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SOME CLASSES OF TWO-PARAMETER MARTINGALES

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A class of two-parameter martingales, named "martingales with orthogonal increments" or "martingales of direction independent variation," is introduced. It is shown that this class, which is characterized by a sample function property, is included in the class of martingales of path independent variation and includes the class of strong martingales. The class of martingales with orthogonal increments is stable under stochastic integration and some results, which were obtained previously for strong martingales, hold also for martingales with orthogonal increments. It is shown that if M_z is a martingale with orthogonal increments on the sigma-fields generated by the Wiener process then there exists a Wiener process such that M_z can be represented as a stochastic integral of first type with respect to it.

1. Introduction. The most natural definition of a two-parameter martingale is, perhaps, the process satisfying $E(X_{(s_2,t_2)} \mid \mathscr{F}_{(s_1,t_1)}) = X_{(s_1,t_1)}$ whenever $s_2 \geq s_1$ and $t_2 \geq t_1$. As is well known not all the properties of one-parameter martingales are inherited by two-parameter martingales under this definition and this leads to the introduction of other classes of martingales with the same partial ordering which are either weaker (e.g., weak martingales, 1- and 2-martingales) or stronger (e.g., strong martingales, martingales of path independent variation) than the natural class of martingales. Strong martingales were introduced in [1] and shown there to play an important role in the theory of two-parameter martingales and stochastic integration; martingales of path independent variation were introduced in [9]. It was shown in [1], that continuous strong martingales are of path independent variation. In the converse direction it was conjectured in [1] that martingales of path independent variation on the sigma fields generated by the two-parameter Wiener processes are strong martingales and it was shown that in a certain special case this is actually so. Weak martingales of path independent variations were introduced and characterized in [4].

This paper was motivated by the relations between strong and path independent martingales; it considers the problem of characterizing strong martingales by sample function properties and gives a partial answer to this problem. A class of martingales is introduced in Section 5 which will be called "martingales with orthogonal increments" ("martingales of direction independent variation" may be more appropriate). It includes the class of strong martingales and is included in the class of martingales of path independent variation. Like the class of path independent martingales, the class of martingales with orthogonal increments is also characterized by a sample function property. On the other hand, martingales of orthogonal increments share with strong martingales several important properties so that results which were obtained for strong martingales hold for direction independent martingales. It is shown that on the sigma fields generated by the Wiener process, martingales with orthogonal increments can be represented as stochastic integrals of the first type with respect to a Wiener process.

In the next section we consider stable subspaces of two parameter square integrable martingales. The main results are similar to those for the one parameter case; these results clarify the stability properties of the classes of martingales considered later. Section 3 gives

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some results on the increasing functions associated with square integrable martingales and follows results of [2] (cf. also [5]). Several characterizations of continuous martingales of path independent variation are given in Section 4. Continuous martingales of orthogonal increments ae introduced in Section 5. Several characterizations of this class are given. It is shown that this class is stable under stochastic integration and it is pointed out that certain results which were obtained for strong martingales hold under the assumption that the martingale involved is of orthogonal increments.

A well-known result of P. Lévy states that if M_t and $M_t^2 - t$ are martingales then M is a Brownian motion. An analogous result is derived for martingales of orthogonal increments. A sufficient condition is given, under which a martingale of direction independent variation is representable as a stochastic integral of the first type with respect to a Wiener process.

Notation and basic assumptions. We follow [1], in particular: $\mathbb{R}_+^2$ will denote the positive quadrant of the plane, if $z_1, z_2 \in \mathbb{R}_+^2$, $z_1 = (s_1, t_1)$, $z_2 = (s_2, t_2)$, then $z_1 \prec z_2$ means that $s_1 \leq s_2$ and $t_1 \leq t_2$; $z_1 \prec\prec z_2$ means that $s_1 < s_2$ and $t_1 < t_2$. Let $z_1 \prec\prec z_2$, $(z_1, z_2]$ denotes the rectangle $\{z: z_1 \prec\prec z \prec z_2\}$, R_z denotes the rectangle $[0, z]$. $(\Omega, \mathscr{F}, P)$ is a complete probability space; $\mathscr{F}_z$, $z \in \mathbb{R}_+^2$ are sub σ-fields of $\mathscr{F}$. Let $z = (s, t)$, $\mathscr{F}_z^1 = \mathscr{F}_{(s,\infty)}$, $\mathscr{F}_z^2 = \mathscr{F}_{(\infty,t)}$. Throughout the paper the sigma fields $\mathscr{F}_z$ are assumed to satisfy: (a) $\mathscr{F}_{z_1} \subset \mathscr{F}_{z_2}$ if $z_1 \prec z_2$ $\mathscr{F}_0$ includes the null sets of $\mathscr{F}$; (b) $\mathscr{F}_z$ is right-continuous, and (c) Condition (F4) of [1] that is: $\mathscr{F}_z^1$ and $\mathscr{F}_z^2$ are conditionally independent given $\mathscr{F}_z$.

Let $z \prec z'$, $z = (s, t)$, $z' = (s', t')$, $M(z, z'] = M_{s',t'} + M_{s,t} - M_{s',t} - M_{s,t'}$; let M_z, $z \in \mathbb{R}_+^2$, be $\mathscr{F}_z$ adapted, $E\,|M_z| < \infty$ for all z then: (a) M_z is a martingale if $E(M_{z'}\,|\,\mathscr{F}_z) = M_z$; (b) M_z is a strong martingale if $E(M(z, z']\,|\,\mathscr{F}_z^1 \vee \mathscr{F}_z^2) = 0$; and (c) M_z is a weak martingale if $E(M(z, z']\,|\,\mathscr{F}_z) = 0$.

$\mathscr{M}^2$ will denote the class of right-continuous martingales, vanishing on the axes such that $\sup_z E\,|M_z|^2 < \infty$. It follows from the Doob-Cairoli maximal inequality that for every martingale M in $\mathscr{M}^2$, $M_{(s,t)}$ converges a.s. to a random variable $\mathscr{M}_\infty$ as $s \to \infty$, $t \to \infty$ and $M_z = \mathscr{E}(\mathscr{M}_\infty\,|\,\mathscr{F}_z)$. $\mathscr{M}_c^2$ will denote all the sample continuous martingales in $\mathscr{M}^2$. $\mathscr{M}_{cc}^2$ will denote the martingales M in $\mathscr{M}_c^2$ for which (a) either $\sup E\,|M_z|^4 < \infty$ or else M is locally L^4 bounded, that is, there exists a sequence of martingales $M^n \in \mathscr{M}_c^2$ such that $\sup_z E\,|M_z^n|^4 < \infty$ and for every finite z for every $\epsilon > 0$ there exists an n_0 such that

$$\operatorname{Prob}\{\sup_{\zeta \in R_z} |M_\zeta - M_\zeta^n| \neq 0\} < \epsilon \qquad n \geq n_0$$

and (b) Let $\langle M\rangle$ be the increasing process associated with $M (M^2 - \langle M\rangle$ is [1] a weak martingale) then $\langle M\rangle$ is assumed to be sample continuous.

REMARK. For the case where the σ-fields $\mathscr{F}_z$ are generated by the Wiener process $\mathscr{M}^2 = \mathscr{M}_c^2$ and it was shown in [2] that $\mathscr{M}_c^2 = \mathscr{M}_{cc}^2$. It is probably true that $\mathscr{M}_c^2 = \mathscr{M}_{cc}^2$ in general but this has not yet been proved.

2. Stable subspaces of two-parameter square integrable martingales. The main results of this section follow along the same lines as the one-parameter case (Chapter II, 1–6, 28, 29 of [7]), with the conclusion "then MN is a martingale" in the one-parameter case becoming, in general, "then MN is a weak martingale", except for the case where M and N are strong martingales (Proposition 2.4).

DEFINITIONS. (a) A random set $D(\omega)$ from ω to the subsets of $\mathbb{R}_+^2$ is *a simple random set* if there exists a partition of $\mathbb{R}_+^2$ into rectangles $(z_{ij}, z_{i+1,j+1}]$ and events $A_{ij} \in \mathscr{F}_{z_{ij}}$ such that

$$D(\omega) = \{\cup_{ij}(z_{ij}, z_{i+1,j+1}]\, I_{A_{ij}}(\omega)\}$$

and there exists a finite z_0 such that $A_{ij} = \varnothing$ for all $z_{ij} \notin R_{z_0}$ (or, otherwise stated, $D(\omega) \subset R_{z_0}$ a.s.). The rectangles $(z_{ij}, z_{i+1,j+1}]$ will be denoted by Δ_{ij} and $M(\Delta_{ij})$ will denote

$M_{z_{i+1,j+1}} + M_{z_{i,j}} - M_{z_{i+1,j}} - M_{z_{i,j+1}}$. For a simple random set D, $M(D)$ will denote $\sum_{ij} M(\Delta_{ij}) \cdot I_{A_{ij}}$.

(b) $D(\omega)$ is *a simple stopping set* if it is a simple random set and in addition $z' \in D(\omega)$ implies $z \in D(\omega)$ for all $z \prec z'$.

LEMMA 2.1. *If M_z, N_z, $z \in \mathbb{R}_+^2$ are square integrable martingales then for every simple random set $D(\omega)$*

$$E(M_\infty - M(D)) \cdot N(D) = 0.$$

PROOF. Notice that

$$(M_\infty - M(D))N(D) = \sum_{i,j,m,n} (1 - I_{A_{ij}}) M(\Delta_{ij}) I_{A_{m,n}} N(\Delta_{m,n}). \tag{2.1}$$

If $i = m$ and $j = n$ then $(1 - I_{A_{ij}}) I_{A_{m,n}} = 0$. If $i \neq m$, let $i > m$ then

$$\begin{aligned} E\{(1 - I_{A_{ij}}) M(\Delta_{ij}) I_{A_{m,n}} N(\Delta_{m,n}) \mid \mathscr{F}^1_{z_{ij}}\} \\ = (1 - I_{A_{ij}}) I_{A_{m,n}} N(\Delta_{m,n}) \cdot EM(\Delta_{ij}) \mid \mathscr{F}^1_{z_{ij}}) \\ = 0 \end{aligned}$$

and by similar arguments for the other cases it follows that the expectation of (2.1) is zero.

DEFINITIONS. (1) $\mathscr{H}$ is said to be a stable collection of martingales in $\mathscr{M}^2$ if $M \in \mathscr{H}$ implies that $M \in \mathscr{M}^2$ and $Y_z = M(R_n \cap D(\omega)) \in \mathscr{H}$ for every simple stopping set D.

(2) $\mathscr{H}$ is said to be a stable subspace of $\mathscr{M}^2$ if it is a closed linear subspace of $\mathscr{M}^2$ and a stable collection.

(3) M, $N \in \mathscr{M}^2$ are said to be weakly orthogonal if $EM_\infty N_\infty = 0$. M, $N \in \mathscr{M}^2$ are said to be orthogonal if $M_z N_z$ is a weak martingale.

(4) Let $\mathscr{H}$ be a stable collection of martingales in $\mathscr{M}^2$, then $\mathscr{H}^\perp$ denotes the collection of all martingales in $\mathscr{M}^2$ which are weakly orthogonal to all the martingales in $\mathscr{H}$:

$$\mathscr{H}^\perp = \{M \in \mathscr{M}^2 : EM_\infty N_\infty = 0 \ \forall N \in \mathscr{H}\}.$$

PROPOSITION 2.2. *Let $\mathscr{H}$ be a stable collection of martingales in $\mathscr{M}^2$ then*

(a) *$\mathscr{H}^\perp$ is a stable subspace*;

(b) *$M \in \mathscr{H}$ and $N \in \mathscr{H}^\perp$ implies that M and N are orthogonal* (i.e., *$M \cdot N$ is a weak martingale*);

(c) *every element M of $\mathscr{M}^2$ can be represented uniquely as $M = N + N'$, $N \in H^\perp$, $N' \in \mathscr{H}^{\perp\perp}$ and $N \cdot N'$ is a weak martingale.*

PROOF. If $N \in \mathscr{H}$ and $M \in \mathscr{H}^\perp$ then for every simple stopping set D $E\, M_\infty N(D) = 0$. By Lemma 1 we also have that $E(M_\infty - M(D)) \cdot N(D) = 0$. Hence $E\, M(D)N(D) = 0$ and, again by Lemma 1, $E\, N_\infty M(D) = 0$; it follows that $M(R_z \cap D) \in \mathscr{H}^\perp$. Therefore $\mathscr{H}^\perp$ is stable, the linearity and closure of $\mathscr{H}^\perp$ are obvious, this proves part (a). Let d denote a rectangle $(z_1, z_2]$, $z_1 \ll z_2$. In order to prove part (b) we have to show that $E(MN(d) \mid \mathscr{F}_{z_1}) = 0$. By Proposition 1.6 of [1], we have

$$E(MN(d) \mid \mathscr{F}_{z_1}) = E(M(d)N(d) \mid \mathscr{F}_{z_1}).$$

Therefore, in order to show that $M_z N_z$ is a weak martingale we have to show that

$$E(M(d)N(d)I_A) = 0 \tag{2.2}$$

for every event $A \in F_{z_1}$. Let

$$D(\omega) = (R_{z_2} \cap d^c) \cup (d \cdot I_{A^c})$$

then D is a simple stopping region. Therefore

$$E\, M_{z_2}\cdot N_{z_2} = 0, \qquad E\, M_{z_2}N(D) = 0, \qquad E\, M(D)N_{z_2} = 0;$$

therefore $E\, M(R_{z_2} - D)N(R_{z_2} - D) = 0$ which is the same as (2.2) proving part (b). Part (c) follows directly from (a) and (b).

PROPOSITION 2.3. *Let $\mathscr{H}$ be a stable subspace of $\mathscr{M}^2$ then if $M \in \mathscr{H}$ and φ is predictable and $E \int_{R_+^2} \varphi^2 d\,\langle M\rangle < \infty$ then $Y_z = \int_{R_z} \varphi\, dM \in \mathscr{H}$.*

The stable subspace generated by M is the class of stochastic integrals $\int \varphi\, dM$ where φ is predictable and $E \int \varphi_\zeta^2\, d\,\langle M_\zeta\rangle < \infty$. Therefore, if $N \in \mathscr{M}^2$ then the projection of N on the stable subspace generated by M is of the form $\int h_\zeta\, dM_\zeta$. Furthermore, $h\cdot$ is a predictable density of $\langle M, N\rangle$ with respect to $\langle M, M\rangle$. The proof is the same as the corresponding one-parameter result (Chapter II, Theorem 28 and 29 of [6]) and Theorem 2.2 of [1] and is, therefore, omitted.

The result of part (b) of Proposition 2.2 that $M\cdot N$ is a weak martingale cannot be improved in general. However, M and N are strong martingales. We have the following result (which is a direct consequence of Theorem 1.9 of [1]).

PROPOSITION 2.4. *If $M, N \in \mathscr{M}_{cc}^2$ are both strong martingales and if $M\cdot N$ is a weak martingale then $M\cdot N$ is a martingale.*

PROOF. Note that both $M + N$ and $M - N$ satisfy the assumptions of Theorem 1.9 of [1]. Therefore $M\cdot N = m + b$ where m_z is a martingale and b_z is the difference of two increasing processes. Therefore b is a weak martingale and since b can be chosen to be continuous and of bounded variation on a countable number of vertical and horizontal lines, it follows by [10] that b_z vanishes which proves the result.

3. Increasing processes. Let $\{\Delta_{ij}^{(n)}\}$ denote a sequence of partitions of R_z into rectangles $\Delta_{ij}^{(n)} = (z_{ij}^{(n)}, z_{i+1,j+1}^{(n)}]$. Let $|\Delta^{(n)}| = \sup_{i,j}\|z_{i+1,j+1}^{(n)} - z_{ij}^{(n)}\|$. The sequence of partitions $\{\Delta^n\}$ will be said to be arbitrary fine if $|\Delta^{(n)}| \to 0$ as $n \to \infty$. We will often write Δ instead of $\Delta^{(n)}$. If M_z, $z \in \mathbb{R}_+^2$ is a right-continuous square integrable martingale, and Δ is an arbitrary fine sequence of partitions, then it was shown in [1] that $\sum_{i,j} E([M(\Delta_{ij}^{(n)})]^2 \mid \mathscr{F}_{z_{ij}}^{(n)})$ converges weakly in L^1 to an increasing function $\langle M\rangle_z$ and $M_z^2 - \langle M\rangle_z$ is a weak martingale.

PROPOSITION 3.1. ([2]). *Let $M \in M_{cc}^2$, then*

$$\sum_{i,j} (M(\Delta_{ij}))^2$$

converges in probability to $\langle M\rangle_z$.

PROOF. This result was derived in Section 4 of [2] for the case where $\mathscr{F}_z$ are the σ-fields generated by the Wiener process W_z. The same proof goes over to the case considered here with minor modifications:

(a) It suffices to prove the result for the case where $E\, M_z^4 < \infty$ since the extension to the case where M is locally in L^4 is straightforward. The next step is to prove that

$$\lim E\{(\langle M\rangle_z - \sum_{i,j} E\, \langle M\rangle\, (\Delta_{ij}^{(n)}) \mid \mathscr{F}_{z_{ij}^{(n)}})^2] = 0 \tag{3.1}$$

as $n \to \infty$. The proof is exactly the same as in [2]. Therefore, since

$$E\{\langle M\rangle\, (\Delta_{ij}) \mid F_{z_{ij}}\} = E\{M(\Delta_{ij})^2 \mid \mathscr{F}_{z_{ij}}\}$$

it follows that $\sum E\{M(\Delta_{ij}^{(n)})^2 \mid \mathscr{F}_{z_{ij}^{(n)}}\}$ converges in L_1 to $\langle M\rangle_z$.

(b) It will be shown that $E(\sum_{ij} d_{ij}^{(n)})^2 \to 0$ as $n \to \infty$ where

$$d_{ij}^{(n)} = M(\Delta_{ij}^{(n)})^2 - E\{M(\Delta_{ij}^{(n)})^2 \mid \mathscr{F}_{z_{ij}^{(n)}}^1\}. \tag{3.2}$$

The proof is very similar to the proof of the final part of the proof of Theorem 1.9 of [1]. It is given here in detail in order to point out that the proof goes over without the requirement that M be a strong martingale. For $i' > i$

$$E(d_{ij}d_{i'j'} \mid \mathscr{F}^1_{z_{i'j'}}) = d_{ij}E(d_{i'j'} \mid \mathscr{F}^1_{z_{i',j'}}) = 0.$$

Therefore,

$$E\left(\sum d_{ij}\right)^2 = E\sum_{ij} d_{ij}^2 + E\sum_{i,j,j';j'>j} d_{ij}d_{ij'}. \tag{3.3}$$

$E\sum d_{ij}^2$ converges to zero exactly as in [1] (the five lines following equation 1.11). Considering the second term of (3.2)

$$\begin{aligned} \left|E\left(\sum_{i,j,j';j'>j} d_{ij}d_{ij'}\right)\right| &= \left|E\sum M(\Delta_{ij})^2 d_{ij'}\right| \\ &\le E\left(\sum_{ij} M(\Delta_{ij})^2 \cdot \sum_{j'>j} M(\Delta_{ij'})^2\right) \\ &\quad + E\left(\sum_{ij} M(\Delta_{ij})^2 \sum_{j'>j} E(M(\Delta_{ij'})^2 \mid \mathscr{F}^1_{z_{i,j'}})\right). \end{aligned} \tag{3.4}$$

Consider the first term and let $\delta_{i,j} = \cup_{j'>j} \Delta_{ij'}$

$$\begin{aligned} &E\left(\sum_{ij} M(\Delta_{ij})^2 \sum_{j'>j} M(\Delta_{ij'})^2\right) \\ &\quad = E\left(\sum_{ij} M(\Delta_{ij})^2 \cdot \sum_{j'>j} E(M(\Delta_{ij'})^2 \mid \mathscr{F}^2_{i,j+1})\right) \\ &\quad = E\left(\sum_{ij} M(\Delta_{ij})^2 \cdot E(M(\delta_{ij})^2 \mid \mathscr{F}^2_{i,j+1})\right) \\ &\quad = E\left(\sum_{ij} M(\Delta_{ij})^2 M(\delta_{ij})^2\right) \\ &\quad \le E\left(\sum_{ij} M(\Delta_{ij})^2 \sup_{i,j} M(\delta_{ij})^2\right) \\ &\quad \le E^{1/2}\left(\sum_{ij} M(\Delta_{ij})^2\right)^2 \cdot E^{1/2}(\sup_{i,j} M(\delta_{ij})^4). \end{aligned}$$

The first expectation is bounded by $c\ E\ M_z^4$ by Burkholder's inequality [6], the second expectation converges to zero since $E \sup M(\delta_{ij})^4 \le E \sup_\zeta M_\zeta^4 \le c\ E\ M_z^4$ and M_ζ is continuous. Therefore, the first term of (3.4) converges to zero. By very similar arguments (and since conditional expectations with respect to $\mathscr{F}^1_\zeta$ and $\mathscr{F}^2_\zeta$ commute), it follows that the second term of (3.4) converges to zero. Therefore (3.3) converges to zero.

(c) Let, now

$$d_{ij}^{(n)} = E(M(\Delta_{ij}^{(n)})^2 \mid \mathscr{F}_{z_{ij}^{(n)}}) - E(M(\Delta_{ij}^{(n)})^2 \mid \mathscr{F}^1_{z_{ij}^{(n)}}). \tag{3.5}$$

In order to complete the proof, it remains to be shown that $E(\sum_{ij} d_{ij}^{(n)})^2 \to 0$ as $n \to \infty$. The proof is the same as the proof of the final part of Theorem 1.9 of [1] and very similar to the proof of part (b) above. It is, therefore, omitted.

In the following corollary, $\langle M\rangle^i$, $i = 1, 2$, will denote the unique $\mathscr{F}_z^i$ predictable process such that $M^2 - \langle M\rangle^i$ is an i-martingale (cf. Proposition 1.8 of [1]). An adapted one-martingale M_z is said to be *proper* ([10]) if for all s, the expectation of the variation of $M_{s,t}$ in the t direction is finite in every finite t interval.

COROLLARY. *Under the assumptions of Proposition* 3.1,

$$\sum_{i,j,j';j'\neq j} M(\Delta_{ij} \cap R_z) M(\Delta_{ij'} \cap R_z) \tag{3.6}$$

converges in probability to a proper two-martingale B_z and $B_z = \langle M\rangle_z^1 - \langle M\rangle_z$. Similarly

$$\sum_{i,j,i';i'\neq i} M(\Delta_{ij} \cap R_z) M(\Delta_{i'j} \cap R_z) \tag{3.6a}$$

converges in probability to a proper one-martingale C_z and $C_z = \langle M\rangle_z^2 - \langle M\rangle_z$. Furthermore,

$$\sum_{i,j,j';j'\neq j} E(M(\Delta_{ij}^{(n)} \cap R_z) M(\Delta_{ij'}^{(n)} \cap R_z) \mid \mathscr{F}^1_{z_{ij}^{(n)}}) \to {}_P B_z \tag{3.7}$$

and

(3.7a) $$\textstyle\sum_{i,j,i';i'\neq i} E(M(\Delta_{ij}^{(n)} \cap R_z)M(\Delta_{i'j}^{(n)} \cap R_z)|\, \mathscr{F}^2_{z_{ij}^{(n)}}) \to {}_P C_z.$$

PROOF. Let $z_o = (s_o, t_o)$. Consider the one-parameter martingale $(M_{u,t_o}, \mathscr{F}^1_u, 0 \leq u \leq s_o)$ then

$$\textstyle\sum_i (M_{s_{i+1},t_o} - M_{s_i,t_o})^2$$

converges in probability to $\langle M \rangle^1_{z_o}$ as the sequence of partitions of $[0, s_o]$ becomes arbitrary fine. However,

$$\textstyle\sum_i (M_{s_{i+1},t_o} - M_{s_i,t_o})^2 = \sum_{i,j} M^2(\Delta_{ij} \cap R_{z_o})^2 + \sum_{i,j,j';j\neq j'} M(\Delta_{ij} \cap R_{z_o})M(\Delta_{ij'} \cap R_{z_o}).$$

Therefore by Proposition 3.1, the last term converges in probability to a limit B_z and $B_z = \langle M \rangle^1_z - \langle M \rangle_z$. Since $M^2 - \langle M \rangle^1$ and $M^2 - \langle M \rangle$ are weak martingales, B is also weak martingale. Since $\langle M \rangle^1$ and $\langle M \rangle$ are of integrable variation in the s direction for every t, so is B, therefore B is a proper two-martingale [10], Equation (3.7) and (3.7a) follow by similar arguments.

4. Martingales of path independent variation. Let $M \in \mathscr{M}^2_c$, M is said to be of path independent variation if the quadratic variation of M, as a one-parameter martingale, along every increasing path depends on the initial and end points of the path only ([9], [1]). Since M is a two-parameter martingale if and only if M is a one-parameter martingale along every increasing path [9] it follows immediately that M is of path independent variation if and only if there exists an order increasing predictable function A such that $M_z^2 - A_z$ is a martingale. Let $\langle M \rangle^i_z$, $i = 1, 2$, be the process such that $M_z^2 - \langle M \rangle^i_z$ is an i-martingale ([1]), *then M is of path independent variation if and only if* $\langle M \rangle^1_z = \langle M \rangle^2_z$, $z \in \mathbb{R}^2_+$ or (since $M^2 = 2 \int M\partial_i M + \langle M \rangle^i$):

$$\int_{H_z} M_\zeta \partial_1 M_\zeta = \int_{V_z} M_\zeta \partial_2 M_\zeta$$

where $z = (s, t)$ and $H_z(V_z)$ denotes the horizontal (vertical)line connecting (s, t) with $(0, t)$ (with (s, o)).

Theorem 1.9 of [1] states that if $M \in \mathscr{M}^2_{cc}$ and M is a strong martingale then M is of path independent variation.

The corollary to Proposition 3.1 leads to the following additional characterizations of path independent variation:

PROPOSITION 4.1 *Let $M \in \mathscr{M}^2_{cc}$. Then the following are equivalent.*

(a) *M is of path independent variation.*

(b) *$\langle M \rangle^1_z = \langle M \rangle^2_z$ for all z in $\mathbb{R}^2_+$ (and then $\langle M \rangle^1_z = \langle M \rangle_z$).*

(c) *For every arbitrary fine sequence of partitions, for all $z \in \mathbb{R}^2_+$ as $n \to \infty$:*

(4.1) $$\textstyle\sum_{i,j,j';j'\neq j} M(\Delta_{ij}^{(n)} \cap R_z)M(\Delta_{ij'}^{(n)} \cap R_z) \to_P 0$$

and

(4.2) $$\textstyle\sum_{i,j,i';i'\neq i} M(\Delta_{ij}^{(n)} \cap R_z)M(\Delta_{i'j}^{(n)} \cap R_z) \to_P 0.$$

(c)

(4.3) $$\textstyle\sum_{i,j,j';j'\neq j} E(M(\Delta_{ij}^{(n)} \cap R_z)M(\Delta_{ij'}^{(n)} \cap R_z)|\, \mathscr{F}^1_{z_{ij}^{(n)}}) \to {}_P 0$$

and

(4.4) $$\textstyle\sum_{i,j,i';i'\neq i} E(M(\Delta_{ij}^{(n)} \cap R_z)M(\Delta_{i'j}^{(n)} \cap R_z)|\, \mathscr{F}^2_{z_{ij}^{(n)}}) \to {}_P 0.$$

PROOF. The equivalence between (a) and (b) has been shown in the beginning of this section. Now, if M is of path independent variation, then $\langle M\rangle_z^1 = \langle M\rangle_z^2$ and, therefore,

$$\langle M\rangle_z - \langle M\rangle_z^1 = \langle M\rangle_z - \langle M\rangle_z^2.$$

Since the left-hand side is a continuous proper two-martingale and the right-hand side is a continuous proper one-martingale, each side of the above equation vanishes and (4.1) and (4.2) follow by the corollary to Proposition (3.1). Conversely, if (4.1) and (4.2) are satisfied then by the corollary to Proposition 3.1, $\langle M\rangle^1 = \langle M\rangle^2$ and therefore M is of path independent variation. The equivalence between (a) and (c) follows by similar arguments.

PROPOSITION 4.2. *Let $M^{(n)}$ be a sequence of martingales of path independent variation, assume that $M^{(n)}$ converge in quadratic mean to a martingale M_z ($\sup_z E\,|M_z^{(n)} - M_z|^2 \to 0$) then M is also of path independent variation.*

PROOF. By the Cairoli-Doob inequality for two-parameter martingales (Theorem 1.2 and Proposition 1.4 of [1]) it follows that M is sample continuous. By the Burkholder inequality for one-parameter martingales $\langle M^{(n)}\rangle^i \to \langle M\rangle^i$, $i = 1, 2$, in L_1. Therefore $\langle M\rangle_z^1 = \langle M\rangle_z^2$ and M is of path independent variation.

5. Martingales with orthogonal increments. Let $M_z \in \mathcal{M}_c^2$. Consider the rectangle $[z_1, z_2]$, $z_1 << z_2$, $z_1 = s_1, t_1$; $z_2 = s_2, t_2$ and

$$m_t = M_{s_2,t} - M_{s_1,t}$$

$$n_s = M_{s,t_2} - M_{s,t_1}.$$

Then $(m_t, \mathcal{F}_t^2, t_1 \le t \le t_2)$ is a continuous one-parameter martingale and so is $(n_s, \mathcal{F}_s^1, s_1 \le s \le s_2)$.

DEFINITION. $M \in \mathcal{M}_c^2$ is said to be a martingale with orthogonal increments if

$$\langle m\rangle_{t_2} - \langle m\rangle_{t_1} = \langle n\rangle_{s_2} - \langle n\rangle_{s_1}$$

for all $z_1 << z_2$ in $\mathbb{R}_+^2$.

PROPOSITION 5.1. *Let $M \in \mathcal{M}_{cc}^2$ then the following are equivalent*

(a) *M is a martingale with orthogonal increments.*

(b) *M is of path independent variation and so are the martingales $Y_z = M(R_z \cap (\alpha, \infty)$ for all $\alpha \in R_+^2$.*

(c) *For every $A = (z_1, z_2]$, $z_1 << z_2 << \infty$, and every arbitrary fine sequence of partitions*

$$\textstyle\sum_{i,j,j';j\neq j'} M(\Delta_{ij} \cap A)M(\Delta_{ij'} \cap A) \to_P 0$$

and

$$\textstyle\sum_{i,j,i';i'\neq i} M(\Delta_{ij} \cap A)M(\Delta_{i'j} \cap A) \to_P 0.$$

(d) *Let $M_z(\omega) = M_s(t, \omega)$ and consider the collection of one-parameter martingales $(M_s(t, \omega), \mathcal{F}_t^2)$ with s as the parameter of the collection then, $s_2 > s_1$*

$$\langle M_{s_1}, (M_{s_2} - M_{s_1})\rangle_t = 0$$

and similarly, setting $M_z(\omega) = M_t(s, \omega)$, $t_2 > t_1$:

$$\langle M_{t_1}, (M_{t_2} - M_{t_1})\rangle_s = 0.$$

(e) *With the same notation as in* (d):

$$M_{s_1}(t, \omega),\ (M_{s_2}(t, \omega) - M_{s_1}(t, \omega))$$

are orthogonal one-parameter martingales, that is,

$$(M_{s_1}(t, \omega) \cdot (M_{s_2}(t, \omega) - M_{s_1}(t, \omega)), \mathscr{F}_t^2)$$

is a martingale and similarly

$$M_{t_1}(s, \omega),\ (M_{t_2}(s, \omega) - M_{t_1}(t, \omega))$$

are orthogonal one-parameter martingales.

(f) *Let* $D_1 = (z_1, z_1']$, $D_2(z_2, z_2']$, $z_i = (s_i, t_i)$ $i = 1, 2$. *Assume that* $D_1 \cap D_2 = \phi$ *and let* $s_o = \min(s_1, s_2)$, $t_o = \min(t_1, t_2)$ *then*

$$E\{M(D_1)M(D_2) | \mathscr{F}_{s_o}^1 = 0$$

and

$$E\{M(D_1)M(D_2) | \mathscr{F}_{t_o}^2\} = 0.$$

(g) *For any rectangle* $D = (z_1, z_2]$

$$E((M)^2(D) | \mathscr{F}_{z_1}^i) = E((M(D))^2 | \mathscr{F}_{z_1}^i) \qquad i = 1, 2.$$

Remarks. (1). The definition of martingales with orthogonal increments and characterizations (b) or (c) of Proposition 5.1 indicate the relation of martingales with orthogonal increments to martingales of path independent variation. On the other hand, (d) to (g) indicate the relation of martingales with orthogonal increments to strong martingales. It follows immediately from (g) and by a direct calculation (or by Proposition 1.7 of [1]), that every strong martingale belonging to M_{cc}^2 is a martingale with orthogonal increments.

(2). When certain results requiring M to be a strong martingale are examined it turns out that the proof depends on the assumption that M is strong only via (f) of Proposition 5.1 (with either $s_1 = s_2$ *or* $t_1 = t_2$) and in these cases the assumption that M is strong can be replaced by the assumption that M is of orthogonal increments. In particular, the stochastic integral of the second type $\int\int \psi(\zeta, \zeta')\, dM_\zeta\, dM_{\zeta'}$ was defined in [1] under the assumptions that M is a right continuous, strong martingale and $EM_{z_o}^4 < \infty$. Going through the details of the proofs (Proposition 2.4 and Theorem 2.5 of [1]), we notice that the assumption that M is strong is used only via property (f) of the last proposition. It, therefore, follows that *the assumption that M is a strong martingale can be replaced by the assumption that M is orthogonal increments without changing the results of Theorem* 2.5 [1] *regarding* $\int\int \psi\, dM\, dM$.

Proof. Let $\alpha = (s_1, t_1)$, let $t > t_1$, $s > s_1$ and let $Y_{(s,t)} = M(R_{s,t} \cap (\alpha, \infty)) = M_{(s,t)} + M_{(s_1,t_1)} - M_{(s,t_1)} - M_{(s_1,t)}$, if Y is of path independent variation for every α then M satisfies the definition of a martingale with orthogonal increments and therefore (b) implies (a). From part (b) of Proposition 4.1 it follows that (a) implies (b). The equivalence between (b) and (c) follows directly from part (c) of Proposition 4.1. Turning to (d), let $A_1 = ((s_1, 0), (s_2, t_1)]$ and $A_2 = ((s_1, t_1), (s_2, t_2)]$, A_1, A_2, and $A_1 \cup A_2$ are rectangles and if M is a martingale with orthogonal increments, then (c) holds for $A = A_1$, for $A = A_2$ and also for $A = A_1 \cup A_2$ and therefore (c) implies (d) and (e). Conversely, if (d) (or (e)) are satisfied then equations (4.3) and (4.4) are satisfied for M_z and for $M(R_z \cap (\alpha, \infty))$ and therefore M is of orthogonal increments. The equivalence between (d) and (f) follows directly from part (d) of Proposition 4.1. Setting $D_1 = A_1$ and $D_2 = A_2$ (where A_1 and A_2 are the rectangles defined in the proof of the equivalence between (c) and (d)) then (g) follows directly from (f) and conversely, setting $D = A_1 \cup A_2$, then it follows by a straightforward calculation that (g) implies (e).

PROPOSITION 5.2. *If $M^{(n)}$ is a sequence of martingales with orthogonal increments and* $\sup_z E\,|M_z^{(n)} - M_z|^2 \to 0$ *as $n \to \infty$ where $M \in \mathscr{M}_{cc}^2$, then M is a martingale with orthogonal increments.*

PROOF. By part 2 of Proposition 5.1, $M_z^{(n)}$ and $\int_{R_z} I_{(a,\infty)}(\zeta)\, dM_z^{(n)}$ are path independent variation. The convergence of $M^{(n)}$ in quadratic mean to M implies the same type of convergence of $M^{(n)}(R_z \cap (a, \infty))$ to $M(R_z \cap (a, \infty))$. It follows therefore by Proposition 4.2 that M and $M(R_z \cap (a, \infty))$ are of path independent variation and therefore by part 2 of Proposition 5.1 M is of orthogonal increments.

PROPOSITION 5.3. *Let $M \in \mathscr{M}_{cc}^2$ be a martingale with orthogonal increments, if φ_z is predictable and $\int_{R_z} \varphi_\zeta\, dM_\zeta \in \mathscr{M}_{cc}^2$ then $\int \varphi\, dM$ is also of orthogonal increments.*

PROOF. Let $\{\Delta_{ij}\}$ be a partition of $\mathbb{R}_+^2$ and let φ be a simple function:

$$\varphi(z) = \textstyle\sum_{ij} I_{\Delta_{ij}}(z)\alpha_{ij}$$

where the α_{ij} are $\mathscr{F}_{z_{ij}}$-adapted and bounded random variables and $I_{\Delta_{ij}}(z)$ is the indicator function of $\Delta_{ij} = (z_{i,j}, z_{i+1,j+1}]$. In this case

$$\int_{R_z} \varphi_\zeta\, dM_\zeta = \textstyle\sum_{ij} \alpha_{ij} M(\Delta_{ij} \cap R_z)$$

and by (3c) of Proposition 5.1, $\int \varphi\, dM$ is also of orthogonal increments. The extension from the case where ϕ is a simple function to the general case follows from the previous proposition.

A characterization of the Wiener process. A well-known result of *P.* Lévy states that if M_t is a continuous (one-parameter) martingale and $M_t^2 - t$ is also a martingale then M_t is a Brownian motion. An analogous result was obtained by *E.* Wong for strong two-parameter martingales (unpublished). A similar result for martingales with orthogonal increments is the following.

PROPOSITION 5.4. *If M_z is a martingale with orthogonal increments and M_z^2 − area $(\zeta < z)$ is a martingale then M_z is a Wiener process.*

PROOF. Let Δ_{ij} be a finite partition of R_{z_0} and consider:

$$m_s^j = \textstyle\sum_i \alpha_{ij} M(\Delta_{ij} \cap R_{s,t_0})$$

where α_{ij} are nonrandom. Then, since M is a martingale with independent increments,

$$\langle m \rangle_s^j = \textstyle\sum_i \alpha_{ij}^2 \quad \text{area} \quad (\Delta_{ij} \cap R_{s,t_0})$$

$$\langle m^j m^k \rangle_s = 0, \qquad j \neq k.$$

By the differentiation formula for one-parameter processes:

$$E\,\mathrm{Exp}\{i \textstyle\sum_j m_{s_0}^j + \tfrac{1}{2} \sum_j \langle m \rangle_{s_0}^j\} = 1$$

and since $\langle m \rangle$ is nonrandom in this case:

$$E\,\mathrm{Exp}\; i \textstyle\sum_{ij} \alpha_{ij} M(\Delta_{ij}) = \mathrm{Exp} - \tfrac{1}{2} \sum_{ij} \alpha_{ij}^2 \cdot \mathrm{area}(\Delta_{ij})$$

and therefore M_z is a Wiener process.

A random function will be said to be predictably absolutely continuous with respect to the Lebesgue measure in the plane if the corresponding Radon-Nykodym derivative is predictable. The following result is a two-parameter counterpart of a result of Doob ([3], page 449).

PROPOSITION 5.5. *If $M \in \mathscr{M}^2_{cc}$ is a martingale with orthogonal increments, and $\langle M\rangle_z$ is predictably absolutely continuous with respect to the Lebesgue measure in the plane then there exists a Brownian motion W^b_z on the original probability space or adjoined to it such that $M_z = \int_{R_z} f_\zeta \, dW^b_\zeta$.*

PROOF. By the assumptions on $\langle M\rangle$; $\langle M\rangle_z = \int_{R_z} \varphi_\zeta \, d\zeta$ where φ is predictable. Set

$$f_\zeta = \begin{cases} 1/\sqrt{\varphi_\zeta} & \text{if } \varphi_\zeta \neq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Let W^a be an adjoined Brownian motion and set

$$W^b_z = \int_{R_z} f_\zeta \, dM_\zeta + \int_{R_z} I(f_\zeta = 0) \, dW^a_\zeta.$$

Then $M_z = \int \varphi \, dW^b$. Moreover, by Proposition 5.3 W^b_z is of direction independent variation and, by Proposition 5.4, W^b_z is a Brownian motion, which completes the proof.

PROPOSITION 5.6. *If (M_z, F_z) is a martingale with orthogonal increments, and $\mathscr{F}_z$ are the σ-fields generated by the two-parameter Wiener process then there exists a Wiener process W^b_z such that $M_z = \int_{R_z} f_\zeta \, dW^b_\zeta$.*

PROOF. By the previous proposition it suffices to show that in the present case M_z is absolutely continuous with respect to the the Lebesgue measure in the plane. Since

$$\langle M\rangle_z = \int_{R_z} \theta^2_\zeta \, d\zeta + \int_{R_z \times R_z}\int \psi^2(\zeta, \zeta') \, d\zeta \, d\zeta'$$

it suffices to consider the second term only:

$$\int_{R_z \times R_z}\int \psi^2(\zeta, \zeta') \, d\zeta \, d\zeta' = \int_{(\sigma',\tau)\in R_z} d\sigma' \, d\tau \int_{(\sigma,\tau')\in R_z} \psi^2(\sigma, \tau; \sigma', \tau') \, d\sigma \, d\tau'.$$

Since $\psi(\zeta,\zeta') = 0$ for $\zeta < \zeta'$ and for $\zeta' < \zeta$, the above integrals become

$$\int_{(\sigma',\tau)\in R_z} d\sigma' \, d\tau \int_{(\sigma,\tau')<(\sigma',\tau)} \psi^2(\sigma, \tau; \sigma', \tau') \, d\sigma \, d\tau'$$

$$= \int_{R_z} \left(\int_{(\alpha,\beta)<\zeta} \psi^2(\alpha, \tau; \sigma, \beta) \, d\alpha\beta \right) d\zeta; \quad \zeta = (\sigma, \tau).$$

Therefore $\langle M\rangle_z$ is absolutely continuous with respect to the Lebesgue measure, the R-N derivative is $\mathscr{F}_z$ adapted and the result follows from Proposition 5.5.

REMARK. An earlier version of this paper contained the following result:

PROPOSITION. *Let W_z be a Wiener process and F_z the σ-fields generated by W_z. Let (M_z, F_z) be a two-parameter martingale, assume that M_z is a Wiener process on its own σ-fields. Then M_z is a strong martingale on F_z, that is: in the representation of M_z as an F_z martingale,*

$$M_z = \int \phi \, dW + \int\int \psi \, dW \, dW', \qquad \textit{we have} \quad |\phi_\zeta| = 1 \quad \textit{and} \quad \psi = 0.$$

It follows immediately from this result that if (M_z, F_z) is a martingale with orthogonal increments, and F_z are the σ-fields generated by the two-parameter Wiener process then M is a strong martingale. However, Professor D. Nualart pointed out to us a serious error in the proof. A correct proof of this result will be published by Prof. Nualart (cf. also part (4) of Theorem 2.1 of [8] for a closely related result).

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