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ON EDGEWORTH EXPANSIONS IN BANACH SPACES[1]

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In this paper we define a generalization of Edgeworth expansions for the expectation of functions of normalized sums of i.i.d. Banach space valued random vectors. These expansions are valid up to $0(n^{-(s-2)/2})$ for functions with $3(s-2)$ bounded Fréchet derivatives and random vectors with finite s^{th} absolute moment.

1. Introduction and Notations. Let $(\Omega, \mathscr{A}, P)$ denote a probability space and let X_i, $i \in \mathbb{N}$ denote a sequence of i.i.d. random vectors with values in a separable Banach space E and distribution Q. Assume $EX_i = 0$. Let $f \colon E \to \mathbb{R}$ denote a measurable function. Assume that there exist an E-valued random vector Y with mean zero and Gaussian distribution Φ fulfilling condition (3.3) (which is related to the condition that Φ has the same covariance operator as Q, i.e.

$$\int x^*(x)\, y^*(x)(Q-\Phi)\,(dx) = 0 \qquad \text{for all} \qquad x^*, y^* \in E^*,$$

for all x^*, $y^* \in E^*$, where E^* denotes the dual space of E). Assume that the symmetric function

$$\underset{\sim}{\eta} \to \varphi(\underset{\sim}{\eta}) := \int f(y + \eta_1 x_1 + \cdots + \eta_q x_q)\Phi\,(dy) Q^q\,(d\underset{\sim}{x}) \tag{1.1}$$

is defined on $\underset{\sim}{\eta} \in [0, n^{-1/2}]^q$ for some integers $q \geq 3(s-2)/2$, $s \geq 3$, and has uniformly bounded derivatives up to the order $3(s-2)$.

Let Q_n denote the distribution of $n^{-1/2}(X_1 + \cdots + X_n)$. Then we define the *formal Edgeworth expansion of order* $s-3$ by

$$\int f\, dQ_n \sim \textstyle\sum_{i=0}^{s-3} n^{-i/2} P_i(D_\eta)\varphi(\underset{\sim}{\eta})|_{\underset{\sim}{\eta}=0} \tag{1.2}$$

where $P_i(D_\eta)$ denotes a sum of partial derivatives with respect to $\underset{\sim}{\eta}$ with orders between i and $3i$. In particular, we have

$$P_0(D_\eta) = 1, \qquad P_1(D_\eta) = \frac{1}{6}\frac{\partial^3}{\partial \eta_1^3} \quad \text{and}$$

$$P_2(D_\eta) = \frac{1}{24}\left(\frac{\partial^4}{\partial \eta_1^4} - 3\,\frac{\partial^4}{\partial \eta_1^2 \partial \eta_2^2}\right) + \frac{1}{72}\,\frac{\partial^6}{\partial \eta_1^3 \partial \eta_2^3}, \qquad \text{see (2.2).}$$

For $E = \mathbb{R}^k$, this yields the usual Edgeworth expansion, provided that differentiation and integration (with respect to Q and the Lebesgue measure) can be interchanged.

In this case, the expansion (1.2) is valid up to a remainder term of order $0(n^{-(s-2)/2})$ if f is the indicator function of an interval in $\mathbb{R}^k$ and Q fulfills Cramér's condition

$$\limsup_{\|t\| \to \infty} |\hat{Q}(t)| < 1 \tag{1.3}$$

where $\hat{Q}(t)$ denotes the characteristic function of Q at $t \in \mathbb{R}^k$. See R. N. Bhattacharya and

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R. Ranga Rao (1976, Corollary 20.5, page 215). A. Bikjalis (1968) proved under the assumption that Q has an absolutely continuous component (which is stronger than (1.3)) that this assertion even holds for every measurable bounded function f.

On the other hand (1.2) holds up to $0(n^{-(s-2)/2})$, if f has $s-2$ derivatives which are bounded by polynomials. See F. Götze and C. Hipp (1978).

For a separable Banach space E we prove with Theorem 3.1 that (1.2) holds with an error $0(n^{-(s-2)/2})$ if the functions

$$\varphi_i(\epsilon_1, \cdots, \epsilon_q \mid \eta_1, \cdots, \eta_q) := \int f(s^{(i)}_{\underline{\epsilon},\underline{\eta}}(\underline{x}, \underline{y}))Q^{n+q}\,(d\underline{x})\Phi^{n+q}\,(d\underline{y}), \tag{1.4}$$

where

$$\begin{aligned} s^{(i)}_{\underline{\epsilon},\underline{\eta}}(\underline{x}, \underline{y}) := n^{-1/2}&(x_1 + \cdots + x_i + y_{i+1} + \cdots + y_n) \\ &+ \epsilon_1 y_{n+1} + \cdots + \epsilon_q y_{n+q} \\ &+ \eta_1 x_{n+1} + \cdots + \eta_q x_{n+q}, \qquad 2q \geq 3(s-2), \end{aligned}$$

admit derivatives of sufficiently high order which are uniformly bounded for $0 \leq i \leq n$, $n \in \mathbb{N}$ and $\epsilon, \eta \in [0, n^{-1/2}]^q$. This condition is fulfilled if f has $3(s-2)$ bounded Fréchet derivatives. See Theorem 3.5. Theorem 3.1 and Theorem 3.5 extend a result of V. I. Paulauskas (1976, Theorem 4, page 393) who proved a Berry-Esséen bound (i.e. the case $s=3$ in Theorem 3.5).

The expansion (1.2) does not apply directly to the distribution of functionals of i.i.d. sums in Banach spaces but it yields an expansion for c.f. of the following type

$$\int \exp(itg[n^{-1/2}(x_1 + \cdots + x_n)])\,Q^n\,(d\underline{x}), \qquad t \in \mathbb{R}, \tag{1.5}$$

where g admits sufficiently many Fréchet derivatives. In F. Götze (1979) we use the asymptotic expansion for (1.5), where g is a quadratic form, in order to derive expansions for the distribution of bivariate von Mises functionals.

2. Notations. Let i, j, k, n, q, r, s denote nonnegative integers and let α, β, γ denote tuples of nonnegative integers of length $q \geq 3(s-2)/2$. For a q-tuple $\alpha = (i_1, \cdots, i_q)$ let $|\alpha| := i_1 + \cdots + i_q$ and denote by D^α_ϵ resp. D^α_η the partial derivatives

$$\frac{\partial^{i_1}}{\partial\epsilon_1^{i_1}} \cdots \frac{\partial^{i_q}}{\partial\epsilon_q^{i_q}} \quad \text{resp.} \quad \frac{\partial^{i_1}}{\partial\eta_1^{i_1}} \cdots \frac{\partial^{i_q}}{\partial\eta_q^{i_1}}$$

of functions $g(\epsilon_1, \cdots, \epsilon_q \mid \eta_1, \cdots, \eta_q)$, symmetric in $\underline{\epsilon}$ and $\underline{\eta}$. For reasons of simplicity we shall write $g(\epsilon \mid \eta)$ resp. $g(\epsilon_1, \epsilon_2 \mid \eta)$ instead of $g(\epsilon, 0, \cdots, 0 \mid \eta, 0, \cdots, 0)$ resp. $g(\epsilon_1, \epsilon_2, 0, \cdots 0 \mid \eta, 0, \cdots, 0)$ etc. Furthermore, we shall use the abbreviation $\tau := n^{-1/2}$ in sections 4 and 5.

Let $u, d_2, d_3, \cdots, d_k$ denote formal variables. Define the r^{th} Edgeworth polynomial (with respect to the moment variables $d_2, \cdots, d_{r+2}$) by means of the formal power series expansion

$$\textstyle\sum_{r=0}^\infty P_r(u; d_2, \cdots, d_{r+2})t^r = \exp(u\varphi(t)) \tag{2.2}$$

where $\varphi(t) = t^{-2}[\log(1 + \sum_{j=2}^\infty d_j j!^{-1} t^j) - d_2 t^2/2]$. See A. Bikjalis (1967), translated (1973, page 153/5).

Replacing the monomials $d_{i_1} d_{i_2} \cdots d_{i_k}$ in $P_r(u; \cdot)$ by the partial derivatives $D_\eta^{(i_1, i_2, \cdots, i_k)}$ applied to *symmetric* functions of the variables $\eta_1, \cdots, \eta_k$ we obtain the differential operator $P_r(u; D_\eta)$. (Notice that this correspondence is well defined since the partial derivative $D_\eta^{(i_1,i_2,\cdots,i_k)}$ of a symmetric function does not depend on the order in the tuple $(i_1, \cdots, i_k)$ so that algebraic identities in the commutable variables $d_2, \cdots, d_s, \cdots$ imply identities for the corresponding partial derivatives).

Define $P_r(D_\eta) := P_r(1; D_\eta)$. Finally, let $\int f(\underline{x}) Q^k(d\underline{x})$, $k \in N$, denote the integral $\int f(x_1, \cdots, x_k) Q(dx_1) \cdots Q(dx_k)$.

3. Results. The following is the main result.

THEOREM 3.1. *Let $s \geq 3$. Assume that the functions $\varphi_i(\underline{\epsilon} \mid \underline{\eta})$, $i = 0, \cdots, n$ defined in (1.4) are differentiable such that*

$$R_n := \sup | D_\epsilon^{(j,0,\cdots,0)} D_\eta^{(i_1,\cdots,i_q)} \varphi_i(\epsilon, 0, \cdots, 0 \mid \eta, 0, \cdots, 0)| \tag{3.2}$$

exists, where the supremum is taken over $0 \leq \epsilon, \eta \leq n^{-1/2}$ with $\epsilon \cdot \eta = 0$, $i = 0, \cdots, n$ and all integers $0 \leq j, i_1, \cdots, i_q \leq s$ such that $s \leq j + i_1 + \cdots + i_q \leq 3(s-2)$. Furthermore assume that

$$D_\epsilon^\alpha D_\eta^\beta \varphi_i(0 \mid 0) = D_\eta^{\alpha+\beta} \varphi_i(0 \mid 0) \tag{3.3}$$

for $\alpha = (i_1, \cdots, i_q)$, $0 \leq i_k \leq 2$, $\beta = (j_1, \cdots, j_q)$, $i_k + j_k \leq s$, for $1 \leq k \leq q$, $0 \leq i \leq n$, $|\alpha| + |\beta| \leq 3(s-2)$ and $\alpha + \beta := (i_1 + j_1, \cdots, i_q + j_q)$. Then

$$\left| \int f dQ_n - \sum_{r=0}^{s-2} n^{-r/2} P_r(D_\eta) \varphi(\underline{\eta}) \right|_{\underline{\eta}=0} \leq c(s) R_n n^{-(s-2)/2}.$$

(For the definition of $P_r(D_\eta)$, see (1.1) − (1.2) and the notations.)

DEFINITION 3.4. A Banach space E is called of *type 2* if there exists a constant $d > 0$ such that for every $m \in N$ and every set of vectors $x_1, \cdots, x_m \in E$

$$2^{-m} \sum{}^* \| \pm x_1 \pm \cdots \pm x_m \|^2 \leq d(\| x_1 \|^2 + \cdots + \| x_m \|^2)$$

where $\sum^*$ denotes summation over the 2^n combinations of + and − signs.

See J. Hoffmann-Jørgensen and G. Pisier (1976). Obviously every Hilbert space is of type 2 and it is well known that all function spaces of the form L^p, $2 \leq p < \infty$ are of type 2.

THEOREM 3.5. *Let $s \geq 3$. Assume that E is a separable Banach space of type 2. Furthermore, assume that*

(i) $$m_s := E \| X \|^s \text{ is finite and } EX = 0.$$

(ii) *The function $f: E \to R$ has $3(s-2)$ Fréchet derivatives (in the sense of J. Dieudonné (1969, page 149)) which are uniformly bounded on E with respect to the supremum norm of multilinear functionals. Then*

$$\int f \, dQ_n = \sum_{r=0}^{s-3} n^{-r/2} P_r(D_\eta) \varphi(\underline{\eta})|_{\underline{\eta}=0} + 0(n^{-(s-2)/2}) \tag{3.6}$$

where $2q \geq 3(s-2)$ and $\varphi(\underline{\eta})$ is defined as in (1.1) with respect to a limiting Gaussian distribution Φ with the same covariance operator as Q.

REMARK 3.7.

(i) The remainder term in (3.6) can be estimated by

$$c(s) \sup\{m_s^{i/s} \| D^i f(x) \| : x \in E, s \leq i \leq 3(s-2)\} \, n^{-(s-2)/2},$$

where $\| D^i f(x) \|$ denotes the supremum norm of the i-linear functional $D^i f(x)$.

(ii) For weaker differentiability assumptions and generalizations to the non i.i.d. case see F. Götze (1978).

EXAMPLE 3.8. Let E denote a separable Hilbert space with norm $\|x\| = \langle x, x\rangle^{1/2}$. Assume $E\|X\|^4 < \infty$. Let c_1, c_2, δ denote positive constants. Then we have uniformly in $a \in E$ with $\|a\| \le c_1$ and r with $\delta \le r \le c_2$

$$\int \exp(-r\|x+a\|^2)Q_n(dx)$$

$$= \chi(r)\exp(-\tfrac{1}{2}\langle B_r a, a\rangle)\left[1 + n^{-1/2}\int (3\langle B_r a, x\rangle\langle B_r x, x\rangle - \langle B_r a, x\rangle^3)Q(dx) + 0(n^{-1})\right], \tag{3.9}$$

where

$$\chi(r) = \textstyle\prod_{k=1}^{\infty}(1+2r\lambda_k)^{-1/2}, \qquad \langle B_r a, a\rangle = 2r\sum_{k=1}^{\infty}\langle a, e_k\rangle^2(1+2r\lambda_k)^{-1}$$

and $\{\lambda_k; e_k\}_{k\in N}$ denotes the set of eigenvalues and corresponding orthonormal eigenvectors of the convariance operator of Q. The expansion (3.9) can be easily extended up to the order $0(n^{-(s-2)/2})$ if condition (i) of Theorem 3.5 is fulfilled. The proof of these results is referred to Section 5.

4. Lemmas. In the following we shall consider the class of functions

$$\mathscr{F}_{n,\beta} := \{D_\eta^\beta \varphi_i(\epsilon \mid \underline{\eta}) : i = 0, \cdots, n\}$$

where φ_i are defined in (1.4). Furthermore, we shall use the abbreviation (i), (i, j) etc. for the q-tuples $(i, 0, \cdots, 0)$, $(i, j, 0, \cdots, 0)$.

LEMMA 4.1. *Let $g \in \mathscr{F}_{n,\beta}$. Then*

(i) $D_\epsilon^\alpha g(0 \mid 0) = 0$ *if one of the components of α is odd.*

(ii) *Let $c_j = (2j)!j!^{-1}2^{-j}$, $j \in N$ and $\alpha_{jk} = (2j, 2(k-j), j_3, \cdots, j_q)$. The numbers $b_j := c_j^{-1}c_{k-j}^{-1}(D_\epsilon^{\alpha_{jk}}g)(0\mid 0)$, $0 \le j \le k$, are equal.*

PROOF. (i) This follows from the symmetry of Φ with respect to the map $x \to -x$, $x \in E$.

(ii) Since Φ has a stable law of index 2, $p^2 + q^2 = 2$, $p, q \in R$ implies

$$(D_\epsilon^{\alpha 00}g)(\epsilon, \epsilon \mid 0) - (D_\epsilon^{\alpha 00}g)(p\epsilon, q\epsilon \mid 0) = 0, \; 0 \le \epsilon \le n^{-1/2}.$$

The $(2k)^{\text{th}}$ derivative with respect to ϵ at $\epsilon = 0$ of this identity yields

$$\textstyle\sum_{j=0}^{k} b_j\binom{k}{j}(1-(p^2)^j(2-p^2)^{k-j}) = 0. \tag{4.2}$$

Since (4.2) is a polynomial in p^2 which vanishes identically for $0 \le p^2 \le 2$ we obtain the relations

$$\textstyle\sum_{j=0}^{k-r} b_{j+r}\binom{k-r}{j}(-1)^j = 0, \quad 0 \le r \le k-1 \qquad \text{and} \qquad \sum_{j=0}^{k} b_j\binom{k}{j} - b_k 2^k = 0$$

which imply part (ii) of Lemma (4.1). As an application of Lemma 4.1 we have

REMARK 4.3. Let $g \in \mathscr{F}_{n,\beta}$.

(i) $$\textstyle\sum_{j=0}^{k}(D_\epsilon^{(2(k-j),2j)}g)(0\mid 0)\binom{k}{j}(-1)^j = 0$$

(ii) $$D_\epsilon^{(2i,2j)}g(0 \mid 0) = c_i c_j D_\epsilon^{\underline{2}} g(0 \mid 0)$$

where $\underline{2}$ denotes the $(i+j)$-tuple $(2, 2, \cdots, 2)$. Notice that condition (3.3) implies $D_\epsilon^{\underline{2}} g(0 \mid 0) = D_\eta^{\underline{2}} g(0 \mid 0)$.

LEMMA 4.4. *Let $p \in N_0$ and denote by β_i, $2 \le 2i \le p$, the $(i+1)$-tuple $(p-2i, 2, \cdots, 2)$. Let $\beta_0 = 0$. For $g \in \mathscr{F}_{n,\beta}$ define*

$$R_p(D_\eta)g := \textstyle\sum_{i=0}^{[p/2]} i!^{-1} 2^{-i} (p-2i)!^{-1} (-1)^i D_\eta^{\beta_i} g.$$

Notice that $R_0(D_\eta) = 1$ and $R_1(D_\eta) = R_2(D_\eta) = 0$. Then

$$\left| \textstyle\sum_{i=0}^{s-1} \tau^i R_i(D_\eta) g(0 \mid 0) - \sum_{i=0}^{s-3} \tau^i P_i(\tau^2; D_\eta) g(0 \mid 0) \right|$$
$$\le c\tau^s \sup\{ | D_\eta^\beta g(0 \mid 0) | : \beta \in \{0, \cdots, s-1\}^q, s \le |\beta| \le 3(s-3)\}$$

for $s \ge 4$ $\quad (\tau = n^{-1/2})$.

PROOF. Assign to each partial derivative $D^{(j_1, \cdots, j_r)}$ the monomial $d_{j_1} \cdots d_{j_r}$, $j_i \in N$, in the formal variables $d_2, \cdots, d_s$. (Compare the remarks following (2.2).) Then $R_i(D_\eta)$ can be defined by means of a formal power series

$$\textstyle\sum_{i=0}^{\infty} t^i R_i(d_2, \cdots, d_{i+1}) = (1 + \sum_{i=2}^{\infty} t^i i!^{-1} d_i) \exp(-t^2 d_2/2).$$

The definition (2.2) of the Edgeworth polynomials $P_i(u; d_2, \cdots, d_{i+2})$, $u \in \mathbb{R}$, implies

$$\textstyle\sum_{i=0}^{\infty} t^i R_i(d_2, \cdots, d_{i+1}) = \sum_{i=0}^{\infty} t^i P_i(t^2; d_2, \cdots, d_{i+2}).$$

Define the polynomials $P_{i,j}(d_2, \cdots, d_{i+2})$, $0 \le j \le i$ by

$$P_i(t^2; d_2, \cdots, d_{i+2}) := \textstyle\sum_{j=0}^{i} t^{2j} P_{i,j}(d_2, \cdots, d_{i+2}).$$

Hence

$$R_i(\cdot) = \textstyle\sum_{r+2j=i, j \ge 1} P_{r,j}(\cdot), \quad i \ge 1$$

implies

$$\textstyle\sum_{i=0}^{s-3} P_i(t^2; \cdot) t^i + \sum_{i=0}^{s-1} R_i(\cdot) t^i = t^s \sum_{(i,j) \in A_s} P_{i,j}(\cdot) t^{2j+i-s}$$

where

$$A_s := \{(i,j) \in N_0^2 : i + 2j \ge s, 0 \le i \le s-3, 1 \le j \le i\}$$

and $s \ge 4$, which proves Lemma 4.4, since $P_{i,j}(D_\eta)$ is a sum of partial derivatives of order $i + 2j$, where $s \le i + 2j \le 3(s-3)$. Notice that we have $\sum_{k=1}^{q} (i_k - 2) = i$, $i_k \ge 2$, $k = 1, \cdots, q$, for any partial derivative $D_\eta^{(i_1, i_2, \cdots, i_q)}$ which occurs in $P_i(t^2; D_\eta)$.

LEMMA 4.5. *Let $g_m := D_\eta^\beta \varphi_m \in \mathscr{F}_{0,\beta}$* for $0 \le m \le n-1$. Let $r \ge 3$. Then

$$| g_{m+1}(0 \mid 0) - g_m(0 \mid 0) - \textstyle\sum_{i=1}^{r-3} \tau^i P_i(\tau^2; D_\eta) g_m(0 \mid 0) |$$
$$\le c\tau^r \sup\{ |(D_\eta^\gamma g_m)(0 \mid \eta\tau)| + |(D_\epsilon^{(i)} D_\eta^\gamma g_{m+1})(\epsilon\tau \mid 0)| :$$
$$\gamma \in \{0, \cdots, r\}^q, r \le i + |\gamma| \le 3(r-2), 0 \le \epsilon, \eta \le 1, 0 \le i \le r\}.$$

PROOF. By Remark 4.3 we have

$$g_{m+1}(0 \mid 0) = \textstyle\sum_{0 \le 2k \le r-1} \tau^{2k} 2^{-k} k!^{-1} \sum_{i+j=k} \binom{k}{i} (-1)^i c_i^{-1} c_j^{-1} D_\epsilon^{(2i, 2j)} g_{m+1}(0 \mid 0)$$
$$= \textstyle\sum_{0 \le 2i \le k-1} \tau^{2i} (2i)!^{-1} (-1)^i [\sum_{0 \le 2j \le r-2i-1} \tau^{2j} (2j)!^{-1} D_\epsilon^{(2i, 2j)} g_{m+1}(0 \mid 0)].$$

By Taylor expansion, the sum over j within the brackets is equal to $(D_\epsilon^{(2i)}g_{m+1})(\tau \mid 0) + 0(\tau^{r-2i})$ (use Lemma 4.1(i)). By definition $(D_\epsilon^{(2i)}g_{m+1})(\tau \mid 0) = (D_\epsilon^{(2i)}g_m)(0 \mid \tau)$. Hence.

$$g_{m+1}(0 \mid 0) = \sum_{0\le 2i\le r-1} \tau^{2i}(2i)!^{-1}(-1)^i(D_\epsilon^{(2i)}g_m)(0 \mid \tau) + 0(\tau^r).$$

Expanding $v \to (D_\epsilon^{(2i)}g_m)(0 \mid \tau v)$, $v \in \mathbb{R}$ in a Taylor series up to a remainder term of order $0(\tau^r)$ we obtain

$$g_{m+1}(0 \mid 0) = \sum_{p=0}^{r-1} \tau^p R_p(D_\eta)g_m(0 \mid 0) + 0(\tau^r).$$

An inspection of the various remainder terms and the application of Lemma 4.4 proves Lemma 4.5.

LEMMA 4.6. Assume that E is a separable Banach space of type 2 (see Definition 3.4). Assume $E\|X\|^2 < \infty$. If Φ has the same covariance operation Q then

$$\int w(x, x)(Q - \Phi)\,(dx) = 0 \tag{4.7}$$

for every continuous bilinear form w: $E^2 \to \mathbb{R}$.

PROOF. For $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ let

$$\chi_t(x) \begin{cases} 0 & x \ge t+1 \\ t - x + 1 & x \in [t, t+1] \\ 1 & x \le t. \end{cases}$$

By M. X. Fernique (1970, page 1698) $\int \|x\|^p\Phi\,(dx)$ exists for every $p > 0$, especially for $p = 2$. Using $\int w(\cdot,\cdot)\,dQ = \int w(\cdot,\cdot)\,dQ_n$ and $|w(x, y)| \le \|w\|\,\|x\|\,\|y\|$ we obtain

$$\left|\int w(\cdot,\cdot)\,d(Q - \Phi)\right| \le \left|\int w(\cdot,\cdot)\chi_t(\|\cdot\|)\,d(Q_n - \Phi)\right| + \|w\|\left[\int \|x\|^2 1_{\{\|x\|>t\}}(Q_n + \Phi)\,dx)\right]. \tag{4.8}$$

By the central limit theorem (see J. Hoffmann-Jørgensen and G. Pisier (1976, Theorem 3.6, page 597)) the first term on the r.h.s. of (4.8) tends to zero for $n \to \infty$. Since the sequence Q_n is relatively compact, we can apply a result of A. de Acosta and E. Giné (1979, Theorem 3.2(3), page 217) with $\Phi = 4(2 + x^2)$ which shows that the second term on the r.h.s. of (4.8) converges to zero for $t \to \infty$ *uniformly* in $n \ge 1$, thus proving (4.7).

5. Proof of the theorems.

Proof of Theorem 3.1. The proof runs by induction on the length of the expansion. We shall prove that for $r \ge 3$, $g_m(\underline{\epsilon} \mid \underline{\eta}) := D_\eta^\beta\varphi_m(\underline{\epsilon} \mid \underline{\eta})$ *and* $0 \le m \le n$

$$|g_m(0 \mid 0) - \sum_{j=0}^{r-3} \tau^j P_j(m\tau^2; D_\eta)g_0(0 \mid 0)| \le c(r)\tau^{r-2}w_{m,n}(r) \tag{5.1}$$

where $w_{m,n}(r) := \sup\{|(D_\epsilon^{(i)}D_\eta^\gamma\varphi_j)(\epsilon\tau \mid \eta\tau)| : \epsilon\cdot\eta = 0, 0 \le i \le r,$

$$\gamma \in \{0, \cdots, r\}^q, r \le |\gamma| + i \le 3(r-2), 0 \le \epsilon, \eta \le 1, j = 0, \cdots, m\}.$$

The assertion (5.1) implies Theorem 3.1 with $m = n$, $r = s$ and $\beta = (0, \cdots, 0)$.

The induction is based on the obvious identity

$$g_m(0 \mid 0) - g_0(0 \mid 0) = \sum_{j=1}^{m} (g_j - g_{j-1})(0 \mid 0). \tag{5.2}$$

We start with $r = 3$. By Lemma 4.5 (with $r = 3$) the r.h.s. of (5.2) can be estimated by

$$c(3)\tau^3 m w_{m,n}(3) \le c(3)\tau w_{m,n}(3). \tag{5.3}$$

Assume that (5.1) has been proved for $r = 3, \cdots, s-1$. We apply Lemma 4.5 to each term of the sum (5.2), expanding up to terms of order $0(\tau^s)$. Hence

$$|(g_m - g_0 - \textstyle\sum_{j=1}^m \sum_{i=1}^{s-3} \tau^i P_i(\tau^2; D_\eta)g_{j-1})(0 \mid 0)| \le \tau^s m\bar{R}_n \tag{5.4}$$

for $0 \le m \le n$, where the remainder term is the supremum over the remainder terms of Lemma 4.5. Application of the inductive assumption (5.1) to $P_i(\tau^2; D_\eta)g_{j-1}$ with $m = j-1$, $r = s-i$, $i \ge 1$, yields

$$\begin{aligned} &| P_i(\tau^2; D_\eta)g_{j-1}(0 \mid 0) - \textstyle\sum_{k=0}^{s-3-i} \tau^k P_i(\tau^2; D_\eta)P_k(\tau^2(j-1); D_\eta)\, g_0(0 \mid 0)| \\ &\quad \le \tau^{s-1}\sup\{|(D_\epsilon^{(l)} D_\eta^\gamma P_{i,k}(D_\eta)g_p)(\epsilon \mid \eta)|\, \tau^{2k-2}\colon \gamma \in \{0, \cdots, s-i\}^q, \\ &\quad\quad l \le k \le i, p = 0, \cdots, j-1, s-i \le |\gamma| + l \le 3(s-i-2), \\ &\quad\quad 0 \le l \le s-i, 0 \le \epsilon, \eta \le \tau, \epsilon\eta = 0\}. \end{aligned} \tag{5.5}$$

Since $P_{i,r}(D_\eta)$ has degree $i + 2r$, we obtain from (5.4) and (5.5) for $0 \le m \le n$

$$g_m(0 \mid 0) = g_0(0 \mid 0) + \textstyle\sum_{j=1}^m \sum_{i=1}^{s-3} \sum_{k=0}^{s-3-i} \tau^i P_i(\tau^2; D_\eta)\, \tau^k P_k(\tau^2(j-1); D_\eta) g_0(0 \mid 0) + \bar{\bar{R}}_n m\tau^s$$

where $|\bar{\bar{R}}_n| \le cw_{m,n}(s)$ and

$$P_k(\tau^2(j-1); D_\eta) = \begin{cases} 0 & \text{if } j = 0, k > 0 \\ 1 & \text{if } j = 1, k = 0. \end{cases}$$

Using the relation

$$\exp(j\varphi(t)) = \exp((j-1)\varphi(t))\exp(\varphi(t))$$
$$\varphi(t) = \log(1 + \textstyle\sum_{k=2}^\infty d_k k!^{-1} t^k) - d_2 t^2/2$$

of formal power series we obtain

$$\textstyle\sum_{k+i=r} P_k(\tau^2(j-1); D_\eta)P_i(\tau^2; D_\eta) = P_r(\tau^2 j;\ D_\eta)$$

for $j \ge 1$, $r \ge 0$. This together with (5.6) implies

$$g_m(0 \mid 0) = g_0(0 \mid 0) + \textstyle\sum_{r=1}^{s-3} \tau^r \sum_{j=1}^m [P_r(j\tau^2; D_\eta) - P_r((j-1)\tau^2; D_\eta)]g_0(0 \mid 0) + \tau^s m\bar{\bar{R}}_n$$

thus concluding the proof of the induction step.

PROOF OF THEOREM 3.5. We show that the conditions of Theorem 3.1 are fulfilled. By Theorem 3.5 in J. Hoffmann-Jørgensen and G. Pisier (1976, page 596) there exists a mean zero Gaussian p-measure Φ with the same covariance functional as Q. Notice that

$$(x_1, \cdots, x_k) \to \frac{\partial^{r_1}}{\partial\epsilon_1^{r_1}} \cdots \frac{\partial^{r_k}}{\partial\epsilon_k^{r_k}} f(y + \epsilon_1 x_1 + \cdots + \epsilon_k x_k)|_{\epsilon_1 = \cdots \epsilon_k = 0}$$

is a multilinear functional bounded by $\omega_{r_1+\cdots+r_k}(f)\|x_1\|^{r_1} \cdots \|x_k\|^{r_k}$, where $\omega_r(f) := \sup\{\|D^r f(x)\|\colon x \in E\}$.

Furthermore, we have $\int \|x\|^P\, \Phi\, (dx) \le c(p)m_2^{p/2}$, $p > 0$. See F. Götze (1979, Lemma 3.30).

Hence,

$$| D_\epsilon^{(j_1,\cdots,j_q)} D_\eta^{(i_1,\cdots,i_q)} \varphi_i(\epsilon \mid \eta)| \le \omega_r(f)m_{i_1} \cdots m_{i_q} m_2^{j_1/2} \cdots m_2^{j_q/2} \tag{5.7}$$

where

$$i_1 + \cdots + i_q + j_1 + \cdots + j_q = r \quad \text{and} \quad i_1, \cdots, i_q \le s$$

follows by interchanging differentiation and integration with the help of Lebesgue's Theorem of Dominated Convergence and Fubini's Theorem. Similar arguments together with Lemma 4.6 prove condition 3.2.

PROOF OF REMARK 3.7(i). The assertion follows from $m_i \leq m_s^{i/s}$, $i \leq s$ together with (5.7) and Theorem (3.1).

PROOF OF EXAMPLE 3.8. The functions $f_{r,a}(x) := \exp(-r \| x + a \|^2)$ fulfill the condition (ii) of Theorem 3.5 uniformly in r and a, for every $s \geq 3$. It remains to show that

$$\int \exp(-r \| x + a \|^2)\Phi\,(dx) = \hat{\chi}(r)\exp\left(-\frac{1}{2}\langle B_r a, a\rangle\right).$$

This relation follows from the corresponding formula for the sequence of subspaces $\mathbb{R}e_1 + \cdots + \mathbb{R}e_k \subset E$, $k \in \mathbb{N}$, see G. V. Martinov (1975, page 790, (18)) and the Theorem of Monotone Convergence.

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