The Annals of Probability
1980, Vol. 8, No. 3, 607–614

LOCAL CONVERGENCE OF A CLASS OF MARTINGALES IN MULTIDIMENSIONAL TIME[1]

BY RICHARD F. GUNDY

Rutgers University

A local convergence theorem for a class of martingales with multidimensional time parameter is proved. This is accomplished by an elaboration of stopping time arguments used in the one-dimensional setting.

The primary purpose of this paper is to prove a local convergence theorem for a class of martingales indexed by the lattice

$$Z_+^3 = \{(s, t, u): s \geqslant 0, t \geqslant 0, u \geqslant 0\}.$$

Roughly speaking, we require the martingale to be a transform of a sequence of uniformly bounded martingale differences. We say that a martingale on Z_+^3 is *regular* if it satisfies Assumptions 1 and 2 below.

THEOREM. *Let f be a regular martingale on Z_+^3. The sets*

$$\{\limsup_{(s,t,u)\to\infty} |f_{(s,t,u)}| < \infty\}$$

and

$$\{\lim_{(s,t,u)\to\infty} f_{(s,t,u)} \text{ exists}\}$$

are equivalent almost everywhere.

REMARKS. For martingales with one-dimensional time parameter, this theorem (with somewhat weaker hypotheses) is found in Burkholder and Gundy (1970), page 285. It should be noted, however, that the hypothesis of the present theorem differs slightly from its one-dimensional time version in that "lim sup" replaces "sup". It can happen that

$$P(\sup_{(s,t,u)} |f_{(s,t,u)}| = \infty) = 1$$

yet

$$P(\limsup_{(s,t,u)} |f_{(s,t,u)}| < \infty) = 1.$$

For example, let $f = \{f_m(k), m = 1, 2, \cdots\}$ be the sequence of partial sums of a Rademacher series on the interval [0, 1], so that $\sup_m |f_m(x)| = \infty$ almost everywhere. Let $g = \{g_n(y); n = 1, 2, \cdots\}$ be a Rademacher martingale of the form

$$g_n(y) = \Sigma_{k=1}^n V_k(y) r_k(y),$$

where $V_0 \equiv 1$, $V_k(y) = g_{k-1}(y)$, and $r_k(y)$, the Rademacher functions, so that

Received March 7, 1978; revised January 29, 1979.

[1]This work was supported in part by National Science Foundation Grant MCS77-12871.

AMS 1970 *subject classifications.* Primary 60G45.

Key words and phrases. Multidimensional time martingales, convergence of martingales.

$g_n(y) = 0$ for all $n \geqslant n_0(y)$ for almost every $y: 0 \leqslant y \leqslant 1$. Finally, let $h_p(z), p = 1, 2, \cdots$ be any Rademacher function martingale such that $h_p(z) > 0$ for every $p = 1, 2, \cdots$. Then

$$f_{mnp}(x, y, z) = f_m(x) g_n(y) h_p(z)$$

is a regular martingale in our sense such that

$$\sup_m |f_{m,n,p}(x, y, z)| \leqslant \sup_{m,n,p} |f_{m,n,p}(x, y, z)| = \infty$$

if $n < n_0(y)$, yet

$$\lim \sup_{(m,n,p)} |f_{(m,n,p)}| = 0$$

almost everywhere.

A similar theorem for strong differentiation is known as Ward's theorem. (See Saks (1937), page 133). However, the strategy for the present theorem owes more to a theorem of Calderón for biharmonic functions. (See Zygmund (1968) Chap. XVII, 4.13.).

The key lemma for this theorem is an elaboration of the usual stopping time argument, splitting the martingale into two martingales, one of which is bounded, the other supported on a small set ("the Calderón-Zygmund decomposition").

The theorem could be stated for martingales on Z_+^n. However, we refrain from doing so in order to keep the notation simple. Also, it is possible to state a version of the theorem for martingales with continuous "paths" (for example, for $f(Z(t), Z(s))$ where f is biharmonic, and $Z(t)$, $Z(s)$ are two independent two dimensional Brownian motions).

Notation. We consider random variable sequences $f_{(s,t,u)}$ where s, t, and u are nonnegative integers. Two triples, $n = (s, t, u)$ and $m = (x, y, z)$, are ordered $n \leqslant m$ if each component is ordered in the usual way: $s \leqslant x$, $t \leqslant y$, and $u \leqslant z$. Let $\mathcal{F}_n$, $n \in Z_+^3$ be an increasing sequence of σ-fields with respect to the above ordering.

A martingale with respect to $\mathcal{F}_n$, $n \in Z_+^3$, is a sequence f_n, measurable $\mathcal{F}_n$, such that

$$E(f_n \| \mathcal{F}_m) = f_m$$

for $m \leqslant n$.

Assumption 1. For each (s, t, u), the σ-fields $\mathcal{F}_{(s,\infty,\infty)} = V_{t,\ u=0}^{\infty} \mathcal{F}_{(s,t,u)}$, $\mathcal{F}_{(\infty,t,\infty)}$, and $\mathcal{F}_{(\infty,\infty,u)}$, defined similarly, are conditionally independent given $\mathcal{F}_{(s,t,u)}$. This means that the conditional expectation of a random variable f relative to $\mathcal{F}_{(s,t,u)}$ may be computed by iteration:

$$E(f \| \mathcal{F}_{s,t,u}) = E(f \| \mathcal{F}_{s,\infty,\infty} \| \mathcal{F}_{\infty,t,\infty} \| \mathcal{F}_{\infty,\infty,u}).$$

Under this assumption, the maximal inequalities for $f^* = \sup_{(s,t,u)} |f_{(s,t,u)}|$ hold for $p > 1$:

$$\|f^*\|_p \leqslant C_p \|f\|_p$$

where

$$\|f\|_p = \sup_{(s,t,u)} \|f_{(s,t,u)}\|_p.$$

This assumption on the sequence $\mathfrak{F}_n$ is given by Cairoli and Walsh (1975), and the maximal inequalities are proved by Cairoli (1968).

These inequalities may be used to prove that a martingale f with

$$\|f\|_p = \sup_{(s,t,u)} \|f_{s,t,u}\|_p < \infty$$

converges almost everywhere provided $p > 1$. For $p = 1$, the same statement is false. (See Cairoli (1968).)

ASSUMPTION 2. Let $n = (s, t, u) \in Z_+^3$ be given. The index n may be *increased by one* in $2^3 - 1$ ways by adding one to each component of a given subset of the components. Let $n + 1$ designate the result of any one of these increases. (For example, $n + 1$ may stand for $(s + 1, t, u)$ or for $(s, t + 1, u + 1)$.) We assume that for every choice, the difference

$$f_{n+1} - f_n = \sum_{j=1}^N V_n^{(j)} r_{n+1}^{(j)}$$

where $V_n^{(j)}$ is measurable $\mathfrak{F}_n$, and $r_{n+1}^{(j)}$ are uniformly bounded, orthonormal martingale differences. That is,

(a) $\|r_{n+1}^{(j)}\|_\infty \leqslant C, \quad j = 1, 2, \cdots, N;$
(b) $E(r_{n+1}^{(j)} r_{n+1}^{(l)} \| \mathfrak{F}_n) = \delta(j, l);$
(c) $E(r_{n+1}^{(j)} \| \mathfrak{F}_n) = 0, \quad j = 1, 2, \cdots, N.$

Here the number N is independent of n and the choice of $n + 1$.

EXAMPLE 1. Let $S_i, i = 0, 1, \cdots,$ $T_j, j = 0, 1, \cdots,$ and $U_k, k = 0, 1, \cdots$ be independent sequences of mutually independent, uniformly bounded mean zero, variance one random variables. Let

$$d_{ikj} = V_{ijk} S_i T_j U_k$$

where V_{ijk} is measurable on the σ-field $\mathfrak{F}_{i-1,j-1,k-1}$ generated by S, T, U with indices less than i, j, and k. If, for example, the S, T, U sequence consists of the Rademacher functions defined on the unit cube in R^3, the resulting martingale concepts are close to those of strong differentiation.

EXAMPLE 2. Let p_1, p_2, and p_3 be primes and consider the sequence of numbers $p_1^i \cdot p_2^j \cdot p_3^k$ where i, j, k are nonnegative integers. Let $\mathfrak{F}_{ijk}$ be the σ-field of sets whose characteristic functions are periodic in θ: $0 \leqslant \theta \leqslant 2\pi$ with period $p_1^i \cdot p_2^j \cdot p_3^k$. The fields $\mathfrak{F}_n$ are *decreasing*, but they do satisfy Assumption 1 with $\mathfrak{F}_{i\infty\infty}$ replaced by $\mathfrak{F}_{i00}$ etc. The resulting martingales satisfy Assumption 2. If, for example, $n = (i, j, k)$ and $n + 1 = (i + 1, j + 1, k + 1)$ then

$$f_n - f_{n+1} = \sum_{j=1}^N V_{n+1}^{(l)} r_n^{(l)}$$

where $V_{n+1}^{(l)}(\theta)$ is periodic with period $p_1^{i+1} \cdot p_2^{j+1} \cdot p_3^{k+1}$. Here the martingale differences $r_n^{(l)}, l = 1, 2, \cdots, N$ consist of the $N = (p_1 \cdot p_2 \cdot p_3 - 1)$ exponentials that are periodic with period $p_1^i \cdot p_2^j \cdot p_3^k$ but are not periodic $p_1^{i+1} p_2^{j+1} p_3^{k+1}$. More details on this example can be found in Gundy and Varopoulos (1976).

PROOF OF THE THEOREM. The following lemma provides the key to the proof.

LEMMA. *Let f be a regular martingale and $\lambda > 0$. Suppose*

$$E = \{\omega\colon \sup_{(s,t,u)}|f(\omega)_{(s,t,u)}| > \lambda\}$$

and $P(E) \leqslant \varepsilon$ for some $\varepsilon < 1$. Then there is a set F with $E \subset F$, constants C_i, $i = 1, 2, 3$, independent of λ, with $P(F) \leqslant C_1\varepsilon$, and a decomposition of f into two martingales

$$f = g + b$$

with

$$\|g\|_\infty \leqslant C_2\lambda$$

and

$$|b_{(s,t,u)}(\omega)| \leqslant C_3 \lambda F_{(s,t,u)}(\omega)$$

for $\omega \notin F$. Here $F_{(s,t,u)}(\omega)$ is the conditional expectation of the characteristic function of F with respect to $\mathcal{F}_{(s,t,u)}$.

Before proving this lemma, we indicate how the convergence theorem follows. First of all, we may replace the probability space Ω by a set G measurable $\mathcal{F}_{(s,t,u)}$ for large (s, t, u) so that on G,

$$\sup_{(m,n,p) > (s,t,u)}|f_{(m,n,p)}| \leqslant \lambda$$

except for a small subset $E' \subset G$,

$$P(E') \leqslant \varepsilon P(G).$$

Since this reduction may be carried out for every $\lambda > 0$, we can replace the assumption of the theorem (concerning lim sup) by the assumption of the lemma. Thus, it suffices to show that f converges almost everywhere on the complement of F. Since $f = g + b$, we first observe that $\lim g_{(m,n,p)}$ exists and is finite almost everywhere, since $\|g\|_\infty = O(\lambda)$. (The convergence of these martingales follows from estimates on the maximal function, as in the one-dimensional parameter case.) Second, observe that

$$\lim_{(m,n,p)} b_{(m,n,p)}(\omega) = 0$$

for almost every $\omega \notin F$ since

$$\lim_{(m,n,p)} F_{(m,n,p)}(\omega) = 0$$

on the complement of F. Therefore, it follows that f converges almost everywhere on the complement of F. With the above reduction, this is enough to prove the theorem.

PROOF OF THE LEMMA. Consider the one-dimensional time martingale

$$h_n = f_{(n,n,n)}, \ n \geqslant 0.$$

Since h_n is regular, its differences

$$h_{n+1} - h_n = \Sigma_{j=1}^N V_n^{(j)} r_{n+1}^{(j)}.$$

Let $d_n = (\Sigma_{j=1}^N [V_n^{(j)}]^2)^{\frac{1}{2}}$ and define the stopping time

$$\tau_1 = \inf\{n: d_n > 4\lambda\}.$$

Now observe that on the set $\{\tau_1 = n\}$,

$$P(|h_{n+1} - h_n| > 2\lambda \| \mathcal{F}_{(n,n,n)}) \geqslant C > 0. \tag{1}$$

This follows from the fact that, relative to the conditional expectation $E(\cdot \| \mathcal{F}_{(nnn)})$, the L^2 and L^∞ norms of $|h_{n+1} - h_n|$ are comparable, since the sequence $r_{n+1}^{(j)}, j = 1, 2, \cdots, N$ is orthonormal and bounded. (See Zygmund (1968), Chapter 5, 8.26).

As a first step toward the construction of the exceptional set F, we include the set $\{\tau_1 < \infty\}$. The above inequality (1) shows that

$$\begin{aligned} P(\tau_1 < \infty) &\leqslant CP(\sup_n |h_{n+1} - h_n| \geqslant 2\lambda) \\ &\leqslant CP(\sup_{(m,n,p)} |f_{(m,n,p)}| > \lambda). \end{aligned}$$

Next, let

$$\tau_2 = \inf(n: |h_n| > \lambda)$$

and note that

$$P(\tau_2 < \infty) \leqslant P(\sup_{(m,n,p)} |f_{(m,n,p)}| > \lambda).$$

Finally, let $\tau_3 = \tau_1 \wedge \tau_2$, the minimum of the two stopping times.

Now let h^{τ_3} be the martingale h, stopped at τ_3. We claim that $\|h^{\tau_3}\|_\infty \leqslant C\lambda$; in fact, on the set where

$$\tau_3 = \tau_1 = k < \tau_2,$$

the random variable $|h_k| \leqslant \lambda$ by definition of τ_2. On the set where

$$\tau_3 = \tau_2 = k \leqslant \tau_1$$

we have

$$\begin{aligned} |h_k - h_{k-1}| &= |\Sigma_{j=1}^N V_{k-1}^{(j)} r_k^{(j)}| \\ &\leqslant NC\big(\Sigma_{k=1}^N (V_{k-1}^{(j)})^2\big)^{\frac{1}{2}} \\ &\leqslant NC\lambda \end{aligned}$$

so that

$$|h_k| \leqslant (NC + 1)\lambda$$

on the set $\tau_3 = \tau_2 = k$. This proves the claim on the set where $\tau_3 < \infty$. When $\tau_3 = \infty$, the claim is automatically fulfilled.

Now define a three-dimensional parameter martingale g as follows: Given (s, t, u) choose an integer n so that $(s, t, u) < (n, n, n)$ in the specified sense. Let

$$g_{(s,t,u)} = E(h_n^{\tau_3} \| \mathcal{F}_{(s,t,u)}).$$

It may be verified that g is well-defined; that is, $g_{(s,t,u)}$ does not depend on the particular value of n chosen, and that g is, in fact, a martingale with respect to $\mathfrak{F}_{(s,t,u)}$. Furthermore, g is uniformly bounded as claimed in the statement of the lemma.

The martingale $b = f - g$; we must show that

$$|b_{(s,t,u)}(\omega)| \leqslant C_3 \lambda F_{(s,t,u)}(\omega)$$

for $\omega \notin F$. The set F is defined as

$$F = E \cup \{\tau_3 < \infty\},$$

so that

$$P(F) \leqslant CP(E)$$

by the argument of the preceding paragraph. If $\omega \notin F$, then for any (s, t, u) we have

$$|b_{(s,t,u)}(\omega)| \leqslant \lambda$$

by definition. The goal is to improve this estimate. Consider a fixed triple (s, t, u) and suppose, for example, that $s \leqslant t \leqslant u$. In this case, we may write $b_{s,t,u}$ as the conditional expectation of a random variable d_u

$$\begin{aligned} b_{(s,t,u)}(\omega) &= E(f_{(u,u,u)} - g_{(u,u,u)} \| \mathfrak{F}_{s,t,u}) \\ &= E(d_u \| \mathfrak{F}_{s,t,u}), \end{aligned}$$

where d_u is measurable $\mathfrak{F}_{u,u,u}$. Now consider the following one dimensional parameter martingale

$$b_k = E(d_u \| \mathfrak{F}_{s+k,t+k,u}),$$

for $k = 0, 1, \cdots, u - t$, and

$$b_k = E(d_u \| \mathfrak{F}_{s+k,u,u})$$

for $k = u - t + 1, \cdots, u - s$. (Notice that this definition makes sense only when there is at least one strict inequality in the chain $s \leqslant t \leqslant u$. If $s = t = u$, then

$$b_{u,u,u}(\omega) = 0$$

since $\tau_3(\omega) = \infty$ for $\omega \notin F$, which satisfies our requirement.)

The martingale b_k, $k = 0, 1, \cdots$ is regular in the sense that

$$b_{k+1} - b_k = \Sigma_{j=1}^{N} V_k^{(j)} r_{k+1}^{(j)}$$

so that we may proceed as in the first part of the argument. Let

$$\gamma_1 = \min_{k<u-s}(k;\ |b_k| > C_4\lambda),$$

$$\gamma_2 = \min_{k<u-s}\left(k\colon \left[\Sigma_{j=1}^{N}\left(V_k^{(j)}\right)^2\right]^{\frac{1}{2}} > 4C_4\lambda\right),$$

and $\gamma_3 = \gamma_1 \wedge \gamma_2$. The constant C_4 will be chosen below. The stopped martingale b^{γ_3} "converges" to a random variable d and

$$b_{s,t,u}(\omega) = E(d \| \mathfrak{F}_{s,t,u}).$$

We now estimate $b_{s,t,u}$ by examining the random variable d. We observe that

$$E(d\|\mathfrak{F}_{s,t,u}) = E(d(\gamma_3 > u)\|\mathfrak{F}_{s,t,u}) + E(d(\gamma_3 \leqslant u)\|\mathfrak{F}_{s,t,u})$$

where $(\gamma_3 > u)$ indicates that set on which the passage across $C\lambda$ is never achieved. Thus, the random variable $d(\gamma_3 > u)$ is measurable $\mathfrak{F}_{(u,u,u)}$ and bounded:

$$|d(\gamma_3 > u)| \leqslant 2C\lambda(\tau_3 < u).$$

Thus,

$$|E(d(\gamma_3 > u)\|\mathfrak{F}_{s,t,u})| \leqslant 2C\lambda E(\tau_3 < u\|\mathfrak{F}_{s,t,u})$$
$$\leqslant 2C\lambda F_{(s,t,u)}.$$

Consider the variable $d(\gamma_3 \leqslant u)$. By the regularity of the martingale b_k, $k = 0, 1, \cdots$ and the definition of γ_3, we know that

$$|d| \leqslant C\lambda(\gamma_3 \leqslant u);$$

this follows by the argument given in the construction of g, using τ_3. To finish the proof, we show that

$$E(\gamma_3 \leqslant u\|\mathfrak{F}_{s,t,u}) \leqslant CF_{s,t,u}.$$

The set $(\gamma_3 \leqslant u)$ consists of two parts, $(\gamma_1 < \gamma_2 \wedge u)$ and $(\gamma_2 \leqslant \gamma_1 \wedge u)$. By choosing C_4 sufficiently large, we insure that a passage across $C_4\lambda$ by the martingale b_k, $k = 0, 1, \cdots$ can take place only on the set E, where $\sup|f_{(s,t,u)}| > \lambda$. With such a choice of C_4 the set $(\gamma_1 \leqslant u)$ is contained in E, and

$$E(\gamma_1 < \gamma_2 \wedge u\|\mathfrak{F}_{s,t,u}) \leqslant F_{s,t,u}.$$

On the set

$$\gamma_2 = k < \gamma_1 \wedge u$$

we have

$$P(|b_{k+1} - b_k| > 2C_4\lambda\|\gamma_2 = k < \gamma_1 \wedge u) \geqslant C > 0$$

by the argument following (1), so that

$$P(|b_{k+1}| > C_4\lambda\|\gamma_2 = k < \gamma_1 \wedge u) \geqslant C > 0.$$

Note that since $B_{u+k+1} - b_{u+k} = 0$, $k = 0, 1, \cdots$, we must have $\gamma_2 < u$, and the above inequality implies

$$P(\gamma_1 = \gamma_2 + 1\|\gamma_2 < \gamma_1 \wedge u) \geqslant C > 0.$$

This, in turn, implies

$$E(\gamma_2 < \gamma_1 \wedge u\|\mathfrak{F}_{s,t,u}) \leqslant CE(\gamma_1 \leqslant u\|\mathfrak{F}_{s,t,u}) \leqslant CF_{s,t,u}.$$

Summing up, we have shown that

$$|b_{(s,t,u)}(\omega)| = |E(d\|\mathfrak{F}_{s,t,u})| \leqslant C\lambda F_{s,t,u}$$

to finish the proof.

REFERENCES

[1] BURKHOLDER, D. L. and GUNDY, R. F. (1970). Extrapolation and interpolation of quasi-linear operators on martingales. *Acta Math.* **124** 249–304.

[2] CAIROLI, R. (1968). Une inégalité pour martingales a indices multiples et ses applications. *Lecture Notes in Mathematics* **124.** Springer-Verlag, New York.

[3] CAIROLI, R. and WALSH, J. B. (1975). Stochastic integrals in the plane. *Acta Math.* **134** 111–183.

[4] GUNDY, R. F. and VAROPOULOS, N. TH. (1976). A martingale that occurs in harmonic analysis. *Arkiv. fur Math.* To appear.

[5] SAKS, S. (1937). *Theory of the Integral. Second revised edition.* Hafner, New York.

[6] ZYGMUND, A. (1968). *Trigonometric Series.* (Vols. I and II combined.) Cambridge.

DEPARTMENT OF STATISTICS
HILL CENTER FOR THE MATHEMATICAL SCIENCES
BUSCH CAMPUS
RUTGERS UNIVERSITY
NEW BRUNSWICK, NEW JERSEY 08903