field X to satisfy [AC-0]:

(23.1) THEOREM. *Suppose $\Gamma_n \uparrow \mathbb{R}^d$* a.e. *and for each n,*

$$\int_T \sup_{\varepsilon > 0} \varepsilon^{-d} \mathbb{P}\{|X_s - X_t| \leq \varepsilon, X_s \in \Gamma_n, X_t \in \Gamma_n\}\, ds < \infty \quad \textit{for} \ \text{a.e. } t. \tag{23.2}$$

Then X is (LT) *and, for* a.e. $x \in \mathbb{R}^d$, *$\alpha(x, dt)$ has no point masses.*

The existence of α follows from (21.4) with $T_n \equiv T$ (the T_n's could, of course, be incorporated into (23.2)). The discussion in § 7 shows that for $B \in \mathcal{B}(T)$:

$$\alpha(X_t, B \cap X^{-1}(\Gamma_n)) = \lim_{\varepsilon \downarrow 0} \frac{1}{c_d \varepsilon^d} \int_B I_{\Gamma_n}(X_s) I_{(0, \varepsilon)}(|X_s - X_t|)\, ds$$

for a.e. $t \in T$, with probability 1. Fatou's lemma yields, for a.e. $t \in T$,

$$\mathbb{E}[\alpha(X_t, B \cap X^{-1}(\Gamma_n)); X_t \in \Gamma_n] \leq \frac{1}{c_d} \int_B \sup_{\varepsilon > 0} \varepsilon^{-d} \mathbb{P}\{|X_s - X_t| \leq \varepsilon, X_s \in \Gamma_n, X_t \in \Gamma_n\}\, ds. \tag{23.3}$$

It follows that, for a.e. t, (23.3) holds simultaneously for all N-dimensional boxes B with rational corners. Fixing one of these t's, choosing $B \downarrow \{t\}$, we find

$$\mathbb{E}\left[\alpha(X_t, \{t\}); X_t \in \Gamma_n\right] = 0.$$

Now let $\Gamma_n \uparrow \mathbb{R}^d$ to obtain $\alpha(X_t, \{t\}) = 0$ a.s. a.e. and then use (6.8) to finish the proof.

We can derive the conclusions of (23.1) under a slightly different hypothesis, viz. that X_s *has an absolutely continuous distribution for each* $s \in T$, *and there is an open strip* K *containing the diagonal in* $\mathbb{R}^d \times \mathbb{R}^d$ *on which each pair* (X_s, X_t) $s \neq t$, *has a continuous density* $P^K(s, t; x, y)$ *with*

$$\int_T \int_T \sup_{x, y \in K} P^K(s, t; x, y)\, ds\, dt < \infty. \tag{23.4}$$

Indeed, these hypotheses imply those of (23.1) as long as the Γ_n's are bounded, and this avoids the Fourier-analytic proof of Geman (1976) that (23.4) implies [AC-0]. Of course, Pitt's (A_2) condition is essentially (23.4), as are the hypotheses in Marcus (1976).

In the Gaussian case, (23.2) reduces to (22.2); whereas, in the (1, 1)-case, (23.4) reduces to (21.4).

In his (1977) paper, the first author gave a stochastic condition for the a.s. validity of the condition [AC-p] with $0 < p < N$.

(23.5) THEOREM. *Suppose* $0 < p < N$ *and*

$$\int_{[0,1]^{N-p}} \sup_{\varepsilon > 0} \varepsilon^{-d} \mathbb{P}\{|X(t_1, \cdots, t_p, s_1, \cdots, s_{N-p}) - X(t_1, \cdots, t_N)| \leq \varepsilon\} \lambda_{N-p}(ds) < \infty \tag{23.6}$$

for λ_N-a.e. $t \in T$. *Then* X *is* (LT) *and* α *satisfies* [AC-p] a.s. (*In* (23.6), $s = (s_1, \cdots, s_{N-p})$.)

When $p = 0$, we are back in the situation of (23.1), without the Γ_n's; we omit the "local version" of (23.5).

To prove (23.5), let us write temporarily $t^* = (t_1, \cdots, t_p)$; this is construed as *a function of* $t \in T$ as well as *a point in* $[0, 1]^p$. We begin observing that, for λ_p-a.e. $t^* \in [0, 1]^p$, the $(N - p, d)$-field $s \mapsto X(t^*, s)$ has an occupation density $\alpha(t^*; x, A)$ $(A \in \mathfrak{B}([0, 1]^{N-p}))$ which satisfies [AC-0]; this is proven in much the same way as is (23.1).

Now let $B = B_p \times B_{N-p}$ be the product of Borel sets in the indicated unit cubes. Using a self-evident notation, we have

$$\begin{aligned}\int_B I_\Gamma(X(t))\lambda_N(dt) &= \int_{B_p}\int_{B_{N-p}} I_\Gamma(X(t^*, s))\lambda_{N-p}(ds)\lambda_p(dt^*)\\ &= \int_\Gamma \int_{B_p} \alpha(t^*; x, B_{N-p})\lambda_p(dt^*)\, dx,\end{aligned}$$

which shows that X is (LT) and that the occupation kernel $\alpha(x, dt)$ has the disintegration

$$\alpha(x, B_p \times B_{N-p}) = \int_{B_p} \alpha(t^*; x, B_{N-p})\lambda_p(dt^*),$$

i.e. satisfies condition [AC-p].

In the Gaussian case, the largest $p < N$ for which (23.6) holds can usually be computed. For example, if X has i.i.d. components and $E|X_s^{(i)} - X_t^{(i)}|^2 = |s - t|^{2\beta}$, $0 < \beta < 1$, $1 \leqslant i \leqslant d$, then it is the largest integer $< N - d\beta$. This example is discussed in § 30.

24. Local nondeterminism. The concept of *local nondeterminism* (LND for short) was introduced by Berman (1973) to unify and extend his earlier work on Gaussian occupation densities ($N = d = 1$). Essentially, a process is LND if it is "locally unpredictable", in that there is an unremovable element of "noise" in the *local* evolution of the process. In his main result (1973, Theorem 8.1), Berman proves that LND, together with lower bounds on the growth of $\Delta(s, t)$, implies that X has a jointly continuous occupation density which satisfies a uniform Hölder condition in t, uniformly in x. Pitt (1978) extended the definition of LND to Gaussian (N, d)-fields and obtained Hölder conditions in the space variable. (Actually, Hölder conditions in x are implicit in Berman's first paper (1969).) We will reproduce Pitt's results and also give a complete extension of Berman's results to Gaussian fields. Further material on LND for Gaussian processes ($N = d = 1$) can be found in Cuzick (1978). We wish to thank L. Pitt for a letter which explains why his formulation is equivalent to Berman's.

In what follows we write $V(\xi)$ for the variance of a random variable ξ and $\langle u, v\rangle$ for the usual inner product of two vectors $u, v \in \mathbb{R}^d$.

From this point on, X denotes a Gaussian (N, d)-field as described in § 22. *We will assume that* $X_0 = 0$; this simplifies the notation below and is no real restriction —cf. § 30. The covariance matrix of $X_t - X_s$ is denoted $\sigma^2(t, s)$, and $\sigma(t, s)$ is now the nonsingular matrix with $\sigma(t, s)\sigma'(t, s) = \sigma^2(t, s)$, $t \neq s$, where the prime denotes transpose. (Hence $\Delta(s, t) = \det \sigma^2(t, s)$.)

X is *locally nondeterministic* (or "is (LND)") if, for each $k \geqslant 2$ and each $(u_1, \cdots, u_k) \in \mathbb{R}^{dk} \setminus \{0\}$, there exist constants $c > 0$ and $\delta > 0$ such that

$$V\left(\textstyle\sum_{j=1}^{k}\langle u_j, \sigma_j^{-1}(X_{t_j} - X_{t_{j-1}})\rangle\right) \geqslant c \tag{24.1}$$

whenever (distinct) points $t_1, \cdots, t_k$ all lie in a cube of edge length at most δ and satisfy

$$|t_{j+1} - t_j| \leqslant |t_{j+1} - t_i| \qquad \text{for all} \quad 1 \leqslant i \leqslant j \leqslant k. \tag{24.2}$$

Here $t_0 = 0$ and $\sigma_j = \sigma(t_j, t_{j-1})$. The inequalities (24.2) are automatically satisifed when $N = 1$ and $t_1 < \cdots < t_k$. In general, for $t_1, \cdots, t_k$ in T, there is a permutation τ of $(1, \cdots, k)$ such that $t_{\tau(1)}, \cdots, t_{\tau(k)}$ satisfies (24.2): choose $\tau(k) = 1$, then $\tau(k-1)$ to satisfy $|t_{\tau(k-1)} - t_{\tau(k)}| \leqslant |t_{\tau(k)} - t_j|$ for all $j \neq \tau(k)$, etc. The point of (24.2) is that, in examples (e.g., Pitt (1978)), one need only verify (24.1) subject to (24.2).

(24.3)) THEOREM. *If X is* (LND), *then for each $k \geq 2$ there exists a $c > 0$ such that*

$$V\left(\Sigma_{j=1}^{k}\langle u_j, \sigma_j^{-1}(X_{t_j} - X_{t_{j-1}})\rangle\right) \geq c \tag{24.4}$$

for every choice of distinct $t_1, \cdots, t_k \in T$ subject to (24.2) *and of $u_1, \cdots, u_k \in \mathbb{R}^d$ subject to $\Sigma|u_j|^2 = 1$.*

PROOF. For brevity, we write $\bar{t} = (t_1, \cdots, t_k) \in T^k$, $\bar{u} = (u_1, \cdots, u_k) \in (\mathbb{R}^d)^k$, $\mathfrak{T}$ for the set of $\bar{t}$ which satisfy (24.2) and S for the unit sphere in $(\mathbb{R}^d)^k$. Were the theorem false, there would be a $k \geq 2$ and sequences $\bar{u}^{(n)} \in S$ and $\bar{t}^{(n)} \in \mathfrak{T}$ such that

$$V\left(\Sigma_{j=1}^{k}\langle u_j^{(n)}, \sigma^{-1}(t_j^{(n)}, t_{j-1}^{(n)})(X_{t_j^{(n)}} - X_{t_{j-1}^{(n)}})\rangle\right) \leq 1/n. \tag{24.5}$$

Extracting a subsequence, if necessary, we may assume $\bar{u}^{(n)} \to \bar{u} \in S$ and $\bar{t}^{(n)} \to \bar{t}^{(\infty)}$. Write $Y_j^{(n)}$ for $\sigma^{-1}(t_j^{(n)}, t_{j-1}^{(n)})(X_{t_j^{(n)}} - X_{t_{j-1}^{(n)}})$ and $\bar{Y}^{(n)}$ for $(Y_1^{(n)}, \cdots, Y_k^{(n)})$. Each $Y_j^{(n)}$ is normal, mean 0 and has covariance matrix the identity. Therefore, the sequence $\{\bar{Y}^{(n)}\}$ is tight and extracting another subsequence we may assume $\bar{Y}^{(n)}$ converges in distribution to a normal vector $\bar{Y}$. An easy argument involving the parameters of the normal distribution now shows

$$\Sigma_{j=1}^{k}\langle u_j^{(n)}, Y_j^{(n)}\rangle \to \Sigma_{j=1}^{k}\langle u_j, Y_j\rangle \tag{24.6}$$

and

$$\Sigma_{j=1}^{k}\langle u_j, Y_j^{(n)}\rangle \to \Sigma_{j=1}^{k}\langle u_j, Y_j\rangle, \tag{24.7}$$

in distribution. By (24.5), the limit $\Sigma\langle u_j, Y_j\rangle$ has variance zero, but then (24.7) shows that (24.1) cannot hold as $\bar{t}$ runs through $\bar{t}^{(n)}$.

25. Basic estimates. Let $\bar{t} = (t_1, \cdots, t_k) \in T^k$ and $D_k(\bar{t})$ be the determinant of the covariance matrix of $(X_{t_1}, \cdots, X_{t_k})$. Since, for distinct t_i, $p_k(\bar{t}, \bar{x})$ is continuous in $\bar{x}$, and $p_k(\bar{t}, \bar{x}) \leq (2\pi)^{-dk/2}D^{-\frac{1}{2}}(\bar{t})$, the condition (A_k) on $H = \mathbb{R}^d$ (§21) becomes

$$\int_{T^k} D_k^{-\frac{1}{2}}(\bar{t})\, d\bar{t} < \infty. \tag{25.1}$$

Now fix an even integer $k \geq 2$ and assume (25.1) *holds* (this remains in force until (25.12)). By (21.17) α exists and is in $L^k(dx)$ a.s.

We wish to estimate $\mathbb{E}[\alpha(x, B) - \alpha(y, B)]^k$ for fixed $B \in \mathfrak{B}(T)$, but first we must specify α more carefully. In §§15 and 21 we mentioned that $\alpha(x, B)$ could be chosen as a kernel on $\mathbb{R}^d \times \Omega \times \mathfrak{B}(T)$, but this "abstractly" given kernel need not have any nice analytical properties as a function of x or B or both. Viewing α as a process with parameter set $\mathbb{R}^d \times \mathfrak{B}(T)$, we now choose a version with nicer properties. Let

$$\alpha_0(x, B) = \liminf_{n\to\infty} \frac{1}{c_d n^{-d}} \int_B I_{(0, 1/n)}(|X_s - x|)\, ds, \tag{25.2}$$

which, for each fixed B, actually exists as a finite *limit* for a.e. x, a.s.; α_0 is jointly

(x, ω)-measurable for each B, and, if $\{B_n\}$ is a disjoint sequence in $\mathfrak{B}(T)$,

$$\alpha_0(\cdot, \Sigma_n B_n) = \Sigma_n \alpha_0(\cdot, B_n) \qquad \lambda_d \times \mathbb{P}\text{-a.e.} \tag{25.3}$$

(In fact, (25.3) holds simultaneously for all finite disjoint sequences $\{B_n\}$ of boxes with rational corners.)

Define

$$v(\bar{x}) = \int_{T^k} p_k(\bar{t}; \bar{x})\, d\bar{t}, \quad \bar{x} = (x_1, \cdots, x_k) \in (R^d)^k. \tag{25.4}$$

Then v is uniformly continuous on $(\mathbb{R}^d)^k$ because, for each $\bar{t} \in T^k$, $p_k(\bar{t}; \bar{x})$ is continuous in $\bar{x}$ and $\to 0$ as $|\bar{x}| \to \infty$, and hence v is also continuous and vanishes as $|\bar{x}| \to \infty$ by dominated convergence, using (25.1).

Let $\alpha_n(x, B)$ be the expression after the "lim inf" in (25.2) and let $\alpha_n(x) = \alpha_n(x, T)$. The uniform continuity of v implies that *$\alpha_n(x)$ is Cauchy in $L^k(\mathbb{P})$, uniformly in x*. (The proof is an elaboration of arguments such as those in Proposition 3.1 of Pitt (1977).) Let (n_p) be a sequence of integers such that

$$\sup_x \mathbb{E}\left[\alpha_{n_{p+1}}(x) - \alpha_{n_p}(x)\right]^k \leq 2^{-p}, \qquad p \geq 1.$$

It follows that $\alpha_{n_p}(x)$ converges uniformly in $L^k(\mathbb{P})$ and converges a.s. for each x fixed. Hence, at least through (n_p), the *limit* exists at (25.2) both uniformly in $L^k(\mathbb{P})$ and a.s. for each x, and there is no problem in then arguing as if (n_p) were the full sequence of integers. Clearly, a similar result obtains for $\alpha_n(x, B)$ for each $B \in \mathfrak{B}(T)$.

Now for any $x_1, \cdots, x_k \in \mathbb{R}^d$ and $B \in \mathfrak{B}(T)$, since $\alpha_n(x_i, B)$ converges to $\alpha_0(x_i, B)$ in $L^k(\mathbb{P})$, $1 \leq i \leq k$, we have $\Pi_1^k \alpha_n(x_i, B)$ converging to $\Pi_1^k \alpha_0(x_i, B)$ in $L^1(\mathbb{P})$. Hence, an easy computation yields:

$$\mathbb{E}\left[\alpha_0(x_1, B) \cdots \alpha_0(x_k, B)\right]^k = \int_{B^k} p_k(\bar{t}; \bar{x})\, d\bar{t} \qquad \text{for all} \quad \bar{x}, \tag{25.5}$$

which leads to

$$\mathbb{E}\left[\alpha_0(x, B) - \alpha_0(y, B)\right]^k = \int_{B^k} \Sigma_{j=0}^k (-1)^j \Sigma_{i=1}^{\binom{k}{j}} p_k(\bar{t}; \bar{z}_{ij})\, d\bar{t} \tag{25.6}$$

where, as i runs from 1 to $\binom{k}{j}$, $\bar{z}_{ij}$ runs through all possible points $(x \cdots y \cdots)$ in $(\mathbb{R}^d)^k$ having an x in exactly j coordinates and y in the remaining $k - j$ coordinates.

Remark. Due to the central role that (25.6) plays in obtaining smooth versions of α, it should be pointed out that (25.6) *is valid for an arbitrary stochastic (N, d)-field such that* (i) *(A_k) holds on $\mathbb{R}^d$ and* (ii) *$v(\bar{x})$ is uniformly continuous.*

We now begin to estimate the right member of (25.6). The characteristic function of the vector $(X_{t_1}, \cdots, X_{t_k})$, construed as $(X_{t_1}^{(1)}, X_{t_1}^{(2)}, \cdots, X_{t_k}^{(d)})$, is

$$\hat{p}(\bar{t}; \bar{u}) = \int_{(\mathbb{R}^d)^k} e^{i\bar{u}\cdot\bar{x}} p_k(\bar{t}; \bar{x})\, d\bar{x} = \exp\left[-\tfrac{1}{2} V\left(\Sigma_{j=1}^k \langle u_j, X_{t_j} \rangle\right)\right]$$

and Fourier inversion gives

$$p(\bar{t}; \bar{x}) = (2\pi)^{-kd} \int_{(\mathbb{R}^d)^k} e^{-i\bar{u}\cdot\bar{x}} \hat{p}(\bar{t}; \bar{u})\, d\bar{u}.$$

Substituting this into (25.6), and changing y to $x + w$, the right member there is

$$(2\pi)^{-kd}\int_{B^k}\int_{(\mathbb{R}^d)^k}\hat{p}(\bar{t};\bar{u})e^{-i\bar{u}\cdot\bar{x}}\sum_{j=0}^{k}(-1)^j\sum_{i=1}^{\binom{k}{j}}e^{-i\bar{u}\cdot\bar{\zeta}_{ij}}\,d\bar{u}\,d\bar{t},$$

where $\bar{x} = (x, \cdots, x)$ and $\bar{\zeta}_{ij}$ has zero and w wherever $\bar{z}_{ij}$ has x and y respectively. The double sum can be written

$$\sum_{j=0}^{k}(-1)^j\sum_{(i_1,\cdots,i_j)}\prod_{q=1}^{k-j}e^{-i\langle u_{i'_q}, w\rangle}$$

where the inner sum is over all samples $(i_1, \cdots, i_j)$ from the integers $\{1, 2, \cdots, k\}$, and $(i'_1, \cdots, i'_{k-j})$ are the "remaining" integers in $\{1, \cdots, k\}$. By induction, the above expression is $\prod_{j=1}^{k}(1 - e^{-i\langle u_j, w\rangle})$, and hence

(25.7) $$\mathbb{E}[\alpha_0(x, B) - \alpha_0(x + w, B)]^k = (2\pi)^{-kd}\int_{B^k}\int_{(\mathbb{R}^d)^k}\hat{p}(\bar{t};\bar{u})e^{-i\bar{u}\cdot\bar{x}}\prod_{j=1}^{k}(1 - e^{-i\langle u_j, w\rangle})\,d\bar{u}\,d\bar{t}.$$

Since $\hat{p}$ is positive, and since $|1 - e^{i\theta}| \leqslant |\theta|^\gamma$ for all θ, for any $0 < \gamma < 1$, we find that $\mathbb{E}[\alpha_0(x, B) - \alpha_0(x + w, B)]^k$ is at most a constant times

(25.8) $$|w|^{k\gamma}\int_{B^k}\int_{(\mathbb{R}^d)^k}\exp\left[-\tfrac{1}{2}V\left(\textstyle\sum_{j=1}^{k}\langle u_j, X_{t_j}\rangle\right)\right]\prod_{j=1}^{k}|u_j|^\gamma\,d\bar{u}\,d\bar{t}.$$

The finiteness of this integral would allow us to use a Kolmogorov type argument to obtain continuity of $\alpha_0(x, B)$ in x.

Let $m_k(\bar{t})$ denote the *inner* integral in (25.8); then $m_k((t_i)) = m_k((t_{\sigma(i)}))$ for any permutation σ of $\{1, 2, \cdots, k\}$. Since every point $\bar{t}$ in T can be permuted into $\mathfrak{T}$, the integral in (25.8) is dominated by a constant times the integral over $(B^k \cap \mathfrak{T}) \times (\mathbb{R}^d)^k$. We will keep writing $B^k \times (\mathbb{R}^d)^k$, assuming when necessary that $\bar{t}$ satisfies (24.2). The rest of the argument is just as in Pitt. Make a change of variables: $u_j = v_j - v_{j+1}$ $(j \leqslant k - 1)$, $u_k = v_k$. Putting $v_{k+1} = 0$, the integrand becomes

$$\exp\left[-\tfrac{1}{2}V\left(\textstyle\sum_1^k\langle v_j, X_{t_j} - X_{t_{j-1}}\rangle\right)\right]\prod_{j=1}^{k}|v_j - v_{j+1}|^\gamma, \qquad \text{still over } B^k \times (\mathbb{R}^d)^k.$$

Since $|a - b|^\gamma \leqslant |a|^\gamma + |b|^\gamma$, the product term is majorized by a finite sum of terms (the number of which depends only on k) each of the form $\prod_{j=1}^{k}|v_j|^{\gamma\beta_j}$ where $\beta_j = 0$, 1, or 2.

If X is (LND), then the integral in (25.8) can be dominated by

(25.9) $$\int_{B^k}\int_{(\mathbb{R}^d)^k}\exp\left(-\tfrac{1}{2}c\textstyle\sum_{j=1}^{k}|\sigma'_j v_j|^2\right)\prod_{j=1}^{k}|v_j|^{\gamma\beta_j}\,d\bar{v}\,d\bar{t},$$

for some c as given in (24.3). The further change of variables $w_j = \sigma'_j v_j$ converts this to

(25.10) $$\int_{B^k}\int_{(\mathbb{R}^d)^k}e^{-\frac{1}{2}c|\bar{w}|^2}\prod_{j=1}^{k}|(\sigma'_j)^{-1}w_j|^{\gamma\beta_j}\prod_{j=1}^{k}\Delta_j^{-\frac{1}{2}}\,d\bar{w}\,d\bar{t}$$

where $\Delta_j = \Delta(t_j, t_{j-1})$. Since each component $X^{(i)}$ is continuous in probability, each $X^{(i)}$ has a continuous covariance function and hence, likewise, the subdeterminants of $\sigma^2(s, t)$. There is, then, a constant B with $|(\sigma'_j)^{-1}w| \leqslant B\Delta_j^{-\frac{1}{2}}|w|$ for all j and all $w \in \mathbb{R}^d$. Moreover, $\Delta_j^{-\gamma\beta_j/2} \leqslant C\Delta_j^{-\gamma}$ where $C = \max(1, (\sup_{s,t}\Delta(s, t)^2)$. Thus, (25.10) is dominated by a constant multiple of

$$\int_{B^k} \prod_{j=1}^k \Delta_j^{-(1+2\gamma)/2}\, d\bar{t} \cdot \int_{(\mathbf{R}^d)^k} e^{-(c/2)|\bar{w}|^2} \prod_{j=1}^k |w_j|^{\gamma\beta_j}\, d\bar{w}.$$

Since the second factor is clearly finite, the issue is whether or not

$$\int_{B^k} \prod_{j=1}^k \Delta_j^{-(\frac{1}{2}+\gamma)}\, d\bar{t} < \infty. \tag{25.11}$$

The utility of these estimates and the concept of (LND) resides in the following:

(25.12) PROPOSITION. *Suppose X is* (LND).

(a) *If* (25.11) *holds for some $k \geqslant 2$ with $\gamma = 0$ and $B = T$, then* (25.1) *holds for* that k.

(b) *If* (25.11) *holds with $B = T$ for some $k \geqslant 2$ and $\gamma \geqslant 0$, then*

$$\int_T \int_T \frac{ds\, dt}{(\Delta(s, t))^{1/2+\gamma}} < \infty.$$

(c) *Suppose, for some $\gamma > 0$,*

$$\sup_s \int_T \frac{dt}{(\Delta(s, t))^{1/2+\gamma}} < \infty. \tag{25.13}$$

Then (25.11) *holds for* every $k \geqslant 2$, *and every $B \in \mathfrak{B}(T)$.*

Beside (LND), (25.13) is the basic assumption in Pitt (1978). Part (a) then justifies Pitt's tacit assumption that his condition A_k holds for *all* $k \geqslant 2$.

PROOF OF (25.12). The proof of (a) is implicit in the arguments above; for any $\bar{x}$ and $\bar{t} \in \mathfrak{T}$:

$$\begin{aligned} p_k(\bar{t}; \bar{x}) &\leqslant (2\pi)^{-kd} \int \hat{p}(\bar{t}; \bar{u})\, d\bar{u} \\ &= (2\pi)^{-kd} \int e^{-\frac{1}{2} V(\Sigma \langle u_j, X_{t_j} \rangle)}\, d\bar{u} \\ &\leqslant (2\pi)^{-kd} \int e^{-\frac{1}{2} c \Sigma |\sigma_j v_j|^2}\, d\bar{v} \\ &= (2\pi)^{-kd} \cdot \prod_1^k \Delta_j^{-\frac{1}{2}} \cdot \int e^{-\frac{1}{2} c |\bar{w}|^2}\, d\bar{w}. \end{aligned}$$

Since this estimate is "permutation invariant" as above, and $D_k(\bar{t}) = \sup_x p_k(\bar{t}; \bar{x})$, (25.1) follows from (25.11) with $\gamma = 0$.

For (b), let $M = \sup_{s, t \in T} \Delta(s, t)$. Then

$$\int_T \cdots \int_T \prod_1^k (\Delta(t_j, t_{j-1}))^{-(\frac{1}{2}+\gamma)}\, dt_1 \cdots dt_k$$
$$\geqslant \left(M^{-(\frac{1}{2}+\gamma)}\right)^{k-2} \int_T \int_T \frac{dt_{k-1}\, dt_k}{(\Delta(t_k, t_{k-1}))^{1/2+\gamma}}.$$

Finally, using iterated integrals as in (b), and doing the dt_k integral first, it follows that the integral in (25.11) is dominated by

$$\left[\sup_{s \in T} \int_T \frac{dt}{(\Delta(s, t))^{1/2+\gamma}}\right]^{k-1} \int_T \frac{dt}{(\Delta(0, t))^{1/2+\gamma}} < \infty.$$

26. Joint continuity (Gaussian case). Let $\mathcal{K}$ be the family of rational boxes $B = \prod_1^N J_i$ (i.e., each J_i a rational interval in [0, 1]) and let Q_t be the "quadrant" in T with upper right corner at t: $Q_t = \{s \in T : s_i \leq t_i, 1 \leq i \leq N\}$. Also, define

$$V_{k,\gamma}(B) = \int_{B^k} \prod_{j=1}^k \Delta_j^{-(\frac{1}{2}+\gamma)} d\bar{t}.$$

The following result is due to Berman (1970, 1973) for $N = d = 1$ and to Pitt (1978) in general. We have made a few improvements (e.g., the "kernel" part) and tried to clarify some points in Pitt's proof (e.g., the "joint continuity" part).

(26.1) THEOREM. *Let X be* (LND) *and suppose $V_{k,\gamma}(T) < \infty$ for some $0 < \gamma \leq 1$ and even integer $k > d/\gamma$. Then X has an occupation kernel $\alpha(x, dt)$ which is* a.s. *jointly continuous in the sense that $(t, x) \mapsto \alpha(x, Q_t)$ is continuous on $T \times \mathbb{R}^d$. Moreover, for each $\beta < \gamma - (d/k)$, $B \in \mathcal{K}$, and compact $U \subset \mathbb{R}^d$*:

$$\sup_{x,y\in U} \frac{|\alpha(x, B) - \alpha(y, B)|}{|x - y|^\beta} < \infty \text{ a.s.} \tag{26.2}$$

Finally, if (25.13) *holds, then we can choose any $\beta < \gamma$ in* (26.2)

PROOF. According to (25.12) (c), the last statement of the theorem will follow from the rest of it. From the material in §25, we know that X is (LT) and for every $B \in \mathcal{B}(T)$:

$$\mathbb{E}|\alpha_0(x, B) - \alpha_0(y, B)|^k \leq C|x - y|^{\gamma k} V_{k,\gamma}(B) < \infty, \tag{26.3}$$

where C is a constant independent of B.

For each $B \in \mathcal{K}$, we choose a measurable, separable version of the process $\alpha_0(x, B)$, $x \in \mathbb{R}^d$, call it $\alpha_1(x, B)$; as shown by Cohn (1972), this works in the same way as in the one-dimensional case. For $\delta > 0$, define

$$K(B, U; k, \delta) = \int_U \int_U \left| \frac{\alpha_0(x, B) - \alpha_0(y, B)}{|x - y|^\delta} \right|^k dx\, dy.$$

Then

$$\mathbb{E}K(B, U; k, \delta) \leq CV_{k,\delta}(T) \int_U \int_U |x - y|^{k(\gamma-\delta)} dx\, dy, \tag{26.4}$$

which is finite as long as $\delta - \gamma < d/k$. According to Garsia's (1971) lemma, for each $B \in \mathcal{K}$,

$$|\alpha_1(x, B) - \alpha_1(y, B)| \leq C_1(K(B, U; k, \delta))^{1/k} |x - y|^{\delta-(2d/k)} \tag{26.5}$$

for all $x, y \in U$ a.s., where $C_1 = C_1(\omega)$ is independent of B. Since this works for *any* $2d/k < \delta < \gamma + (d/k)$, we see that $\alpha_1(\cdot, B)$ satisfies a Lipschitz condition on U of any order $< \gamma - (d/k)$, the Lipschitz constant depending on B. In particular, since U is arbitrary, and $\mathcal{K}$ is countable, $\alpha_1(\cdot, B)$ is continuous on $\mathbb{R}^d$ for every $B \in \mathcal{K}$, a.s. We still need to find an occupation *kernel* on $\mathbb{R}^d \times \mathcal{B}(T)$ with these properties and which is jointly continuous.

Let $\alpha(x, dt)$ be any version of the occupation kernel. Then for each $B \in \mathcal{K}$,

$$\alpha_1(x, B) = \alpha_0(x, B) = \alpha(x, B) \quad \lambda_d \times \mathbb{P} - \text{a.e.},$$

and it follows that we can choose an $x_0 \in \mathbb{R}^d$ such that, with probability one, $\alpha_1(x_0, B) = \alpha(x_0, B)$ for all $B \in \mathcal{K}$: thus, $\alpha(x_0, dt)$ is the extension of $\alpha_1(x_0, \cdot)$ to a *measure* on $\mathcal{B}(T)$.

(26.6) LEMMA. *The measure $\alpha(x_0, dt)$ has no mass on any hyperplane in T of dimension less than N which is parallel to the "coordinate hyperplanes"*, a.s.

This is similar to, but weaker than, the [AC-p] condition for $p = N - 1$. The proof is easy to describe when $N = 2$: divide the unit square into vertical strips I_{nj} of width 2^{-n} and let $\alpha_n = \max\{\alpha(x_0, I_{nj}) : 1 \leqslant j \leqslant 2^n\}$. Now the argument given by Pitt (1978), in the proof of Proposition 3.2, shows that $\mathbb{E}\alpha_n^k \to 0$ as $n \to \infty$, which implies that there are no vertical line masses. The same type of argument gives the general result. We note, for use below, that *$s \mapsto \alpha(x_0, \{t : t_j \leqslant s\})$ is uniformly continuous on* $[0, 1]$ *for each* j, a.s.

Fix a (large) closed set $U \subset \mathbb{R}^d$ containing x_0 and define, for rational $s \in [0, 1]$,

$$\alpha_j(s, x) = \alpha_1(x, T \cap \{t : t_j \leqslant s\}).$$

From the discussion in §25 we know that $\alpha_0(x, T \cap \{t : t_j \leqslant s\})$ is increasing (as a function of rational s) a.s. for a.e. x, and hence $\alpha_j(s, x)$ has the same property since $\alpha(x, B) = \alpha_0(x, B)$ for all $B \in \mathcal{K}$ a.s., for a.e. x. It then follows from the continuity in x of $\alpha_j(s, x)$ (for each rational s) that $\alpha_j(\cdot, x)$ is $\uparrow$ for every $x \in \mathbb{R}^d$, a.s. Moreover, with probability 1,

$$\alpha_1(x, B_1 \cup B_2) = \alpha_1(x, B_1) + \alpha_1(x, B_2) \qquad \text{for every} \quad x,$$

whenever B_1, B_2 (disjoint) and $B_1 \cup B_2$ are in $\mathcal{K}$. This is due again to the fact that $\alpha_0(x, \cdot)$ is finitely additive on $\mathcal{K}$, for a.e. x, a.s.

Let $I_{ni} = \{s : (i - 1)/n < s_j \leqslant i/n\}$. Then

$$\alpha_j\left(\frac{i}{n}, x\right) - \alpha_j\left(\frac{i-1}{n}, x\right) = \alpha_1(x, I_{ni}) \qquad \text{for all} \quad x, \text{ a.s.},$$

for any (appropriate) i and n. Writing $\|f\|$ for $\sup_{x \in U}|f(x)|$ and using (26.5) with $\delta = \gamma$, we find that a.s.

$$\|\alpha_1(\cdot, I_{Ni})\| \leqslant \alpha(x_0, I_{ni}) + c(K(I_{ni}, U))^{1/k} \qquad \text{for all} \quad n, i,$$

the constant c being independent of n and i. Raising both sides to the kth power and using the inequality $(a + b)^k \leqslant 2^k a^k + 2^k b^k$, we have

(26.7)

$$\Sigma_{i=1}^n \left\| \alpha_j\left(\frac{i}{n}, \cdot\right) - \alpha_j\left(\frac{i-1}{n}, \cdot\right) \right\|^k \leqslant 2^k \Sigma_{i=1}^n (\alpha(x_0, I_{ni}))^k + 2^k c^k \Sigma_{i=1}^n K(I_{ni}, U).$$

The first term on the right is dominated by

$$2^k(\max_{1 \leqslant i \leqslant n} \alpha(x_0, I_{ni}))^{k-1} \alpha(x_0, T),$$

which $\to 0$ as $n \to \infty$ in view of the uniform continuity of $s \mapsto \alpha(x_0, \{t : t_j \leqslant s\})$. In addition, the expected value of the second term $\to 0$ as $n \to \infty$; for example, with $k = 2, j = 1$ and letting $(\Delta_1\Delta_2)^{-(\frac{1}{2}+\delta)} = \zeta(t_1, t_2) = \zeta(r, s; u, v)$, we have

$$\mathbb{E}\Sigma_{i=1}^{n} K(I_{ni}, U) \leqslant C \textstyle\int_T \int_T \left(\int\int_{D_n} \zeta(r, s; u, v)\, dr\, du\right) ds\, dv,$$

where $D_n = \cup_1^n((i-1)/n, i/n) \times ((i-1)/n, i/n)$, which tends to zero by dominated convergence. Thus, with probability one, the left term in (26.7) converges to zero through some subsequence (n_p).

It is now easy to see that, a.s., $\alpha_j(\cdot, x)$ is *uniformly continuous on the rationals, uniformly in* $x \in U$, and so has a jointly continuous extension to $T \times U$. A simple argument then shows that, a.s., $\alpha_1(x, Q_t)$ is a uniformly continuous function of t_j for $t = (t_1, \cdots, t_N)$ rational, uniformly for $x \in U$ *and* for t_i rational $(i \neq j)$, and a slight elaboration proves that $\alpha_1(x, Q_t)$ is a uniformly continuous function of rational $t \in T$, uniformly in $x \in U$. This extends to a function $\phi(x, t)$ on $U \times T$, which is continuous in x for each t, and a continuous (multivariate) distribution function in t for each x, and (hence) also jointly continuous. Replacing U with $\mathbb{R}^d$ is now easy, and the occupation kernel described in (26.1) is just the measure on $\mathfrak{B}(T)$ which corresponds to $\phi(x, \cdot)$.

27. Hölder continuity in the set variable. We now extend the principal result of Berman (1973) to (N, d)-fields. Let $\mathfrak{D}_0 = \{T\}$, and for each $n \geqslant 1$, let $\mathfrak{D}_n$ be the family of 2^{nN} "dyadic cubes" in T, each of measure 2^{-nN}, obtained by successive subdivision of T. Also, put $\mathfrak{D} = \cup_n \mathfrak{D}_n$.

(27.1) THEOREM. *Let X be* (LND) *and suppose that, for some $0 < \gamma \leqslant 1$ and even $k > d/\gamma$, there is a $\xi = \xi(k, \gamma) > 1$ for which*

$$V_{k,\gamma}(B) \leqslant C_1(\lambda_N(B))^{\xi} \quad \textit{for every} \quad B \in \mathfrak{D}_n, \qquad n \geqslant n_0, \tag{27.2}$$

for some constants C_1 and n_0. Then, for each compact $U \subset \mathbb{R}^d$ and each $\zeta < (\xi - 1)/k$, there is a constant C_2 and a random variable $\varepsilon = \varepsilon(\omega)$ such that, a.s.,

$$\alpha(x, B) \leqslant C_2(\lambda_N(B))^{\zeta} \quad \textit{for all} \quad x \in U \tag{27.3}$$

for any cube $B \subset T$ of edge length smaller than ε.

NOTE. The hypotheses of (27.1) imply those of (the first part) of (26.1), and the α in (27.3) is the good version guaranteed by (26.1). In certain cases (see §30), (27.2) is valid for every k (or every large k) for some γ, C_1, and n_0, in which case (27.3) is true for any

$$\zeta < \limsup_{k\to\infty} \frac{\xi(k, \gamma)}{k},$$

which can be computed explicitly.

PROOF. We begin by noting that all the computations made in the proof of (26.1) are valid. Recall that

$$\mathbb{E}|\alpha_0(x, B) - \alpha_0(y, B)|^k \leqslant C|x - y|^{\gamma k} V_{k,\gamma}(B) \tag{27.4}$$

for all $x, y \in \mathbb{R}^d$ and $B \in \mathfrak{B}(T)$. Here and below, C is a constant, which may change from line to line, but *does not depend on* B. Let $K(B) = K(B; U, k, \gamma)$. Since $\alpha(x, B) = \alpha_0(x, B)$ a.e. a.s. for all B, we have by (26.4) (with $\delta = \gamma$):

$$\mathbb{E}K(B) \leqslant CV_{k,\gamma}(B) \qquad \text{for all} \quad B \in \mathfrak{B}(T). \tag{27.5}$$

From (25.5),

$$\mathbb{E}\alpha^k(0, B) = \mathbb{E}\alpha_0^k(0, B) = \int_{B^k} p_k(\bar{t}; 0)\, d\bar{t},$$

and the estimates in the proof of (25.12) (a) then give

$$\mathbb{E}\alpha^k(0, B) \leqslant CV_{k0}(B) \leqslant C' V_{k\gamma}(B), \qquad B \in \mathfrak{B}(T). \tag{27.6}$$

Let $B_1, B_2, \cdots$ be the sets in $\mathfrak{D}$, where $B_1 = T$, $\mathfrak{D}_1$ is comprised of $B_2, \cdots, B_{2^N+1}$, etc. Also, let $\xi' < \xi - 1$. By (27.2) and (27.5), for sufficiently large m,

$$\Sigma_{n=m}^{\infty}\mathbb{P}\{K(B_n) > (\lambda_N(B_n))^{\xi'}\} \leqslant C\Sigma_m^{\infty}(\lambda_N(B_n))^{\xi-\xi'} < \infty.$$

Hence, according to the Borel-Cantelli lemma, there is a random variable $\nu_1(\omega)$ such that, with probability 1,

$$K(B) \leqslant (\lambda_N(B))^{\xi'} \qquad \text{for all} \quad B \in \mathfrak{D}_n, \qquad n \geqslant \nu_1. \tag{27.7}$$

Similarly, for large m,

$$\Sigma_m^{\infty}\mathbb{P}\{\alpha(0, B_n) > (\lambda_N(B_n))^{\xi'/k}\} \leqslant \Sigma_m^{\infty}\frac{\mathbb{E}\alpha^k(0, B_n)}{(\lambda_N(B_n))^{\xi'}} < \infty,$$

and, consequently,

$$\alpha(0, B) \leqslant (\lambda_N(B))^{\xi'/k} \qquad \text{for all} \quad B \in \mathfrak{D}_n, \qquad n \geqslant \nu_2, \tag{27.8}$$

for some random variable $\nu_2(\omega)$.

Next, since $\mathfrak{D} \subset \mathcal{H}$ and $\alpha(x, dt)$ is just the extension of $\alpha(x, \cdot)$ to a measure, (26.5) yields

$$\begin{aligned} |\alpha(x, B) - \alpha(y, B)| &\leqslant C(K(B))^{1/k}|x - y|^{\gamma - (2d/k)} \\ &\leqslant C(\lambda_N(B))^{\xi'/k}|x - y|^{\gamma-(2d/k)} \end{aligned} \tag{27.9}$$

for all $x, y \in U$ and $B \in \mathfrak{D}_n$, $n \geqslant \nu_1(\omega)$.

Let us now take $U = [0, 1]^d$ for simplicity and assume that the random point ω is in all the a.s. sets mentioned above. If $x \in U$, we may write

$$x = \Sigma_{j=0}^{\infty}2^{-j}(\xi_j^{(1)}, \cdots, \xi_j^{(d)}), \qquad \xi_j^{(i)} = 0 \text{ or } 1,$$

and, since $\alpha(x, B)$ is continuous in x for every $B \in \mathcal{H}$,

$$\alpha(x, B) = \alpha(0, B) + \Sigma_{n=1}^{\infty}\Big[\alpha\big(\Sigma_{j=0}^{n}2^{-j}(\xi_j^{(1)}, \cdots, \xi_j^{(d)}), B\big) - \alpha\big(\Sigma_{j=0}^{n-1}2^{-j}(\xi_j^{(1)}, \cdots, \xi_j^{(d)}), B\big)\Big].$$

If $B \in \mathfrak{D}_n$ where $n \geqslant \nu \equiv \max(\nu_1, \nu_2)$, then (27.8) and (27.9) yield

$$\alpha(x, B) \leqslant (\lambda_N(B))^{\xi'/k} + (\lambda_N(B))^{\xi'/k} c \Sigma_{n=1}^{\infty} (d^{\frac{1}{2}} 2^{-n})^{\gamma - (2d/k)}$$
$$\leqslant C(\lambda_N(B))^{\xi'/k}.$$

This shows that the theorem holds for dyadic cubes.

Now let $B \subset T$ be *any* cube of edge length $b \leqslant \varepsilon = 2^{-\nu}$. Using the material in Billingsley (1965), page 140, it is not hard to show that B can be covered by at most $L_N = 16^N$ *dyadic* cubes $C_1, \cdots, C_{L_N}$ of edge length $\leqslant b$; thus each $C_i \in \cup_{\nu}^{\infty} \mathfrak{D}_n$ and $\lambda_N(C_i) \leqslant \lambda_N(B)$. Thus

$$\alpha(x, B) \leqslant \Sigma_1^{L_N} \alpha(x, C_i) \leqslant c \Sigma_1^{L_N} (\lambda_N(C_i))^{\xi'/k}$$
$$\leqslant 16^N c (\lambda_N(B))^{\xi'/k},$$

which finishes the proof.

Another way of stating (27.1) is that, a.s.,

$$\alpha(x, B) \leqslant g(x)(\lambda_N(B))^{\xi}$$

for some continuous function $g(x)$ (depending on ω), and all balls B of radius $< r(x, \omega)$, where, for each ω, $r(x, \omega)$ is bounded away from zero on compact sets in $\mathbb{R}^d$. This suffices for the proof of (10.1) to work, yielding

(27.10) COROLLARY. *With probability* 1, *for any* $r > d^{-1}N(1 - k^{-1}(\xi - 1))$,

$$\text{(27.11)} \qquad \text{ap} - \lim_{s \to t} \frac{|X_s - X_t|}{|s - t|^r} = \infty \qquad \textit{for every} \quad t.$$

There are two other consequences of (27.1) that should be mentioned. The first is that, using (27.7), *the conclusion of* (26.2) *can be strengthened to read*: *for each* $\xi < \gamma - (d/k)$ *and each* $U \subset \mathbb{R}^d$:

$$\text{(27.12)} \qquad \sup_{B \in \mathfrak{D}} \sup_{x, y \in U} \frac{|\alpha(x, B) - \alpha(y, B)|}{|x - y|^{\xi}} < \infty \text{ a.s.,}$$

and this can no doubt be jacked up to "sup"$_{B \in \mathcal{K}}$, although we will not pursue this point.

The other consequence is that estimates of the form (27.3) will imply that $\alpha(x, B)$ is dominated for every set $B \subset T$, by an appropriate Hausdorff measure. Specifically, if $h(u)$ is increasing on $u \geqslant 0$, $h(0) = 0$, and if

$$\alpha(x, B) \leqslant ch(\lambda_N(B)) \qquad \text{for all small cubes,}$$

then it is easy to show that

$$\text{(27.13)} \qquad \alpha(x, B) \leqslant c\Lambda(B) \qquad \text{for all} \quad B \in \mathfrak{B}(T),$$

where Λ is the Hausdorff measure based on the function $u \mapsto h((u/N^{\frac{1}{2}})^{1/N})$, $u \geqslant 0$. The problem is that, in general, the estimates on $\alpha(x, \cdot)$ are not sharp enough to give $\Lambda(M_x) < \infty$, without which (27.13) is meaningless.

We conclude this section with some recent results of Davies (1976, 1977) and Kôno (1977) which go well beyond (27.1) in the case of certain real, stationary Gaussian processes on $t \in [0, 1]$. To state Davies' results, suppose X has a spectral density

$$f(\lambda) = a^{2\beta}\frac{\Gamma(\beta + 1/2)}{\Gamma(1/2)\Gamma(\beta)}(\lambda^2 + a^2)^{-(\beta+\frac{1}{2})}$$

where $0 < \beta < \frac{1}{2}$, $a > 0$. This guarantees the existence and joint continuity of $\alpha(x, dt)$ (details aside) and Davies' (1976) result is

(27.14) THEOREM. *There exist constants* $0 < C_1 < C_2 < \infty$ *such that for each* t,

$$C_1 \leqslant \limsup_{h\downarrow 0}\frac{\alpha(X_t, [t, t + h])}{h^{1-\beta}(\log(\log 1/h))^\beta} \leqslant C_2 \text{ a.s.}$$

It is an open problem to find a single constant C equal to the above "lim sup". For the bearing of (27.14) on the Hausdorff measure of M_x, see §29.

Davies' paper is very difficult; the proof of the upper bound was simplified and extended by Kôno (1977). Let X be a real, mean 0, Gaussian process on $0 \leqslant t \leqslant 1$ with stationary increments: $\Delta(s, t) = \mathbb{E}(X_s - X_t)^2 = \sigma^2(|s - t|)$. Under a variety of conditions involving the growth of $\sigma(s)$ and $\sigma'(s)$ near $s = 0$, Kôno proves that the following upper limits are finite a.s.:

$$\limsup_{h\downarrow 0}\frac{\alpha(X_t, [t, t + h])}{\psi_1(h)}, \qquad \limsup_{|t-s|\downarrow 0;\ 0\leqslant s,\, t\leqslant 1}\frac{|\alpha_t(x) - \alpha_s(x)|}{\psi_2(|t - s|)},$$

where $\psi_1(h) = h/\sigma(h/\log\log 1/h)$, $\psi_2(h) = h/\sigma(h/\log 1/h)$. In particular, Kôno's hypotheses are satisfied whenever $\sigma^2(s)$, $s > 0$, is differentiable, concave, and "nearly" regularly varying with index $0 < \beta \leqslant \frac{1}{2}$. Berman's results (1970) imply the existence of a jointly continuous $\alpha_t(x)$ under these assumptions. The question of the lower bound remains open for these processes.

28. Differentiability in the space variable. As a function of the "space" variable x, the (one-dimensional) Brownian occupation density satisfies a Hölder condition of any order $\beta < \frac{1}{2}$, but *not* of order $\beta = \frac{1}{2}$. This precludes it from being differentiable, a condition we believe to be typical of standard Markov occupation densities.

The situation is quite different for Gaussian processes, and we give several results concerning the higher smoothness of $\alpha(x, B)$ as a function of x, which originate in Berman's work. The results in part 1° amount to standard Fourier analysis put in a convenient form for present purposes; for $N = d = 1$, these concern conditions for $\alpha_t(x)$ to be smooth in x a.s. for each fixed t, whereas in part 2° we require that $\alpha_t(x)$ be smooth in x for *all* t, a.s. We then conclude with a new result of Berman's about the quadratic mean analyticity of $\alpha(x)$ which we include with his permission.

1°. *Via Fourier analysis.* Let $\hat{\mu}_B(\theta)$, $\theta \in \mathbb{R}^d$, be the Fourier transform of the occupation measure μ_B of X:

$$\hat{\mu}_B(\theta) = \int_{\mathbb{R}^d} e^{i\theta\cdot y}\mu_B(dy) = \int_T e^{i\theta\cdot X_t}\,dt.$$

Of course, if α exists, then $\hat{\mu}_B(\theta)$ is the Fourier transform $\hat{\alpha}(\theta, B)$ of $\alpha(\cdot, B)$. We now extend Lemma 5.1 of Berman (1969a) to (N, d)-fields.

(28.1) THEOREM. *Suppose*

$$\int_T\int_T \frac{ds\,dt}{\Delta^{p+1/2}(s,t)} < \infty \tag{28.2}$$

for some $p > 0$. Then, for each $B \in \mathfrak{B}(T)$, the function $x \mapsto \alpha(x, B)$ has (mixed) partial derivatives of all orders up to $[p]$, all of which are in $L^2(dx)$, a.s.

The proof is a trivial application of well-known results in Fourier analysis, *once we have shown that, for each $1 \leqslant i \leqslant d$,*

$$\mathbb{E}\int_{\mathbb{R}^d}|\theta_i|^{2p}|\hat{\mu}_B(\theta)|^2\,d\theta = C_p\int_B\int_B \frac{ds\,dt}{\Delta^{p+1/2}(s,t)} \tag{28.3}$$

for some constant C_p. The expected value in (28.3) is easily seen to be

$$\int_B\int_B\int_{\mathbb{R}^d}|\theta_i|^{2p}e^{-\frac{1}{2}\theta\cdot\sigma^2(s,t)\cdot\theta'}\,d\theta\,ds\,dt.$$

Now fix $s, t \in B$ ($s \neq t$) and let $A = (a_{ij})$ be the inverse matrix of $\sigma(s, t)$. Writing $\theta\sigma^2\theta'$ as $(\theta\sigma)\cdot(\theta\sigma)'$ and making the change of variables $\lambda = \theta\sigma$ converts the inner integral above to

$$\Delta^{-\frac{1}{2}}\int_{\mathbb{R}^d}\left|\sum_{j=1}^d a_{ij}\lambda_j\right|^{2p}e^{-\frac{1}{2}|\lambda|^2}\,d\lambda = \frac{(2\pi)^{d/2}}{\Delta^{\frac{1}{2}}}\mathbb{E}\left|\sum_1^d a_{ij}Y_j\right|^{2p}$$

where $Y_1, \cdots, Y_d$ are independent standard normal. It now follows that

$$\mathbb{E}\int_{\mathbb{R}^d}|\theta_i|^{2p}|\hat{\mu}_B(\theta)|^2\,d\theta = (2\pi)^{(d-1)/2}2^{p+\frac{1}{2}}\Gamma\left(p+\tfrac{1}{2}\right)\int_B\int_B\left(\sum_1^d a_{ij}^2\right)^p\Delta^{-\frac{1}{2}}\,ds\,dt.$$

Finally, recall that $a_{ij} = \tilde{\sigma}_{ij}/\Delta^{\frac{1}{2}}(s,t)$ where $\tilde{\sigma}_{ij}$ is the (j, i)-cofactor of $\sigma(s, t)$. Since the numbers $\tilde{\sigma}_{ij}(s, t)$ are uniformly bounded, we obtain (28.3)

For $N = d = 1$, the following result, also due to Berman (1969), is a kind of limiting case of (28.1).

(28.4) THEOREM. *Suppose, for some $b > 0$,*

$$\int_0^1\int_0^1 e^{b^2/\sigma^2(s,t)}\,ds\,dt < \infty.$$

Then $x \mapsto \alpha_t(x)$ is (real) analytic for a.e. t, a.s.

The proof of this is of the same Fourier analytic genre as that of (28.1). As a consequence, in this case, *the trajectories are Carathéodory functions*, and, in particular, cannot be continuous or even right continuous (contrast the Markov case!). Now it is possible, on the other hand, to construct an X with continuous trajectories such that $\alpha_t(\cdot)$ is C^∞ for a.e. t, and this leads us to wonder whether *the discontinuous Gaussian processes are* exactly *those with analytic occupation densities*?

2°. *Smoothness in x, for all t.* The results below are more subtle than those in 1° and were developed in order to produce examples of functions which satisfy the hypotheses of the "perturbation" results discussed in §12.

(28.5) THEOREM. *Suppose $\sigma(s, t) \sim |s - t|^\gamma$ as $|s - t| \to 0$. Then*:

(a) *if $0 < \gamma < \frac{1}{3}$, X is* (LT), *$\alpha_t(x)$ is jointly continuous, and $x \mapsto \alpha_t(x)$ is absolutely continuous for every t and is in $L^2(dx)$, all with probability one*;

(b) *if $0 < \gamma < \frac{1}{5}$, then, in addition, $\alpha_t'(x)$ $(= (\partial/\partial x)\alpha_t(x))$ can be chosen jointly continuous.*

By making γ successively smaller, we get the existence of higher derivatives

$$\frac{\partial^n \alpha_t(x)}{\partial x^n} = \alpha_t^{(n)}(x)$$

from (28.1), and the form of σ^2 (viz. $|s - t|^\gamma$) allows us to obtain the existence for every t, as well as the joint continuity of $\alpha_t^{(n)}(x)$. Using noninteger values of p in (28.3) one can slightly improve the numbers $\frac{1}{3}$, $\frac{1}{5}$ in (28.5) and obtain Hölder conditions in x for $\alpha_t(x)$. The details will appear elsewhere.

Let X be an $(N, 1)$-field which is (LT), and write Q_t for the quadrant in T with upper right corner at t. The following is an interesting (real-variable) complement to (28.5).

(28.6) THEOREM. *Suppose $\alpha(\cdot, Q_t)$ is continuous and in $L^2(dx)$ for a given $t \in T$ and that, for some $p > \frac{1}{2}$,*

$$\int_{-\infty}^{\infty} |\theta|^{2p} |\hat{\alpha}(\theta, Q_t)|^2 \, d\theta < \infty;$$

then

$$\alpha(x, Q_t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\theta x} \int_{Q_t} e^{i\theta X_s} \, ds \, d\theta \qquad \text{for all } x. \tag{28.7}$$

If, as in (28.5b), $\alpha_t'(x)$ exists and is in $L^2(dx)$ for *every* t, we may take $p = 1$ and so have the Fourier representation (28.7) for *all t and x* (a.s.). Obviously the finiteness of the right member of (28.3) suffices for (28.6) if t is fixed, but other examples can be constructed (for $N > 1$) in which the conclusion holds for all t. Equation (28.7) has been used by Tran (1976) and Adler (1977b) in the case of Brownian sheets, i.e., N-parameter Wiener process, but the interpretation there is as a quadratic mean limit. We have not been able to decide whether (28.7) holds literally in this case.

To prove (28.6) it suffices to show

$$\int_{-\infty}^{\infty} |\hat{\alpha}(\theta, Q_t)| \, d\theta < \infty. \tag{28.8}$$

Let $f(\theta) = 1$ for $|\theta| < 1$, $= |\theta|^p$ for $\theta > 1$. By Schwarz' inequality, the integral is dominated by the square root of

$$\int_{-\infty}^{\infty} |f(\theta)|^2 |\hat{\alpha}(\theta, Q_t)|^2 \, d\theta \int_{-\infty}^{\infty} \frac{d\theta}{|f(\theta)|^2} \tag{28.9}$$

which is easily checked to be finite. We note, finally that, in (28.7), we *cannot* replace Q_t by an arbitrary set $B \in \mathfrak{B}(T)$: take $B = M_x$. This is of measure zero for each x, but, in general, $\alpha(x, M_x) > 0$ for at least one x.

3°. *Quadratic mean analyticity* ($N = d = 1$). A process $\xi(x, \omega)$, $x \in \mathbb{R}$, is q.m. *analytic* if it is the sum (limits in q.m. relative to $\mathbb{P}$) of a random power series in x.

Let $R(t, t) \equiv 1$ and assume that (25.1) holds, i.e.,

$$\int_0^1 \int_0^1 (1 - R^2(s, t))^{-\frac{1}{2}} \, ds \, dt < \infty. \tag{28.10}$$

Also, let $\alpha_0(x) = \alpha_0(x, T)$ be as in §25:

$$\alpha_0(x) = \liminf_{n \to \infty} n \int_0^1 I_{(0, 1/n)}(|X_s - x|) \, ds.$$

Then (25.5) gives, after some manipulations,

$$\mathbb{E}\alpha_0(x)\alpha_0(y) = \phi(y) \int_0^1 \int_0^1 (1 - R^2(s, t))^{-\frac{1}{2}} \phi\left(\frac{x - R(s, t)y}{(1 - R^2(s, t))^{\frac{1}{2}}} \right) ds \, dt < \infty, \tag{28.11}$$

where ϕ is the standard normal density.

(28.12) THEOREM. *The process* $\alpha_0(x)$ *is* q.m. *analytic if and only if*

$$\int_0^1 \int_0^1 \exp\left(\frac{b}{1 - R^2(s, t)} \right) ds \, dt < \infty \qquad \textit{for some} \quad b > 0. \tag{28.13}$$

Since $\sigma^2(s, t) = 2(1 - R(s, t))$, (28.13) implies the hypothesis in (28.4), and the two are equivalent if $R(s, t) \neq -1$ for all $0 \leqslant s, t \leqslant 1$, in which case α (or a version thereof) is real analytic iff it is q.m. analytic. We also note that (28.13) remains unchanged if the factor $(1 - R^2(s, t))^{-\frac{1}{2}}$ is added to the integrand.

First, $\mathbb{E}\alpha_0(x) = \phi(x)$, and thus, from the criterion in Loève (1963), section 34.2, we find that α is q.m. analytic iff the function in (28.11) is analytic on the diagonal. If this is so, then, in particular, (28.11) will be *finite* for $y = 0$ and $x = i(2b)^{\frac{1}{2}}$ for $b > 0$, small, whence (28.13) will hold.

Now assume (28.13) holds and split the right member of (28.11) into two integrals, viz. over the sets

$$\{(s, t) : R(s, t) < 1 - \varepsilon\} \qquad \text{and} \qquad \{(s, t) : R(s, t) \geqslant 1 - \varepsilon\}$$

where $\varepsilon > 0$. Clearly only the second piece is at all questionable. But this will be analytic as long as $|x - y| < (b/2)^{\frac{1}{2}}$ and $|y| < (b/2)^{\frac{1}{2}}/\varepsilon$, hence for all $|x| < 2(b/2)^{\frac{1}{2}}/\varepsilon$, and then for all x since ε is arbitrary.

29. Level sets. Most of the applications of occupation densities hitherto have been to such things as the Hausdorff dimension or "exact Hausdorff measure" of the level sets $M_x = \{t : M_t = x\}$ or "progressive level sets" $L_t = \{s : X_s = X_t\}$. The literature on Markov processes is replete with this type of information; see §16 and Fristedt (1973).

Lately, there has been considerable activity in finding the dimension of M_x for various (N, d) Gaussian fields, as well as the dimension of the *graph* and *image* (or *range*) of X, defined, respectively, as

$$\operatorname{Gr} X = \{(t, X_t) : t \in T\}, \qquad \operatorname{Im} X = \{X_t : t \in T\}.$$

We will focus on results which use local time methods whether explicitly or in disguise, as in Marcus (1976) and Kahane (1968).

Cardinality. Perhaps the earliest contribution here is Rice's formula: let X be a real, stationary Gaussian process on the line with covariance function $r(t)$; then

$$\mathbb{E}\#(M_x \cap B) = \frac{\lambda(B)}{\pi}\left(\frac{-r''(0)}{r(0)}\right)^{\frac{1}{2}} e^{-x^2/2r(0)} \leqslant \infty.$$

It remains an open question (e.g., Dudley (1973)) whether M_x is infinite for a.e. x in $\operatorname{Im} X$ when $-r''(0) = \infty$. Using some classical real variable results from Saks, Klein (1976) provides a partial answer: *if X is* (LT), *then for every x, M_x is infinite with positive probability*. The best known sufficient condition for (LT) is (22.7), but there are processes for which (22.7) fails and yet $-r''(0) = \infty$. Indeed, once we *assume* (22.7), we know that M_x is actually uncountable with positive probability for all x in a set of positive Lebesgue measure: combining (13.1) and (23.1) shows that *for any real Gaussian process on the line, L_t is uncountable for* a.e. t, a.s. *whenever*

(29.1) $$\int_0^1(\sigma(s, t))^{-1}\, ds < \infty \qquad \textit{for} \quad \text{a.e. } t.$$

In the stationary case, (29.1) and (22.7) are equivalent. Other results on cardinality are in Berman (1969, 1969a, 1970), Brillinger (1972), Marcus (1977), and in §13.

Hitting points. (a) Let X be an Gaussian (N, d)-field. As Cuzick (1977) points out, if $\dim(\operatorname{Im} X) = d$ a.s., then it is of interest to know whether $\lambda_d(\operatorname{Im} X) > 0$ and whether X hits fixed points. Of course, (LT) implies the former. Cuzick shows that, *if the components of X have stationary increments, and satisfy* (29.7) *below and* (25.11) *for $k = 2$, $\gamma = 0$, and $B = T$, then each $y \in \mathbb{R}$ is hit with positive probability* (*probability* 1 *if X is ergodic and $T = \mathbb{R}_+^N$*).

(b) Let $W_{N,d}$ be the N-parameter Wiener process in $\mathbb{R}^d$: $W_{N,d}$ has i.i.d. components, each being Gaussian, mean 0, with covariance $\prod_{i=1}^N(s_i \wedge t_i)$, $s = (s_1, \cdots, s_N)$, $t = (t_1, \cdots, t_N)$. First, *if $d \geqslant 2N$, then* $\lambda_d(\operatorname{Im} X) = 0$ a.s. (Orey and Pruitt (1973)). Tran (1977) then showed that $\operatorname{Im} X$ actually has Hausdorff dimension $2N$.

On the other hand, if $d < 2N$, Orey and Pruitt (1973) raise the question of whether $\operatorname{Im} X = \mathbb{R}^d$ a.s. (here $T = \mathbb{R}_+^N$) having shown that $\lambda_d(\mathbb{R}^d \setminus \operatorname{Im} X) = 0$ a.s. Tran (1976) had apparently established that $\operatorname{Im} X = \mathbb{R}^d$ a.s. by showing that $W_{N,d}$

has a jointly continuous occupation density (which is the crux of the matter) and that the occupation density $\alpha(x, B)$ is eventually positive as $B \uparrow \mathbb{R}_+^N$. However, Pruitt (*Math. Reviews*, **56**, 207 (1978)) spotted an error in the proof of the central estimate (see (30.8)) used by Tran to get the joint continuity of α. As far as we know (via private communication) the validity of this estimate, which is very plausible in view of what is known for fields similar to $W_{N,d}$, is still in doubt.

In the special case $d = 1$, the joint continuity of α was done for $N = 2$ by Cairoli and Walsh (1975, Theorem 6.4) and recently for general N by Davydov (1978).

Hausdorff dimension. Upper bounds on dim M_x are generally found by appealing to the *regularity* of X; for example, if $X(t)$ is a (nonrandom) (N, d)-field which satisfies a (uniform) Lipschitz condition of order β, and if $N - d\beta \geqslant 0$, then dim $M_x \leqslant N - d\beta$ for a.e. $x \in \mathbb{R}^d$ (Kahane (1968) for $N = 1$, and Adler (1977) in general). Lower bounds are more difficult to come by and the method is usually this: fix an x and construct a measure $\nu_x(dt) \neq 0$ concentrated on M_x (either a.s. or with positive probability) for which

$$\mathbb{E}\int_T \int_T |s - t|^{-\beta} \nu_x(ds) \nu_x(dt) < \infty,$$

for some $\beta > 0$. It is then standard fare that dim $M_x \geqslant \beta$ a.s. on $\{\nu_x \neq 0\}$. In every case we know of (at least in the Gaussian literature), $\nu_x(dt)$ turns out to be $\alpha(x, dt)$ (up to a constant multiple). For instance, Marcus (1976) considers the measure ν_x which corresponds to the increasing function of t (here $N = d = 1$) given by

$$\lim_{p\to\infty} \frac{1}{2\varepsilon_{n_p}} \int_0^t I_{[0, \varepsilon_{n_p})}(|X_s - x|)\, ds,$$

where $\varepsilon_n = 2^{-n}$ and (n_p) is a fixed subsequence; the hypothesis here is essentially the one at (23.4), and the fact that $\nu_x(dt) = \alpha(x, dt)$ is then obvious. Kahane (1968) and Adler (1977), working in the (N, d) case, define the measure

$$\nu_x^\varepsilon(dt) = \left(\frac{2\pi}{\varepsilon}\right)^{d/2} \exp - \left(\frac{(X_t - x)^2}{2\varepsilon}\right) dt$$

and let $\varepsilon \downarrow 0$. This amounts to simply replacing the "approximate identity" $\phi_\varepsilon(y) = \varepsilon^{-d} I_{(0,\varepsilon)}(y)$ that we have used throughout with the Gaussian kernel $\phi_\varepsilon(y) = \varepsilon^{-d/2} e^{-y^2/2\varepsilon}$; everything important remains the same.

As concerns the dimension of Im X and Gr X, the natural measures to use are the distributions of $t \mapsto X_t$ and $t \mapsto (t, X_t)$ respectively, i.e., the occupation measure $\mu(dy)$ and $\lambda_N\{t \in T : (t, X_t) \in dy\}$. This leads one to investigate the finiteness of integrals of the form

$$\int_T \int_T |X_s - X_t|^{-\beta}\, ds\, dt \quad \text{and} \quad \int_T \int_T |(s - t)^2 + (X_s - X_t)^2|^{-\beta}\, ds\, dt.$$

From here on, we will focus on results concerning dim M_x, as these are the ones involving occupation densities. For results on dim Im X and dim Gr X, see Orey

(1970), Cuzick (1977), Hawkes (1977), Tran (1977), Yoder (1975), and Adler (1977, 1977a).

dim $M_x : N = d = 1$. The situation for Gaussian Fourier series is fully recounted in Kahane (1968) and we will not repeat those results here. However, many of the results below, as well as the methods of proof, are generalizations or adaptations of Kahane's work.

The earliest general results are due to Orey (1970) for the so-called "index β" processes. These processes have stationary increments, and

$$\sup\{\gamma : \sigma(t) = o(t^\gamma), t\downarrow 0\} = \inf\{\gamma : t^\gamma = o(\sigma(t)), t\downarrow 0\}, \tag{29.2}$$

where σ^2 is the incremental variance and β is the common value in (29.3). Thus, roughly speaking,

$$\sigma(t) \sim t^\beta \quad \text{as} \quad t\downarrow 0 \qquad (0 < \beta \leqslant 1), \tag{29.3}$$

meaning that $\lim_{t\downarrow 0}\sigma(t)t^{-\beta}$ exists, finite. Essentially, what Orey found was that the right dimension number for M_x was $1 - \beta$, as one might expect. Orey's result was improved or elaborated by Berman (1970, 1972), Marcus (1976), and Hawkes (1977) as follows:

(29.4) THEOREM. (a) (Berman). *Under* (29.3), *for* a.e. t, dim $L_t = 1 - \beta$ a.s.

(b) (Berman). *Suppose X is stationary and ergodic, $\sigma(t) \leqslant B|t|^\beta$ near $t = 0$, and $|\lambda|^{1+\beta/2}F'(\lambda) \geqslant C$ for all large $|\lambda|$, where F is the spectral distribution*; *then* a.s.,

$$\dim M_x = 1 - \beta \qquad \textit{for all} \quad x.$$

Also, this is valid for Brownian motion with $\beta = \frac{1}{2}$.

(c) (Marcus). *Let X be stationary*, (29.3) *hold, and assume $r(t) \to 0$ as $t \to \infty$. Then, for each fixed $x \in \mathbb{R}$*, dim $M_x = 1 - \beta$ a.s.

(d) (Hawkes). *Assume σ is monotone*; *then for* a.e. x,

$$\text{ess sup}_\omega \dim M_x = 1 - \beta,$$

where β here is the "inf" *in* (29.2).

Perhaps the most delicate result along these lines is that of Davies (1977) giving the exact Hausdorff measure function for the class of stationary processes described in §27, just prior to (24.14).

(29.5) THEOREM. *Let Ψ be the Hausdorff measure based on the function $h^{1-\beta}(\log(\log 1/h))^\beta$. Then*

$$0 < \Psi\{s : X_s = 0 \textit{ and } \alpha_s(0) \leqslant t\} < \infty \textit{ for all } t > 0, \text{ a.s.}$$

dim M_x : *N, d arbitrary*. First, as concerns $W_{N,d}$, Adler (1977) shows that *if $N - (d/2) \geqslant 0$, then* dim $M_0 = N - (d/2)$ *with positive probability*. However, when $d = 1$, a much stronger result is possible (Adler (1977a)): dim $M_x = N - \frac{1}{2}$ *for every x in the interior of* Im $W_{N,1}$, a.s.

For more general fields, Adler (1977) proves:

(29.6) THEOREM. *Let X be stationary with* i.i.d. *components, each of index β. If $N - \beta d \geq 0$, then for* a.e. $x \in \mathbb{R}^d$,

$$\dim M_x = N - \beta d$$

with positive probability.

Cuzick (1977) notes that Adler's proofs can be adapted to cover certain processes with dependent components of varying indices: let each X_i, $1 \leq i \leq d$, have stationary increments, let σ_i be of index β_i, and suppose that

$$\frac{D_2(t, 0)}{\prod_{i=1}^N \sigma_i(t)} \geq \varepsilon > 0 \qquad \text{for all} \quad t \in T. \tag{29.7}$$

(Recall $D_2(s, t) = \det \operatorname{Cov}(X_s, X_t)$). *Then, for* a.e. $x \in \mathbb{R}^d$,

$$\begin{aligned} \dim M_x &= 0 && \textit{if} \quad \dim \operatorname{Gr} X \leq d \\ &= N - \Sigma_1^d \beta_i && \textit{if} \quad \dim \operatorname{Gr} X > d \end{aligned}$$

with positive probability.

Finally Cuzick computes dim Gr X via standard arguments, the result being that if $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_d$, then a.s.

$$\dim \operatorname{Gr} X = \min\left(\frac{N + \Sigma_1^d(\beta_d - \beta_i)}{\beta_d}, N + \Sigma_1^d(1 - \beta_i) \right).$$

(Information of this nature for *smooth* vector fields is also provided in Cuzick's paper.)

30. Index β processes and Brownian sheets. We now apply the results in §§27 and 28 to a specific class of Gaussian fields. Afterwards, we will collect our findings for $N = d = 1$ in Table 2. This provides examples of functions having the properties listed in Table 1, direct construction of which might be very difficult. The situation is similar to that in Fourier analysis where, for example, one proves the existence of sets of nonspectral synthesis rather easily by a probabilistic argument, as in Kahane (1968).

Aside from the examples in §29 and those here, other sources are the (1972) and (1973) papers of Berman, Davydov's papers, and Pitt's paper, especially for examples of Gaussian fields which are (LND).

Let $X = (X^{(1)}, \cdots, X^{(d)})$ have independent and identically distributed components, each having zero mean, stationary increments and incremental variance

$$\sigma^2(t) = \mathbb{E}(X_{t+s}^{(i)} - X_s^{(i)})^2, \qquad 1 \leq i \leq d, \tag{30.1}$$

where, for some "index" $0 < \beta < 1$,

$$|t|^{-\beta}\sigma(t) \to c \qquad \text{as} \quad t \to 0, \tag{30.2}$$

for some constant $0 < c < \infty$. Here are two specific examples.

(1) Let each $X^{(i)}$ have the covariance function

$$R(s, t) = \mathbb{E}X_t^{(i)}X_s^{(i)} = \frac{c}{2}\{|t|^{2\beta} + |s|^{2\beta} - |t - s|^{2\beta}\}.$$

(When $\beta = \frac{1}{2}$, this is Lévy's multiparameter (isotropic) Brownian motion.) Here $\sigma^2(t) = c|t|^{2\beta}$ and $X^{(i)}$ is (LND) on the set $\{t \in \mathbb{R}^N : \varepsilon \leqslant |t| \leqslant \varepsilon^{-1}\}$ for any $\varepsilon > 0$; see Pitt (1978). The condition that $|t| \geqslant \varepsilon$ can be circumvented by slightly adjusting (see Pitt) the definition of (LND); the results in §§26–28 go through as before, and, of course, with no changes at all for the index set $T = [\varepsilon, 1]^N$.

(2) Let each $X^{(i)}$ be the stationary process on $\mathbb{R}$ with spectral density function $|y|^{1-2\beta}(1 + y^2)^{-1}$; then $\sigma^2(t) = K|t|^{2\beta}$ where

$$K = \int_{-\infty}^{\infty}|e^{i\lambda} - 1|^2|\lambda|^{-2\beta-1}\,d\lambda.$$

(See Berman (1972).)

To simplify matters we will also assume

$$\mathbb{E}|X_t|^2 = dE\big(X_t^{(i)}\big)^2 \geqslant \delta > 0 \qquad \text{for all} \quad t \in T. \tag{30.3}$$

This is no loss of generality because it is true (even if $X_0 = 0$) for the process $X + Y$, where $Y = (Y_1, \cdots, Y_d)$ has independent, standard normal components and is independent of X, and because the occupation density of $X + Y$ has the same local properties as that of X. Thus, we can ignore the Δ_1 term in bounding $V_{k,\gamma}(B)$.

(30.4) Theorem. *Let X be as above and suppose X is* (LND) *and that $N - d\beta > 0$. Then X has a jointly continuous occupation kernel $\alpha(x, dt)$ such that for any compact $U \subset \mathbb{R}^d$:*

(a) *If $\delta < \min(1, N/2d\beta - \frac{1}{2})$, then*

$$\sup_{B \in \mathcal{K}} \sup_{x, y \in U} \frac{|\alpha(x, B) - \alpha(y, B)|}{|x - y|^\delta} < \infty \text{ a.s.} \tag{30.5}$$

(b) *If $\zeta < 1 - (d\beta/N)$, then*

$$\sup_{x \in U} \sup_B \frac{\alpha(x, B)}{(\lambda_N(B))^\zeta} < \infty \text{ a.s.} \tag{30.6}$$

where the inner supremum is over all cubes $B \subset T$ of sufficiently small edge length.

Finally, for any $r > \beta$,

$$\text{ap} - \lim_{s \to t} \frac{|X_s - X_t|}{|s - t|^r} = \infty \qquad \textit{for every} \quad t \in T, \text{a.s.} \tag{30.7}$$

Remark. The assumption that X is (LND) is superfluous in (1) and (2), and whenever $\sigma(t) = c|t|^\beta$ and $N = 1$; for the latter, see Berman (1973).

Proof. First

$$\Delta(s, t) = \det \operatorname{Cov}(X_t - X_s) = (\sigma^2(s, t))^d \sim |t - s|^{2d\beta}.$$

Hence, for any $u > 0$,

$$\begin{aligned}\sup_{s\in T}\int_T(\Delta(s,t))^{-u}\,dt &\leqslant c\sup_{s\in T}\int_T|s-t|^{-2d\beta u}\,dt\\ &\leqslant c\sup_{s\in T}\int_{B_N(s,N^{1/2})}|s-t|^{-2d\beta u}\,dt\\ &\leqslant c\int_0^{N^{1/2}}t^{-2d\beta u}t^{N-1}\,dt\end{aligned}$$

which is finite if $2d\beta u < N$. Hence, (25.13) holds whenever $d\beta(1+2\gamma) < N$ *and* $0<\gamma\leqslant 1$, i.e., whenever

$$\gamma < \min\left(1, \frac{N}{2d\beta} - \frac{1}{2}\right).$$

To obtain the uniformity in B, and to prove (b), we need to examine $V_{k,\gamma}(B)$ for balls $B \subset T$. Consider the integral

$$\int_{B_N(a,r)}\cdots\int_{B_N(a,r)}\frac{dt_1\cdots dt_k}{|t_k-t_{k-1}|^u\cdots|t_2-t_1|^u}. \tag{30.8}$$

Since

$$\int_{B_N(a,r)}\frac{dt}{|t-s|^u} \leqslant \int_{B_N(s,2r)}\frac{dt}{|t-s|^u} = c(2r)^{N-u},$$

the integral in (30.8) is dominated by a constant (independent of $B_N(a,r)$) times $(2r)^{(N-u)k}$. Using (30.2) and (30.3) then yields

$$V_{k,\gamma}(B) \leqslant c(\lambda_N(B))^{(N-u)k/N}$$

for all balls $B \subset T$, where $u = 2d\beta(\frac{1}{2}+r)$. It follows that (27.1) is in force for each $0<\gamma\leqslant 1$ and $k > d/\gamma$, with

$$\xi(k,\gamma) = k\left[1 - \frac{d\beta}{N}(1+2\gamma)\right],$$

provided this is positive, i.e., $(d\beta/N)(1+2\gamma) < 1$; this is certainly true for all small $\gamma > 0$ if $N - d\beta > 0$. Hence for each small γ, (27.3) works for each

$$\zeta < \sup_{k:k>d/\gamma}\frac{\xi(k,\gamma)}{k} = \left[1 - \frac{d\beta}{N}(1+2\gamma)\right].$$

Letting $\gamma\downarrow 0$ now gives (b), and (a) follows from (27.12).

Finally, if $0<\gamma\leqslant 1$ and $k > \gamma/d$, (27.10) implies that (30.6) works for any

$$\begin{aligned} r &> \frac{N}{d}(1 - k^{-1}(\xi(k,\gamma)-1))\\ &= \frac{N}{d}\left(\frac{d\beta}{N}(1+2\gamma) + \frac{1}{k}\right) \geqslant \beta(1+2\gamma).\end{aligned}$$

Letting $\gamma\downarrow 0$ again completes the proof.

NOTES. (1) If $r = \beta$, then (30.8) is false; indeed, the first author and J. Zinn (1978) have shown that for a class of (N, d)-fields including those here,

$$\operatorname{ap\,lim\,inf}_{s\to t}\frac{|X_s - X_t|}{\sigma(s,t)} = 0 \quad\text{and}\quad \operatorname{ap\,lim\,sup}_{s\to t}\frac{|X_s-X_t|}{\sigma(s,t)} = \infty \text{ for a.e. } t, \text{ a.s.}$$

(2) In (a), we will have $N/2d\beta - \frac{1}{2} \leq 1$ iff $N \leq 3\, d\beta$, which is consistent with the fact (see (28.5)) that if $N = d = 1$ and $\beta < \frac{1}{3}$, then $\alpha_t(x)$ is absolutely continuous in x, for every t.

(3) We saw in §23 that X satisfies the condition [AC-p] for $p = [N - d\beta]$, the largest integer *less than* $N - d\beta$. However, the best result of the form (30.6) that can be obtained via (10.5) is that r can be chosen larger than $d^{-1}(N - [N - d\beta])$, which is $> \beta$ and again consistent with (30.4).

(4) If we allow the components of X to have different indices, say $\beta_1, \beta_2, \cdots, \beta_d$, then everything in (30.4) goes through with β replaced by $d^{-1}(\beta_1 + \cdots + \beta_d)$.

N-parameter Wiener process in $\mathbb{R}^d$. This process, denoted $W_{N,d}$, is described in §29, and is (LT) provided $N - (d/2) > 0$. Since $W_{N,d}$ may not be (LND), the estimates in §25 following (25.8) do not apply. The estimate that Tran attempted to prove (see §29: *hitting points*) is the following: let $\varepsilon > 0$ and let $\alpha(x, dt)$ be the occupation kernel of the process $(t_1, \cdots, t_N) \mapsto W_{N,d}(t_1 + \varepsilon, \cdots, t_N + \varepsilon)$. Then for any $B = \Pi_1^N[t_i, t_i + h_i]$, $0 < \gamma < 1$, and $x, u \in \mathbb{R}^d$:

$$(30.9)\quad \mathbb{E}|\alpha(x+u, B) - \alpha(x, B)|^k$$
$$\leq C|u|^{k\gamma}\left(\int_0^{h_N} \cdots \int_0^{h_1}[s_1 + \cdots + s_N]^{-d(\frac{1}{2}+\gamma)}\, ds_1 \cdots ds_N\right)^k.$$

Assuming (30.9), the far right term would then play the role of our $V_{k,\gamma}(B)$ and the analogue of (30.9), for example, would obtain as follows. We need only consider the case $h_i = h$, $1 \leq i \leq N$. Then

$$\int_0^h \cdots \int_0^h \left(\Sigma_1^N s_i\right)^{-d(\frac{1}{2}+\gamma)} ds_1 \cdots ds_N \leq \int \cdots \int_{B_N(0,\, N^{1/2}h)} \left(\Sigma_1^N s_i^2\right)^{-d(\frac{1}{2}+\gamma)} ds_1 \cdots ds_N$$

which is less than a constant times $h^{N-d(\frac{1}{2}+\gamma)}$ by a change to polar coordinates. Here we take γ small enough that $N - d(\frac{1}{2} + \gamma) > 0$. Since $\lambda_N(B) = h^N$, we can choose

$$\xi(k, \gamma) = k\left(1 - \frac{d(\frac{1}{2} + \gamma)}{N}\right)$$

in (27.1), and hence

$$\lim_{\gamma\downarrow 0} \limsup_{k\to\infty} \frac{\xi(k, \gamma)}{k} = 1 - \frac{d}{2N}.$$

Indeed, it is now apparent that if (30.9) holds, then all the conclusions of (30.4) for $\beta = \frac{1}{2}$ would be valid for $W_{N,d}$.

The idea of using Tran's estimates to obtain a result of the form (30.6) for $W_{N,1}$ is due to Adler (1978). There is a small mistake there which is corrected in Adler (1978b); however, the problem about the validity of (30.9) remains.

$N = d = 1$: *summary*. Berman provides conditions for the index β processes to be (LND), in which case they all have jointly continuous occupation densities with the additional features listed in Table 2 below. The last entry does not apply to the

TABLE 2

$\sigma(t)$	$\alpha_t(\cdot)$	$\alpha_{\cdot}(x)$
$\beta = 1$	Discontinuous	Pure Jump
$\frac{1}{3} < \beta < 1$	Hölder continuous	Hölder continuous
$BM : \beta = \frac{1}{2}$	$\frac{1}{2\beta} - \frac{1}{2}$ ↓	$1 - \beta$ ↓
$\beta < \frac{1}{3}$	Absolutely continuous (for all t)	↓
$\beta < \frac{1}{5}$	$C^{(1)}(\mathbb{R})$ (for all t)	↓
.	$C^{(k)}$	↓
.	⋮	↓
.		↓
$\sigma^2(t) \sim (\log 1/t)^{-1}$	Real analytic	Hölder Continuous any order < 1.

index β processes; it derives from (28.4) and arguments similar to those above and is included to display the full range of possible behavior.

31. Postscript: additional work. To make our survey as complete as possible, we mention here some of the most recent work on occupation densities which came to our attention too late to be included in the main body of the paper; indeed, we have not yet had a chance to fully assimilate most of these new results.

(a) Davydov and Rozin (1978) consider occupation densities of smooth random fields, and also establish smoothness in the space variable for the occupation densities of the Brownian sheet and the Lévy multiparameter Brownian motion. The results on smooth fields should be compared with Cuzick (1977); the others are related to the material in §28. The same authors (1978a) consider conditions on a family of measures under which it is the occupation measure of a function.

Davydov (1978) deals with $(N, 1)$-fields from a general point of view and proves the joint continuity of the occupation density of the Brownian sheet $W_{N,1}$ (see §29).

Finally, Y. Davydov has informed us that, in addition to the result of (22.5), M. Lifschitz has produced a stationary Gaussian process which is not (LT) and for which (22.7) of course fails. Lifschitz' results will appear in *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov* **73** 1978 and *Teor. Verojatnost. i Primenen* 1979.

(b) A recent issue of *Astérisque* **52–53** (1978), which is not accessible to us, is devoted to the subject of local times for semimartingales and related processes. Much of this work stems from Meyer's generalization of Tanaka's formula mentioned in §20. Among the papers in this issue that we have seen are Yor (1978, 1978a, b), Jeulin and Yor (1978), Azéma and Yor (1978), and Yeourp (1978). The approach in these works is largely through the "théorie générale des processus."

There are also four papers of Walsh (1978, 1978a, b, c) on Brownian local times, diffusion local times, Brownian sheets, and the Ray-Knight theory.

(c) Adler (1978a, c) gives further information on asymptotic distributions of occupation measures for the Brownian sheet and the Lévy Brownian motion, and compares the local behavior of the trajectories of these two processes.

(d) Wolpert (1978) uses an argument similar to one in Marcus (1976) to construct the occupation density at $x = 0$ of the Gaussian vector field

$$(t_1, \cdots, t_k) \mapsto (W_1(t_1) - W_2(t_2), \cdots, W_{k-1}(t_{k-1}) - W_k(t_k)),$$

where $W_1, \cdots, W_k$ are independent, planar Wiener processes. The idea is to measure the "amount of time" these k Wiener processes spend intersecting each other. This is applied in Wolpert (1978a) to the $(\phi^4)_2$ model in Euclidean quantum field theory.

REFERENCES

ADLER, R. J. (1977). Hausdorff dimension and Gaussian fields. *Ann. Probability* **5** 145–151.

ADLER, R. J. (1977a). Hausdorff dimension and Brownian sheets. Unpublished manuscript.

ADLER, R. J. (1978). The uniform dimension of the level sets of a Brownian sheet. *Ann. Probability* **6** 509–515.

ADLER, R. J. (1978a). Distribution results for the occupation measures of continuous Gaussian fields. *Stochastic Processes Appl.* **7** 299–310.

ADLER, R. J. (1978b). Correction to: The uniform dimension of the level sets of a Brownian sheet. Unpublished manuscript.

ADLER, R. J. (1978c). A note on some differences between two multiparameter Brownian motions. Unpublished manuscript.

AZÉMA, J. and YOR, M. (1978). En guise d'introduction. *Astérisque* **52–53** 3–16.

BERMAN, S. M. (1969). Local times and sample function properties of stationary Gaussian processes. *Trans. Amer. Math. Soc.* **137** 277–299.

BERMAN, S. M. (1969a). Harmonic analysis of local times and sample functions of Gaussian processes. *Trans. Amer. Math. Soc.* **143** 269–281.

BERMAN, S. M. (1970) Gaussian process with stationary increments: local times and sample function properties. *Ann. Math. Statist.* **41** 1260–1272.

BERMAN, S. M. (1970a). Occupation times of stationary Gaussian processes. *J. Appl. Probability* **7** 721–733.

BERMAN, S. M. (1972). Gaussian sample functions: uniform dimension and Hölder conditions nowhere. *Nagoya Math. J.* **46** 63–86.

BERMAN, S. M. (1973). Local nondeterminism and local times of Gaussian processes. *Indiana Univ. Math. J.* **23** 69–94.

BERMAN, S. M. (1978). Gaussian processes with biconvex covariances. *J. Multivariate Anal.* **8** 30–44.

BERMAN, S. M. (1978a). A new kind of tightness of probability measures, and its applications to integral functionals of stochastic processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **42** 155–165.

BILLINGSLEY, P. (1965). *Ergodic Theory and Information*. Wiley, New York.

BINGHAM, N. (1975). Fluctuation theory in continuous time. *Advances in Appl. Probability* **7** 705–766.

BLUMENTHAL, R. M. and GETOOR, R. K. (1964). Local times for Markov processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **3** 50–74.

BLUMENTHAL, R. M. and GETOOR, R. K. (1968). *Markov Processes and Potential Theory*. Academic Press, New York.

BOYLAN, E. S. (1964). Local times for a class of Markov processes. *Illinois J. Math.* **8** 19–39.

BRETAGNOLLE, J. (1971). Resultats de Kesten sur les processus à accroisements indépendents. *Sem. de Prob. V. Lect. Notes in Math.* **191** 21–36.

BRILLINGER, D. (1972). On the number of solutions of systems of random equations. *Ann. Math. Statist.* **43** 534–540.

BRUNEAU, M. (1974). Une propriété des trajectoires d'un mouvement brownien. *C. R. Acad. Sci. Paris. Ser. A-B* **278** 509–511.

CAIROLI, R. and WALSH, J. B. (1975). Stochastic integrals in the plane. *Acta Math.* **134** 111–183.

ÇINLAR, E. (1975). A local time for a storage process. *Ann. Probability* **3** 930–950.

COHN, D. L. (1972). Measurable choice of limit points and the existence of separable and measurable processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **22** 161–165.

CRAMÉR, H. and LEADBETTER, M. R. (1967). *Stationary and Related Stochastic Processes.* Wiley, New York.

CUZICK, J. M. (1977). Some local properties of Gaussian vector fields. *Ann. Probability* **6** 984–994.

CUZICK, J. M. (1978). Local nondeterminism and the zeros of Gaussian processes. *Ann. Probability* **6** 72–84.

DAVIES, P. L. (1976). Local Hölder conditions for local times of certain stationary Gaussian processes. *Ann. Probability* **4** 277–298.

DAVIES, P. L. (1977). The exact Hausdorff measure of the zero set of certain stationary Gaussian processes. *Ann. Probability* **5** 740–755.

DAVYDOV, Y. (1976). On local times for random processes. *Theor. Probability Appl.* **21** 171–178.

DAVYDOV, Y. (1976a). Local times for random fields. *Teor. Verojatnost. i Primenen.* **21** 670–671.

DAVYDOV, Y. (1978). Local times for multiparameter stochastic processes. *Teor. Verojatnost. i Primenen.* **23** 594–605.

DAVYDOV, Y. and ROZIN, A. L. (1978). On the characterization of certain families of measures. *Teor. Verojatnost. i Primenen* **23** 134–136.

DAVYDOV, Y. and ROZIN, A. L. (1978a). On occupation times for functions and random processes. *Teor. Verojatnost. i Primenen* **23** 650–654.

DUFF, G. F. D. (1970). Integral inequalities for equimeasurable rearrangements. *Canad. J. Math.* **22** 408–430.

FEDERER, H. (1969). *Geometric Measure Theory.* Springer-Verlag, Berlin.

FREEDMAN, D. (1971). *Brownian Motion and Diffusion.* Holden-Day, San Francisco.

FRISTEDT, B. (1973). Sample functions of stochastic processes with stationary, independent increments. *Advances in Probability* **3** 241–396.

FRISTEDT, B. and PRUITT, W. (1972). Uniform lower functions for subordinators. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **24** 63–70.

GARSIA, A. M. (1971). Continuity properties of Gaussian processes with multidimensional time parameter. *Proc. 6th Berkeley Symp. Math. Statist. Probability* **II** 369–374.

GEMAN, D. (1976). A note on the continuity of local times. *Proc. Amer. Math. Soc.* **57** 321–326.

GEMAN, D. (1977). On the approximate local growth of multidimensional random fields. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **38** 237–251.

GEMAN, D. (1977a). Local times for vector functions: energy integrals and local growth rates. *Houston J. Math.* **3** 195–206.

GEMAN, D. and HOROWITZ, J. (1973). Occupation times for smooth stationary processes. *Ann. Probability* **1** 131–137.

GEMAN, D. and HOROWITZ, J. (1974). Local times and supermartingales. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **29** 273–293.

GEMAN, D. and HOROWITZ, J. (1976). Local times for real and random functions. *Duke Math J.* **43** 809–828.

GEMAN, D. and HOROWITZ, J. (1976a). Occupation times for functions with countable level sets and the regeneration of stationary processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **35** 189–211.

GEMAN, D. and HOROWITZ, J. (1979). Smooth perturbations of a function with a smooth local time. To appear in *Trans. Amer. Math. Soc.*

GEMAN, D. and ZINN, J. (1978). On the increments of multidimensional random fields. *Ann. Probability* **6** 151–158.

GETOOR, R. K. and KESTEN, H. (1972). Continuity of local times for Markov processes. *Compositio Math.* **24** 277–303.

GETOOR, R. K. and MILLAR, P. W. (1972). Some limit theorems for local time. *Compositio Math.* **25** 123–134.

GREENWOOD, P. (1975). Extreme time of stochastic processes with stationary independent increments. *Ann. Probability* **3** 664–676.

GRIEGO, R. J. (1967). Local times as a derivative of occupation times. *Illinois J. Math.* **11** 54–63.

GRIEGO, R. J. (1967a). Local times and random time changes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **8** 325–331.

GRIEGO, R. J. (1968). Local times for transforms of Markov processes. *J. Math. Kyoto Univ.* **8** 131–139.

HAWKES, J. (1971). A lower Lipschitz condition for the stable subordinator. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **17** 23–32.

HAWKES, J. (1973). Local times and zero sets for processes with infinitely divisible distributions. *J. London Math. Soc.* **8** 517–525.

HAWKES, J. (1977). Local properties of some Gaussian processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **40** 309–315.

HOBSON, E. W. (1927). *The Theory of Functions of a Real Variable.* Reprinted in 1957 by Dover, New York.

HOROWITZ, J. (1968). The Hausdorff dimension of the sample path of a subordinator. *Israel J. Math.* **6** 176–182.

HOROWITZ, J. (1972). Semilinear Markov processes, subordinators, and renewal theory. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **24** 167–193.

ITÔ, K. and MCKEAN, H. P. (1963). Brownian motion on a half-line. *Illinois J. Math.* **7** 181–231.

ITÔ, K. and MCKEAN, H. P. (1965). *Diffusion processes and their sample paths.* Academic Press, New York.

JEULIN, T. and YOR, M. (1978). Autour d'un théorèm de Ray. *Astérisque* **52–53** 23–75.

KAHANE, J. P. (1968). *Some Random Series of Functions.* D. C. Heath, Lexington, MA.

KAHANE, J. P. (1971). The technique of using random measures and random sets in harmonic analysis. *Adv. Probability* **1** 65–101. Dekker, New York.

KAHANE, J. P. (1976). Sur les zéros et les instants de ralentissement du mouvement Brownien *C. R. Acad. Sci. Paris Ser. A-B* **282** 431–433.

KAHANE, J. P. (1976a). Brownian motion and classical analysis. *Bull. London Math. Soc.* **8** 145–155.

KAUFMAN, R. (1975). Fourier analysis and paths of Brownian motion. *Bull. Soc. Math. France* **103** 427–432.

KESTEN, H. (1965). An iterated logarithm law for local time. *Duke Math J.* **32** 447–456.

KESTEN, H. (1969). Hitting probabilities of single points for processes with stationary independent increments. *Mem. Amer. Math. Soc.* **93**.

KESTEN, H. (1976). Lévy processes with nowhere dense range. *Indiana Univ. Math. J.* **25** 45–64.

KHINCHIN, A. Y. (1949). *Mathematical Foundations of Statistical Mechanics.* Dover, New York.

KINGMAN, J. F. C. (1964). The stochastic theory of regenerative events. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **2** 180–224.

KINGMAN, J. F. C. (1973). An intrinsic description of local time. *J. London Math. Soc.* **6** 725–731.

KLEIN, R. (1976). A representation theorem on stationary Gaussian processes and some local properties. *Ann. Probability* **6** 844–849.

KNIGHT, F. (1963). Random walks and a sojourn density process of Brownian motion. *Trans. Amer. Math. Soc.* **109** 56–86.

KNIGHT, F. (1969). Brownian local times and taboo processes. *Trans. Amer. Math. Soc.* **143** 173–185.

KNIGHT, F. (1971). The local time at zero of the reflected symmetric stable process. *Z. Wahrsheinlichkeitstheorie und Verw. Gebiete* **19** 180–190.

KNIGHT, F. (1974). Existence of small oscillations at zeros of Brownian motion. *Sem. de Prob. VIII, Lecture Notes Math.* **381** 134–149.

KÔNO, N. (1977). Hölder conditions for the local times of certain Gaussian processes with stationary increments. *Proc. Japan Acad. Sci. Ser. A* **53** 91–94.

LÉVY, P. (1965). *Processus Stochastiques et Mouvement Brownien.* Gauthier Villars, Paris.

LOÈVE, M. (1963). *Probability Theory, 3rd ed.* Van Nostrand, Princeton.

MAISONNEUVE, B. (1970). Temps local d'un fermé droit aléatoire. *C. R. Acad. Sci. Paris Ser. A-B* **270** 1526–1528.

MAISONNEUVE, B. (1972). Processus d'excursions et d'incursions. *C. R. Acad. Sci. Paris Ser. A-B* **274** 497–500.

MAISONNEUVE, B. and MEYER, P. A. (1974). Ensembles aléatoires Markoviens homogènes. *Sem. de Prob. VIII. Lecture Notes in Math.* **381** 172–261.

MAISONNEUVE, B. and MORANDO, P. (1969). Temps locaux pour les ensembles régénératifs. *C. R. Acad. Sci. Paris Ser. A-B* **269** A523–A525.

MARCUS, M. B. (1976). Capacity of level sets of certain stochastic processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **34** 279–284.

MARCUS, M. B. (1977). Level crossings of a stochastic process with absolutely continuous sample paths. *Ann. Probability* **5** 52–71.

MCKEAN, H. P. (1961/62). A Hölder condition for Brownian local time. *J. Math. Kyoto Univ.* **1** 195–201.

MCKEAN, H. P. (1975). Brownian local time. *Advances in Math.* **15** 91–111.

MEYER, P. A. (1966). Sur les lois de certaines fonctionelles additives; applications aux temps locaux. *Pub. Inst. Statist. Univ. Paris* **15** 295–310.

MEYER, P. A. (1975). Un cours sur les intégrales stochastiques. *Sem. de Prob. X, Lecture Notes in Math.* **511** 245–400.

MEYER, P. A., SMYTHE, R. and WALSH, J. (1971). Birth and death of Markov processes. *Proc. 6th Berkeley Symp. Math. Statist. Probability* **III** 295–305.

MILLAR, P. W. (1972). Modulus of continuity for continuous additive functionals. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **24** 71–78.

MILLAR, P. W. and TRAN, L. T. (1974). Unbounded local times. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **30** 87–92.

OREY, S. (1970). Gaussian sample functions and the Hausdorff dimension of level crossings. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **15** 249–256.

OREY, S. and PRUITT, W. E. (1973). Sample functions of the N-parameter Wiener process. *Ann. Probability* **1** 138–163.

PITT, L. (1978). Local times for Gaussian vector fields. *Indiana Univ. Math. J.* **27** 309–330.

PITTENGER, A. O. and SHIH, C. T. (1972). Coterminal families and the strong Markov property. *Bull. Amer. Math. Soc.* **78** 439–443.

RAY, D. (1963). Sojourn times of diffusion processes. *Illinois J. Math.* **7** 615–630.

REVUZ, D. (1970). Mesures associées aux fonctionelles additives de Markov, I. *Trans. Amer. Math. Soc.* **148** 501–531.

SAKS, S. (1937). *Theory of the Integral.* Reprinted in 1964 by Dover, New York.

SARKHEL, D. N. (1971). On metric density of functions. *Rev. Roumaine Math. Pures Appl.* **16** 965–976.

SATO, K. and TANAKA, H. (1962). Local times on the boundary for multidimensional reflecting diffusion. *Proc. Japan Acad. Ser. A Math. Sci.* **38** 699–702.

SHEPP, L. A., EZAWA, H. and KLAUDER, J. R. (1974). On the divergence of certain integrals of the Wiener process. *Ann. Inst. Fourier (Grenoble)* **24** 189–193.

SILVERSTEIN, M. L. (1967/68). A new approach to local times. *J. Math. Mech.* **17** 1023–1054.

STERNBERG, S. (1964). *Lectures on Differential Geometry.* Prentice-Hall, Englewood Cliffs, N.J.

STONE, C. (1963). The set of zeros of a semistable process. *Illinois J. Math.* **7** 631–637.

TAYLOR, S. J. (1955). The α-dimensional measure of the graph and the set of zeros of a Brownian path. *Proc. Cambridge Phil. Soc.* **51** 265–274.

TAYLOR, S. J. (1973). Sample path properties of processes with stationary independent increments. In *Stochastic Analysis.* Wiley, New York.

TAYLOR, S. J. and WENDEL, J. G. (1966). The exact Hausdorff measure of the zero set of a stable process. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **6** 170–180.

TRAN, L. T. (1976). On a problem posed by Orey and Pruitt related to the range of the N-parameter Wiener process in $\mathbb{R}^d$. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **37** 27–33.

TRAN, L. T. (1977). The Hausdorff dimension of the range of the N-parameter Wiener process. *Ann. Probability* **5** 235–242.

TROTTER, H. F. (1958). A property of Brownian motion paths. *Illinois J. Math.* **2** 425–432.

WALSH, J. B. (1972). Markov processes and their functionals in duality. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **24** 229–246.

WALSH, J. B. (1978). The local time of the Brownian sheet. *Astérisque* **52–53** 47–61.
WALSH, J. B. (1978a). A diffusion with a discontinuous local time. *Astérisque* **52–53** 37–45.
WALSH, J. B. (1978b). Downcrossings and the Markov property of local time. *Astérisque* **52–53** 89–115.
WALSH, J. B. (1978c). Excursions and local time. *Astérisque* **52–53** 159–192.
WILLIAMS, D. (1967). Local times at fictitious states. *Bull. Amer. Math. Soc.* **73** 542–544.
WILLIAMS, D. (1969). Fictitious states, coupled laws and local time. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **11** 288–310.
WILLIAMS, D. (1969a). Markov properties of Brownian local time. *Bull. Amer. Math. Soc.* **75** 1035–1036.
WILLIAMS, D. (1970). Decomposing the Brownian path. *Bull. Amer. Math. Soc.* **76** 871–873.
WILLIAMS, D. (1974). Path decomposition and continuity of local time for one-dimensional diffusions, I. *Proc. London Math. Soc.* **28** 738–768.
WOLPERT. R. L. (1978). Wiener path intersections and local time. *J. Functional Anal.* **30** 329–340.
WOLPERT, R. L. (1978a). Local time and a particle picture for Euclidean field theory. *J. Functional Anal.* **30** 341–357.
YODER, L. (1975). The Hausdorff dimensions of the graph and range of the N-parameter Brownian motion in d-space. *Ann. Probability* **3** 169–171.
YOEURP, C. (1978). Compléments sur les temps locaux et les quasi-martingales. *Astérisque* **52–53** 197–218.
YOR, M. (1978). Sur la continuité des temps locaux associés à certaines semi-martingales. *Astérisque* **52–53** 23–75.
YOR, M. (1978a). Rappels et préliminaires généraux. *Astérisque* **52–53** 17–22.
YOR, M. (1978b). Un example de processus qui n'est pas une semi-martingale. *Astérisque* **52–53** 219–221.

DEPARTMENT OF MATHEMATICS & STATISTICS
GRC TOWER
UNIVERSITY OF MASSACHUSETTS
AMHERST, MASSACHUSETTS 01003