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UNIMODALITY OF PASSAGE TIMES FOR ONE-DIMENSIONAL STRONG MARKOV PROCESSES[1]

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Let τ_x be the first passage time of x for a diffusion or birth-death process. If one starts in a reflecting state, say 0, then the distribution $P_0(\tau_x \leqslant \cdot)$ is strongly unimodal. Here we show for an arbitrary state 0 the distribution $P_0(\tau_x \leqslant \cdot)$ is unimodal. Further we give a discrete analogue for the random walk.

1. Introduction. For a continuous birth-death process with 0 reflecting, the distribution $P_0(\tau_x \leqslant \cdot)$ is (if not degenerate) a convolution of exponential distributions (Kielson 1971). As a consequence, $P_0(\tau_x \leqslant \cdot)$ is strongly unimodal. This result extends partially to diffusions with 0 reflecting, using an approximation by birth-death processes (Stone 1963). What happens if 0 is not a reflecting, but an arbitrary state? For Brownian motion (Keilson 1971) or an Ornstein-Uhlenbeck process we can explicitly calculate $P_0(\tau_x \leqslant \cdot)$. This distribution is not strongly unimodal, but is nevertheless unimodal. In this paper we prove unimodality in general.

THEOREM 1.1. *Let $X(t)$ be a continuous birth-death process. Then the distribution function $P_i(\tau_j \leqslant \cdot)$ is unimodal for all $i, j \in \mathbb{Z}$. If $|i-j| = 1$ then the density of $P_i(\tau_j \leqslant \cdot)$ is decreasing on $t \geqslant 0$,* a.e.

By approximating a one-dimensional diffusion by a sequence of birth-death processes (Stone (1963)) we obtain

THEOREM 1.2. *For any one-dimensional diffusion process the distribution function $P_i(\tau_j \leqslant \cdot)$ is unimodal, $i, j \in I$.*

THEOREM 1.3. *Let $X(t)$ be a discrete birth-death process. Then $P_i(\tau_j = 2k)$, $P_i(\tau_j = 2k+1)$ are first increasing in $k \in \mathbb{Z}$, then decreasing. If $|i-j| = 1$ then $P_i(\tau_j = 2k+1)$ is decreasing on $k \geqslant 0$, $k \in \mathbb{N}$, and 0 otherwise.*

The three strong Markov processes mentioned above enjoy the common property that the basic state space is a subset of the (extended) reals and the trajectories do not jump over points in the state space. This is the reason (not obvious) for basically the same proof of Theorems 1.1, 1.2, 1.3. We will give the proof of Theorem 1.1 in detail in Sections 3 and 4, and the idea in the beginning of Section 3. In order to avoid technical difficulties and different cases, we get Theorem 1.2 by an approximation of the diffusion by birth-death processes. Theorem 1.3 is the discrete analogue of Theorem 1.1. We omit the proof and mention only the result.

An independent analytic proof of Theorem 1.1 for ergodic birth-death processes and Theorem 1.2 was given by Keilson (1979).

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The distribution $P_0(\tau_x \leqslant \cdot)$ for diffusions as well as for birth-death processes is known to be infinitely divisible. It would be interesting to know whether they belong to class L or not. If they belong to class L, unimodality follows already by a paper of Wolfe (1971).

2. Notations and definitions. Let $X(t)$, $t \in T \subset \mathbb{R}^+$ be a stationary strong Markov process with state space $I \subset R \cup \{-\infty, +\infty\}$ on a probability space $(\Omega, \mathfrak{F}, P)$. For a discrete birth-death process (random walk) we have $T = N$, $I \subset \mathbb{Z}$. The transition probabilities for a jump from x to $x+1$, $x-1$ are p_x, q_x. For a continuous birth-death process we have $T = \mathbb{R}^+$, $I \subset \mathbb{Z} \cup \{-\infty, +\infty\}$. For certain birth-death rates λ_x, μ_x the process will converge to $\pm\infty$ in a finite time (Karlin, McGregor 1957). We include these processes by adding two absorbing states $\pm\infty$ in an obvious way.

For a diffusion we have $T = \mathbb{R}^+$ and $I \subset \mathbb{R} \cup \{\pm\infty\}$. The trajectories are a.s. continuous relative to I.

In this paper $\tau_y, y \in I$ denotes the stopping time $\tau_y = \inf\{t\, X_t = y\}$ relative to the σ-algebras generated by the process X_t. We use here the convention $\inf \varnothing = \infty$.

In general $P_\mu(\cdot)$, $P_x(\cdot)$ denotes the probability measures given by the transition probabilities and the initial measure (pointmeasure in x). For birth-death processes we further use the functions

$$P(x,y,t) = P_x(X(t) = y)$$

$$P(\mu,y,t) = P_\mu(X(t) = y).$$

A function F is called unimodal if there exists at least one value b such that $F(x)$ is convex for $x < b$ and concave for $x > b$ (Ibragimov 1956). Every unimodal distribution function has left and right derivatives, except perhaps at the point b. These derivatives increase monotonously for $x < b$ and decrease for $x > b$. A distribution function is called strongly unimodal, if any convolution with a unimodal distribution is unimodal. Strong unimodality implies unimodality (convolution with pointmeasure). Ibragimov (1956) showed that a unimodal distribution function is strongly unimodal iff F is continuous and $\ln F'(x)$ is concave on $\{F' \neq 0\}$. This leads to: the derivative F' of a strongly unimodal function is a Polya frequency function of order two (Schoenberg 1951).

3. Idea. Let J be an ordered set, $f\colon J \to \mathbb{R}$ a function. The sign sequence for f is the sequence of ± 1's generated by $\operatorname{sgn} f(x)$ as x runs through J with the given order. Zeros are neglected. The function $S_J^-(f)$ counts the sign changes of the sign sequence of f. (The index $-$ is only used to have the same notation as Karlin (1968)).

The following theorem is in Karlin (1968). For an introduction to totally positive matrices of order $r \in \mathbb{N}$ (TP_r matrices) and the variation diminishing property see Karlin (1968). We will not use these properties, except in the Theorem 3.1.

THEOREM 3.1. *Let $K(x,y)$, $x, y \in J$, be a TP_r matrix, J a finite subset of $\mathbb{Z}$. Define as usual $\mu K = \Sigma_x K(x, \cdot)\mu(x)$ for μ a signed finite measure on J. Then*

$$S_J^-(\mu K) \leqslant S_J^-(\mu)$$

holds for $S_J^-(\mu) \leqslant r - 1$. Furthermore the sign sequence of μK is contained in the sign sequence of μ.

We will apply this theorem to the transition matrix P^t of a birth-death process. Let $I = \{0, 1, \cdots, x_0 - 1, x_0\}$ with x_0 absorbing and J be the state space I without x_0. It is well known (Karlin 1968) that these matrices P^t for fixed t, defined by

$$P^t(x,y) = P(x,y,t) \qquad x,y \in J$$

are totally positive of all orders.

The following picture shall help to understand the next two lemmas and the underlying structure of the proof. For simplicity we take a discrete time.

Every point represents a + or − according to sgn $\partial/\partial t P(\mu,\cdot,\cdot)$, sign changes of a row (excluding x_0) is decreasing. This is the main statement of Theorem 3.1. Then the Kolmogorov backward equations give us the upper line, the sign of the second derivative to the time of $P(\mu, x_0, t)$. This line starts with a + switches ones to − and then remains −. A discussion of this behaviour will finally show the unimodality of $P_x(\tau_{x_0} \leqslant \cdot)$, $x \in I$.

LEMMA 3.2. *Let $X(t)$ be a continuous birth-death process on $I = \{0, 1, \cdots, x_0\}$. The state x_0 is absorbing. For simplicity we may assume all birth-death rates $\lambda_0, \lambda_i, \mu_i$*

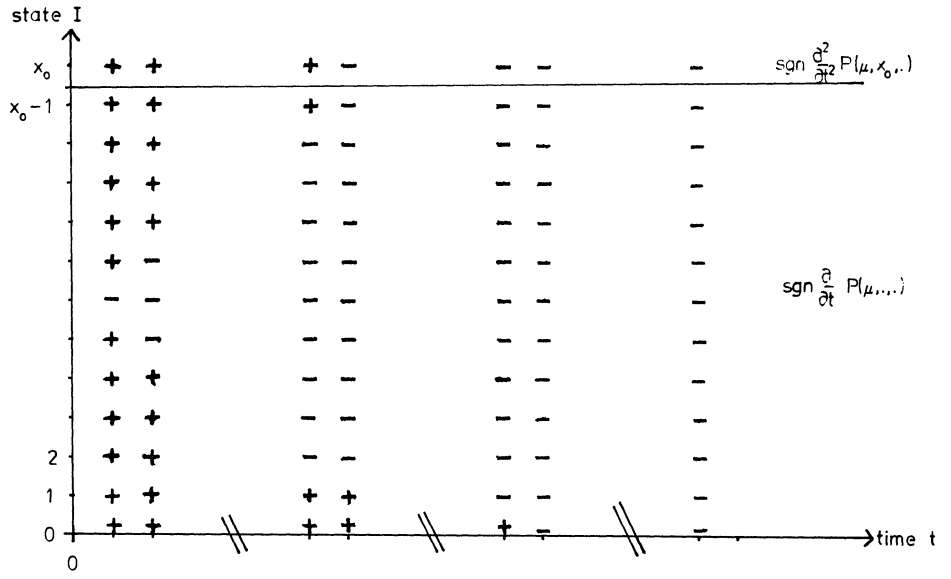

FIG. 1

are positive for $i = 1, 2, \cdots, x_0 - 1$. *Let* N *be* $\{0, 1, \cdots, x_0 - 1\}$ *then*

(a) $$S_N^- \left(\frac{\partial}{\partial t} P(\mu, \cdot, t)\right) \quad \text{is decreasing in } t \geqslant 0.$$

(b) $$S^-_{[0,t_1]}\left(\frac{\partial}{\partial t} P(\mu, x_0 - 1, \cdot)\right) - S_{[0,t_0]}\left(\frac{\partial}{\partial t} P(\mu, x_0 - 1, \cdot)\right) = 1, t_1 > t_0$$

implies

$$S_N^- \left(\frac{\partial}{\partial t} P(\mu, \cdot, t_0)\right) > S_N^- \left(\frac{\partial}{\partial t} P(\mu, \cdot, t_1)\right).$$

(c) $$S_N^- \left(\frac{\partial}{\partial t} P(\mu, \cdot, 0)\right) \leqslant n$$

implies

$$S^-_{\mathbf{R}^+} \left(\frac{\partial}{\partial t} P(\mu, x_0 - 1, \cdot)\right) \leqslant n.$$

PROOF. Take the derivative to t of the Chapman-Kolmogorov equation to get

$$\frac{\partial}{\partial t} P(\mu, x, t_1) = \Sigma_{y \in \bar{N}}\left(\frac{\partial}{\partial t} P(\mu, y, t_0)\right) P(y, x, t_1 - t_0).$$

Now apply Theorem 2.1 to obtain

$$S_N^- \left(\frac{\partial}{\partial t} P(\mu, \cdot, t_0)\right) \geqslant S_N^- \left(\frac{\partial}{\partial t} P(\mu, \cdot, t_1)\right).$$

(b) Suppose equality above holds. The sign rule of Theorem 3.1 $\frac{\partial}{\partial t} P(\mu, \cdot, t_0)$ and $\frac{\partial}{\partial t} P(\mu, \cdot, t_1)$ possess an identical sign sequence.

Our assumption in b implies $\frac{\partial}{\partial t} P(\mu, x_0 - 1, t_1)$ has a different sign from that of $\frac{\partial}{\partial t} P(\mu, x_0 - 1, t_0)$(both without loss of generality are non-zero). This is a contradiction.

(c) Suppose $S^-_{\mathbf{R}^+}\left(\frac{\partial}{\partial t} P(\mu, x_0 - 1, \cdot)\right) > n$. Choose a sequence $t_0 < t_1 < t_2 \cdots < t_{n+1}$ such that $\operatorname{sgn} \frac{\partial}{\partial t} P(\mu, x_0 - 1, t_i)$ alternates and

$$S^-_{[0,t_i]}\left(\frac{\partial}{\partial t} P(\mu, x_0 - 1, \cdot)\right) - S^-_{[0,t_{i-1}]}\left(\frac{\partial}{\partial t} P(\mu, x_0 - 1, \cdot)\right) = 1,$$

$i = 1, \cdots, n + 1$ holds. An inductive application of b gives us $S_N^-\left(\frac{\partial}{\partial t} P(\mu, \cdot, 0)\right) > n$. This proves (c).

LEMMA 3.3. *Take the same assumptions as in Lemma* 3.2. *Let* μ *be a point measure on* $x_1 \in N$. *Then*

$$S_N^-\left(\frac{\partial}{\partial t} P(x_1, \cdot, 0)\right) = 2 \qquad 0 < x_1 < x_0 - 1$$

$$S_N^-\left(\frac{\partial}{\partial t} P(x_1, \cdot, 0)\right) = 1 \qquad x_1 = x_0 - 1$$

$$\text{or} \qquad x_1 = 0$$

$$S_{\mathbb{R}^+}^-\left(\frac{\partial}{\partial t}P(x_1, x_0 - 1, 0)\right) = 1 \qquad x_1 < x_0 - 1$$

$$S_{\mathbb{R}^+}^-\left(\frac{\partial}{\partial t}P(x_1, x_0 - 1, \cdot)\right) = 0 \qquad x_1 = x_0 - 1.$$

PROOF. The Kolmogorov backward equations give the first two statements. The sign sequence looks like $\cdots 1, -1, 1, \cdots$. For the third statement, Lemma 3.2 says

$$S = S_{\mathbb{R}^+}^-\left(\frac{\partial}{\partial t}P(x_1, x_0 - 1, \cdot)\right) \leqslant 2 \qquad x_1 < x_0 - 1.$$

We will show $S = 0$ and $S = 2$ are impossible.

(i) Suppose $S = 0$. $\frac{\partial}{\partial t} P(x_1, x_0 - 1, t)$ is always greater than or equal to 0 since $P(x_1, x_0 - 1, 0) = 0$, $P(x_1, x_0 - 1, t) \geqslant 0$ and $S = 0$. This is impossible by reason of $\lim_{t\to\infty} P(x_1, x_0 - 1, t) = 0$ (x_0 is absorbing) and the existence of at least a t_0 satisfying $P(x_1, x_0 - 1, t_0) > 0$.

(ii) Suppose $S = 2$. By $P(x_1, x_0 - 1, 0) = 0$ and $P(x_1, x_0 - 1, t) > 0$ the function $\frac{\partial}{\partial t} P(x_1, x_0 - 1, \cdot)$ is initially positive. $S = 2$ implies the sequence positive, negative, positive. There exists a t_0 satisfying

$$P(x_1, x_0 - 1, t_0) > 0 \quad \text{and} \quad \frac{\partial}{\partial t}P(x_1, x_0 - 1, t) \geqslant 0 \qquad \text{for all } t \geqslant t_0.$$

This implies

$$\lim_{t\to\infty} P(x_1, x_0 - 1, t) \geqslant P(x_1, x_0 - 1, t_0) > 0$$

in contradiction to x_0 absorbing and $P(x_1, x_0, t) \to 1$ as $x \to \infty$.

We now prove the fourth statement. We know already by Lemma 3.2 that $S \leqslant 1$ holds.

Suppose $S = 1$. We have $P(x_0 - 1, x_0 - 1, 0) = 1$ and $P(x_0 - 1, x_0 - 1, t) \leqslant 1$. Therefore $\frac{\partial}{\partial t} P(x_1, x_0 - 1, \cdot)$ is first negative, then by $S = 1$ always positive or zero. Choose a t_0 big enough in the positive part. We have

$$\lim_{t\to\infty} P(x_0 - 1, x_0 - 1, t) \geqslant P(x_0 - 1, x_0 - 1, t_0) > 0.$$

This is a contradiction to $P(x_0 - 1, x_0, t) \to_t 1$.

LEMMA 3.4. *Let $X(t)$ be a continuous birth-death process on $I = \{0, 1, \cdots, x_0\}$. The state x_0 is absorbing. All birth-death rates $\lambda_0, \lambda_i, \mu_i$, $i = 1, \cdots, x_0 - 1$ are strictly positive. Then the distribution function $P_j(\tau_{x_0} \leqslant \cdot)$, $j \in I$, is unimodal. For $j = x_0 - 1$ is the distribution function $P_{x_0-1}(\tau_{x_0} \leqslant \cdot)$ concave on $\{t \geqslant 0\}$.*

PROOF. Without loss of generality we may assume $0 \leqslant i < j$ and $j = x_0$. We will first prove the theorem for $\bar{N} = \{0, 1, \cdots, i, \cdots, j = x_0\}$ and λ_i, μ_i are positive for $i \in \bar{N} \setminus \{0, x_0\}$.

(1) Let $i < x_0 - 1$. According to Lemma 2.3 $\frac{\partial}{\partial t} P(i, x_0 - 1, \cdot)$ possesses one sign change. The backward equations imply one sign change of $\frac{\partial^2}{\partial t^2} P(i, x_0, \cdot)$.

Therefore $\frac{\partial}{\partial t} P(i, x_0, \cdot)$ has at most two extrema. By reason of $P(i, x_0, 0) = 0$ and $P(i, x_0, t) \geqslant 0$ we first have a local maximum. A further local minimum is not possible, because otherwise $P(i, x_0, t)$ would be unbounded. Therefore $\frac{\partial}{\partial t} P(i, x_0, \cdot)$ possesses only one maximum, is increasing until this maximum and decreasing afterwards.

(ii) Let $i = x_0 - 1$. Now $\frac{\partial^2}{\partial t^2} P(i, x_0, \cdot)$ has no sign change. Thus $\frac{\partial}{\partial t} P(i, x_0, \cdot)$ has at most one extremum. Assume one extremum. This extremum is a local minimum since $\frac{\partial^2}{\partial t^2} P(i, x_0, 0)$ is negative. Hence $\frac{\partial}{\partial t} P(i, x_0, t)$ is increasing for t big enough. This contradicts $P(i, x_0, t) \leqslant 1$ and $\frac{\partial}{\partial t} P(i, x_0, t) \geqslant 0$. Therefore $\frac{\partial}{\partial t} P(i, x_0, t)$ has no extremum and is monotone decreasing on $t \geqslant 0$.

4. Results. We will now extend the result of Lemma 3.4. The main tool thereby is the fact, the weak limit of unimodal distributions is unimodal (Ibragimov 1956).

1. Let $X(t)$ be a birth-death process in Lemma 3.4, but some λ_i, μ_i may be zero. (This includes 0 absorbing by $\lambda_0 = 0$). Construct a sequence X^n of birth and death processes with λ_0^n, λ_i^n, μ_i^n converging to λ_0, λ_i, μ_i and λ_0, λ_i, μ_i strictly positive for $i = 1, \cdots, x_0 - 1$. Then $\tau_{x_0}^n$ will converge weakly to τ_{x_0}. This implies τ_{x_0} has a unimodal distribution.

2. *Proof of Theorem* 1.1. Let $X(t)$ be an arbitrary birth-death process. Without loss of generality it suffices to show $P_0(\tau_{x_0} \leqslant \cdot)$ is unimodal for $x_0 > 0$ absorbing. Define a sequence X^n of birth-death processes by the same rates. Change only the state $-n$ into an absorbing one. Now $\tau_{x_0}^n$ converges weakly to τ_{x_0}. This finishes the proof of Theorem 1.1.

3. *Proof of Theorem* 1.2. Let $X(t)$ be an arbitrary diffusion. We will show $P_0(\tau_{x_0} \leqslant \cdot)$ is unimodal, $x_0 > 0$. Without loss of generality the diffusion is on the natural scale. The process is completely determined by the speedmeasure $m(\cdot)$. (Our processes all have infinite lifetime by the introduction of $\pm\infty$ and therefore the killing measure is 0.) Following Stone (1963) we construct a sequence of birth-death processes satisfying some conditions ($i - ix$ on page 643 of Stone (1963)). These processes X^n converge to X. More precisely: Let ρ be a pseudo metric on $\mathbb{R}$ (we allow $\rho(x, y) = 0$ for $x \neq y$) such that if $x_n \in \mathbb{R}$ for $n \geqslant 0$ and $x_n - x_0 \to 0$, then $\rho(x_n, x_0) \to 0$. A sequence $x^n(t)$ is said to be J convergent to $x_0(t)$, if there is a sequence of continuous one-to-one mappings $\lambda_n(t)$ of $[0, \infty]$ onto itself such that for every $T > 0$

$$\sup_{0 < t < T} |\lambda_n(t) - t| \to 0$$

and

$$\sup_{0 < t < T} \rho(x_n(t), x(\lambda_n(t))) \to 0.$$

It remains to show $\tau_{x_0}^n$ converges in distribution to τ_{x_0}. Choose an increasing sequence $y_i \in I$, $i \in N$, with $\rho(y_i, x_0)$ strict decreasing to 0. We may assume x_0, y_i are elements of I^n for all i and n. By the Stone convergence we get

$$\tau_{y_{i-1}} - 2\epsilon \leqslant \tau_{y_i}^n - \epsilon \leqslant \tau_{x_0}$$

for $n = n(\omega, \epsilon)$ big enough. With $\epsilon \to 0$, $i \to \infty$ we get $\tau_{x_0}^n$ converging weakly to τ_{x_0}. With the arguments as before, we get the desired unimodality.

4. *Proof of Theorem* 1.3. The proof is analogus to that of Theorem 1.1. It is easy to show the matrix

$$P = (P_{i,j})_{i,j \in \mathbf{Z}} \qquad \begin{aligned} p_{ij} &= q_i & j &= i - 1 \\ &= p_i & j &= i + 1 \\ &= 0 & &\text{otherwise} \end{aligned}$$

is total positive for every order on $N = \{0, 1, \cdots, x_0 - 1\}$. 0 is absorbing or reflecting. The composition of TP_r matrices is again TP_r (Karlin 1963). For reason of the periodicity 2 it is appropriate to use the matrix $P^2 = P \cdot P$ instead of P. Furthermore the functions

$$F(\mu, i, n) = P(\mu, i, n) - P(\mu, i, n - 2)$$

replace $\dfrac{\partial}{\partial t} P(\mu, \cdot, \cdot)$. With these aids we can first prove the analog to Lemma 3.4 and then extend the result to a general discrete birth-death process.

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