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# THE SPECTRAL DECOMPOSITION OF A DIFFUSION HITTING TIME

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All first hitting times for a one-dimensional diffusion belong to the *Bondesson class* of infinitely divisible distributions on $[0, \infty]$. A distribution in this class can be conveniently represented in terms of its *canonical measure*. In this paper we establish a link between the canonical measure of a hitting time and the *spectral measure* of the differential generator of the diffusion. In particular, it is shown that the derivative of the canonical measure with respect to natural scale (as a function of the point being hit) equals the spectral measure of the differential generator on a restricted interval. The canonical measure is then calculated for several examples arising from the Bessel diffusion process, including the inverse of a gamma variate and the Hartman-Watson mixing distribution.

**1. Introduction.** Consider a nonsingular diffusion on an interval $[r_0, r_1]$ and let $\tau_{ab}$ denote the first time the diffusion hits $b$ starting at $a$, $r_0 < a < b < r_1$. In Kent (1980) and Bondesson (1981) it was shown that $\tau_{ab}$ lies in the *Bondesson class* of infinitely divisible distributions on $[0, \infty]$. Properties of this class of distributions are described in Section 2. In particular, it follows that associated with $\tau_{ab}$ is a *canonical measure* $Q_{ab}(d\sigma)$ on $(0, \infty)$ which can be used to give an integral representation for the moment generating function (m.g.f.) of $\tau_{ab}$. The precise form of the measure $Q_{ab}(d\sigma)$ is derived in Section 3.

There is also an infinitesimal version of a hitting time distribution as $a \uparrow b$ and we shall denote the canonical measure corresponding to this limiting distribution by $\Omega_b(d\sigma)$. It turns out (see Section 4) that $\Omega_b(d\sigma)$ is related to $Q_{ab}(d\sigma)$ by

$$\partial Q_{ab}(d\sigma)/\partial s(b) = \Omega_b(d\sigma)$$

where $s(\cdot)$ is the natural scale of the diffusion.

The canonical measure $\Omega_b(d\sigma)$ is important because it also plays a role in the *spectral theory* associated with the diffusion. The following theorem forms the core of this paper and is proved in Section 5.

THEOREM 1.1. *The canonical measure $\Omega_b(d\sigma)$ can be identified with a* **spectral measure** *corresponding to the differential generator of the diffusion on the interval up to $b$, with a Dirichlet boundary condition at $b$.* □

Recall that a complete description of a spectral measure depends upon a choice of normalization for the eigenfunctions. To make the above theorem valid we choose the eigenfunctions to have constant derivative at $b$ (see Section 5). The above theorem provides a "probabilistic motivation" for this normalization. Moreover, since the *support* of a spectral measure does not depend on the normalization, there really is a significant connection between probability theory and spectral theory here (although there seems to be no simple intuitive way to "explain" this connection).

The important special cases of *discrete* and *continuous* spectral measures are covered in Sections 6–7. Sections 8–10 describe some bounds on the magnitude of diffusion hitting

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time densities near $t = 0$, the effect on the canonical measure from augmenting the killing measure, and some implications for the spectral measure when the hitting times lie in the *Thorin* class of distributions. Consideration of only those diffusion hitting times which are finite with probability one can be achieved by a conditioning argument (Section 11). More general *additive functionals* up to hitting times can be analyzed by means of *time changes* (Section 12).

Two examples based on the Bessel diffusion process are examined in Sections 13–14. In particular it is shown that the Hartman-Watson mixing distribution can be given a meaningful probabilistic interpretation as a diffusion hitting time.

**2. The Bondesson class of distributions.** An important class of infinitely divisible distributions on $[0, \infty]$ was introduced by Bondesson (1981) who used the somewhat awkward name *generalized convolutions of mixtures of exponential distributions.* We shall use the term *Bondesson class* and the following characterization is central to our purposes.

DEFINITION 2.1. Say that a distribution function $F(t)$ on $[0, \infty]$ comes from the *Bondesson class* if its moment generating function (m.g.f.) $G(\lambda) = \int_{[0,\infty]} e^{\lambda t} F(dt)$, $\lambda < 0$, can be analytically continued to the cut complex plane $C\backslash[0, \infty)$ and given the representation

$$G(\lambda) = \exp\{-\alpha + \delta\lambda + \int_{(0,\infty)} \left(\frac{1}{\sigma - \lambda} - \frac{1}{\sigma}\right) Q(d\sigma)\}, \qquad \lambda \in C\backslash[0, \infty), \tag{2.1}$$

where $\alpha \geq 0$, $\delta \geq 0$ and $Q(d\sigma)$ is a non-negative measure on $(0, \infty)$ satisfying

$$\int_{(0,1)} \sigma^{-1} Q(d\sigma) < \infty \qquad \int_{[1,\infty)} \sigma^{-2} Q(d\sigma) < \infty. \tag{2.2}$$

(For simplicity, we exclude the trivial distribution concentrated at $\infty$.) Further, we shall call $\alpha$, $\delta$ and $Q(d\sigma)$ the *deficiency*, *infimum* and *canonical measure*, respectively, of the distribution.

The deficiency and infimum are so named because

$$e^{-\alpha} = F(\infty -), \qquad \delta = \sup\{t : F(t) = 0\}.$$

They and the canonical measure can be found by the inversion formulae

$$e^{-\alpha} = G(0-), \qquad \delta = \lim_{\lambda\downarrow-\infty}\{\log G(\lambda)\}/\lambda, \tag{2.3}$$

and

$$Q(\sigma) = \int_0^\sigma Q(d\rho) = \frac{1}{\pi} \lim_{\eta\downarrow 0} \int_0^\sigma \arg G(\rho + i\eta)\, d\rho \tag{2.4}$$

at continuity points $\sigma$ of $Q(\sigma)$.

It is easy to see any function $G(\lambda)$ of the form (2.1) is an m.g.f. of a distribution on $[0, \infty]$, so that *every* function of the form (2.1) is the m.g.f. of a distribution from the Bondesson class. Further, it is clear that if $G_1(\lambda)$ and $G_2(\lambda)$ are m.g.f.'s from the Bondesson class, so are $G_1(\lambda)G_2(\lambda)$ and $G_1(\lambda)^c$, where $c > 0$. The Bondesson class is also closed under weak limits as the following "continuity" theorem shows.

THEOREM 2.1. (Bondesson, 1981) *Let $F_n$, $n = 1, 2, \cdots$, be a sequence of distributions in the Bondesson class with* m.g.f.'s $G_n(\lambda)$.

(i) *If $F_n$ converges weakly to a distribution function $F$ with* m.g.f. $G(\lambda)$, *then $F$ lies in the Bondesson class and $G_n(\lambda) \to G(\lambda)$ for all $\lambda \in C\backslash[0, \infty)$.*

(ii) *If for some function $G(\lambda)$, $G_n(\lambda) \to G(\lambda)$ for $\lambda \in \Lambda$ where $\Lambda$ is a set of points in $C\backslash[0, \infty)$ containing a cluster point, then $G(\lambda)$ can be analytically continued to $C\backslash[0, \infty)$ as the* m.g.f. *of a distribution function $F$ in the Bondesson class, and $F_n$ converges weakly to $F$.*

(iii) *If* (i) (*or equivalently* (ii)) *holds, then in an obvious notation*

$$Q(d\sigma) = \lim_{n\to\infty} Q_n(d\sigma)$$

$$\delta = \lim_{K\to\infty} \lim_{n\to\infty}\left\{\delta_n + \int_{\sigma\geq K} \sigma^{-2}Q_n(d\sigma)\right\}, \qquad \alpha = \lim_{K\to 0} \lim_{n\to\infty}\left\{\alpha_n + \int_{\sigma\leq K} \sigma^{-1}Q_n(d\sigma)\right\}.$$

REMARKS. (1) Part (iii) of the above theorem says that the deficiencies and canonical measures behave "continuously" as $n \to \infty$. Further, if the canonical measures can be uniformly bounded, $Q_n(d\sigma) \leq P(d\sigma)$, for some measure $P(d\sigma)$ satisfying the regularity conditions (2.2), then the deficiencies and infima also behave "continuously."
(2) The convergence of canonical measures in the above theorem is also "weak convergence;" that is, $Q_n(d\sigma) \to Q(d\sigma)$ means $Q_n(\sigma) \to Q(\sigma)$ at all continuity points $\sigma$ of $Q(\sigma)$. With this notion of convergence it is also possible to consider the continuity or even differentiability of a collection of canonical measures $\{Q_x(d\sigma)\}$ with respect to a real-valued parameter $x$.

**3. Diffusion hitting times.** A diffusion with possible killing on an interval $[r_0, r_1]$ can be described in terms of a generalized second order differential operator A. Alternatively, the diffusion can be described in terms of its *speed measure* $m(dx)$, *natural scale* $s(dx)$ and *killing measure* $k(dx)$. (For background information on diffusions, see for example Ito and McKean (1965) or Mandl (1968).)

There are four types of boundary for a diffuson: *natural, exit, entrance* and *regular*; and regular boundaries subdivide into *absorbing* and *reflecting.* In the case of a regular reflecting boundary $r_i$, we must also assign speed and killing measure to the boundary point, $0 \leq m\{r_i\}$, $k\{r_i\} < \infty$. We shall adopt the convention that $m\{r_i\} = k\{r_i\} = 0$ for all other types of boundary.

To facilitate discussion of right- and left-hand derivatives with respect to natural scale on $(r_0, r_1)$, we shall adopt throughout the paper the notation

$$f^+(x) = d^+f(x)/ds(x), \qquad f^-(x) = d^-f(x)/ds(x), \qquad r_0 < x < r_1$$

when these limits exist. Interpret such functions and their derivatives at boundary points by

$$f(r_i) = \lim_{x\to r_i} f(x), \qquad f^+(r_i) = \lim_{x\to r_i} f^+(x), \qquad f^-(r_i) = \lim_{x\to r_i} f^-(x).$$

Also associated with the diffusion are *boundary conditions* $B_0$ and $B_1$ at $r_0$ and $r_1$. For suitable functions $f(x)$, the boundary condition at $r_0$ takes the form $B_0(f) = 0$, where $B_0(f) = f(r_0)$ for natural, absorbing regular and exit boundaries; $B_0(f) = f^+(r_0)$ for an entrance boundary; and $B_0(f) = f^+(r_0) - [k\{r_0\} - \lambda m\{r_0\}]f(r_0)$ for a regular reflecting boundary.

Note that the description of the diffusion is unaltered if the speed measure, natural scale and killing measure are rescaled by

$$m(dx) \to cm(dx), \qquad s(dx) \to c^{-1}s(dx), \qquad k(dx) \to ck(dx) \tag{3.1}$$

where $c > 0$.

Let $\tau_{ab}$ denote the first time the diffusion hits $b$, starting at $a$. Denote its m.g.f. by $G_{ab}(\lambda) = E\{\exp(\lambda\tau_{ab})\}$. To find $G_{ab}(\lambda)$, where $r_0 < a < b < r_1$ and $\lambda < 0$, it is sufficient to solve the generalized differential equation

$$A\psi + \lambda\psi = 0 \tag{3.2}$$

together with the boundary condition

$$B_0(\psi) = 0 \tag{3.3}$$

for a continuous function $\psi(x) = \psi(x, \lambda)$, which is not identically zero. Then

$$G_{ab}(\lambda) = \psi(a, \lambda)/\psi(b, \lambda). \tag{3.4}$$

Equation (3.2) can also be written in the equivalent integral form

$$\psi^+(x) - \psi^+(y) = \int_{(y,x]} \psi(w)\{k(dw) - \lambda m(dw)\} \qquad r_0 < y < x < r_1. \tag{3.5}$$

From (3.5) we see that $\psi^+(x)$ $(\psi^-(x))$ is right- (left-) continuous in $x$ and that

$$\psi^+(x) - \psi^-(x) = \psi(x)[k\{x\} - \lambda m\{x\}], \qquad r_0 < x < r_1. \tag{3.6}$$

In particular, if $\psi(x) = 0$ or if $x$ is *not* an atom of the speed or killing measure, then $\psi^+(x) = \psi^-(x)$.

Since for $\lambda < 0$, $\psi$ satisfies the boundary condition (3.3), we can let $y \downarrow r_0$ in (3.5) to get

$$\psi^+(x) - \psi^+(r_0) = \int_{(r_0,x]} \psi(w)\{k(dw) - \lambda m(dw)\}, \qquad r_0 < x < r_1, \lambda < 0. \tag{3.7}$$

Note here that $\psi^+(r_0) = \psi^-(r_0)$.

In Kent (1980) it was shown that $G_{ab}(\lambda)$ lies in the Bondesson class. Thus $G_{ab}(\lambda)$ can be analytically continued as an analytic *nonzero* function of $\lambda \in C\backslash[0, \infty)$. Now for $\lambda < 0$, $\psi(x, \lambda)$ is determined by (3.2)–(3.3) only up to multiplication by an arbitrary nonzero function of $\lambda$. Hence without loss of generality, we may suppose that for each $x \in (r_0, r_1)$,

$$\psi(x, \lambda) \quad \text{is an analytic nonzero function of } \lambda \in C\backslash[0, \infty), \tag{3.8}$$

$$\psi(x, \lambda) \quad \text{is real-valued and positive for } \lambda < 0, \tag{3.9}$$

and

$$\psi(x, 0-) \in (0, \infty). \tag{3.10}$$

(Further, an analytic continuation argument in (3.5) shows that any function which satisfies (3.2) for $\lambda < 0$ and (3.8)–(3.10) for $\lambda \in C\backslash[0, \infty)$ must in fact satisfy (3.2) for *all* $\lambda \in C\backslash[0, \infty)$, including the limiting case $\lambda \uparrow 0$.)

We shall suppose throughout the paper that any solution $\psi(x, \lambda)$ of (3.2)–(3.3) for $\lambda < 0$ is also chosen to satisfy (3.8)–(3.10). Then $\arg \psi = \operatorname{Im} \log \psi$ is well-defined (with $\arg \psi(x, \lambda) = 0$ for $\lambda < 0$) and the inversion formula (2.4) for the canonical measure $Q_{ab}(d\sigma)$ of $\tau_{ab}$ takes the form

$$Q_{ab}(\sigma) = \frac{1}{\pi} \lim_{\eta \downarrow 0} \int_0^\sigma [\arg \psi(a, \rho + i\eta) - \arg \psi(b, \rho + i\eta)]\, d\rho \tag{3.11}$$

at continuity points $\sigma$ of $Q_{ab}(\cdot)$. Further $\tau_{ab}$ has infimum $\delta = 0$ because $P(\tau_{ab} < \varepsilon) > 0$ for all $\varepsilon > 0$ (Ito and McKean, 1965, page 157), and the deficiency is given by $e^{-\alpha} = \psi(a, 0-)/\psi(b, 0-)$.

**4. Infinitesimal hitting times.** By letting $c \uparrow b$ in $\tau_{cb}$ and scaling appropriately, we can obtain further distributions in the Bondesson class. Define

$$\begin{aligned} H_b^L(\lambda) &= \lim_{c \uparrow b} \exp\{[s(b) - s(c)]^{-1}[\log \psi(c) - \log \psi(b)]\} \\ &= \exp\{-\psi^-(b, \lambda)/\psi(b, \lambda)\} \end{aligned} \tag{4.1}$$

to be the m.g.f. of the *left-infinitesimal left-hitting time* at $b$. It roughly represents the distribution of time taken to move from $b - 0$ to $b$, convoluted with itself infinitely often. Similarly, we can define the *right-infinitesimal left-hitting time* at $b$ by

$$H_b^R(\lambda) = \exp\{-\psi^+(b, \lambda)/\psi(b, \lambda)\}, \tag{4.2}$$

to describe the time taken to move from $b$ to $b + 0$.

Both of these infinitesimal hitting times are termed *left-hitting times* because the relevant boundary condition is at the *left*-hand endpoint $r_0$. Note that although these infinitesimal hitting times describe well-defined distributions on $[0, \infty]$, they have no meaning in terms of the sample paths of the diffusion. (However, see Section 15.)

By Theorem 2.1, it follows that $H_b^L(\lambda)$ and $H_b^R(\lambda)$ are m.g.f.'s from the Bondesson class. From (3.6) we see that the deficiencies and infima are related by

$$\alpha_b^R = \alpha_b^L + k\{x\}, \qquad \delta_b^R = \delta_b^L + m\{x\}, \tag{4.3}$$

in accordance with our intuition about the effect of speed and killing measure, and further, $H_b^L(\lambda)$ and $H_b^R(\lambda)$ have the *same* canonical measure, which we shall denoted by $\Omega_b(d\sigma)$. (In fact, if $b$ is *not* an atom of the speed or killing measure, $H_b^L(\lambda)$ and $H_b^R(\lambda)$ are identical.)

From (2.3), (3.6) and (3.7), it is not difficult to show that for all types of boundary

$$\delta_b^L = 0,$$

thus simplifying (4.3). Further, the inversion formula for the canonical measure $\Omega_b(d\sigma)$ takes the form

$$\Omega_b(\sigma) = \frac{1}{\pi} \lim_{\eta \downarrow 0} \int_0^\sigma - \operatorname{Im}[\psi^-(b, \rho + i\eta)/\psi(b, \rho + i\eta)]\, d\rho \tag{4.4}$$

at continuity points $\sigma$ of $\Omega_b(\sigma)$.

Theorem 2.1 (iii) applied to (4.1) and (4.2) respectively shows that $\Omega_b(d\sigma) = \partial^- Q_{ab}(d\sigma)/\partial s(b) = \partial^+ Q_{ab}(d\sigma)/\partial s(b)$ for all $r_0 < a < b < r_1$. Further, since $H_b^L(\lambda)$ ($H_b^R(\lambda)$) is left-(right-) continuous in $b$, so is $\Omega_b(d\sigma)$. Putting these one-sided pieces together yields the following result.

THEOREM 4.1. *The* m.g.f.'*s* $H_b^L(\lambda)$ *and* $H_b^R(\lambda)$ *(for the left- and right-infinitesimal left-hitting times, respectively) have the same canonical measure* $\Omega_b(d\sigma)$, *which is related to* $Q_{ab}(d\sigma)$ *by*

$$\partial Q_{ab}(d\sigma)/\partial s(b) = \Omega_b(d\sigma), \tag{4.5}$$

*or in integral form,*

$$Q_{ab}(d\sigma) = \int_a^b \Omega_x(d\sigma) s(dx). \tag{4.6}$$

*Further,* $\Omega_b(d\sigma)$ *is continuous in b.*

In the next section we shall show that the *canonical measure* $\Omega_b(d\sigma)$ can be identified with an appropriate *spectral measure.*

Note that if we are dealing with *right*-hitting times instead of *left*-hitting times, then certain obvious changes must be made to the formulae given here.

**5. Eigenfunction expansions.** Consider the differential operator A on the restricted interval $[r_0, b]$, $r_0 < b < r_1$. Let $\phi(x, \lambda)$ be the solution of (3.2) on $(r_0, b]$ which at $b$ satisifies the Dirichlet boundary condition

$$\phi(b, \lambda) = 0, \tag{5.1}$$

and the initial condition

$$\phi^-(b, \lambda) = 1. \tag{5.2}$$

For existence of $\phi(x, \lambda)$, see Theorem 4.1 in Kent (1980), and in particular note that $\phi(x, \lambda)$ is an entire function of $\lambda$ for each fixed $x \in (r_0, b]$ (including $x = r_0$ if $r_0$ is regular).

In the context of the following theorem, the *canonical measure* $\Omega_b(d\sigma)$ can also be interpreted as a *spectral measure* for the differential operator A on the interval $(r_0, b]$, with the diffusion boundary condition $B_0$ at $r_0$ and a Dirichlet boundary condition at $b$.

THEOREM 5.1. *There is a Hilbert space isomorphism* $\Phi: L^2([r_0, b), m(dx)) \to L^2((0, \infty), \Omega_b(d\sigma))$ *given by*

$$\Phi f(\sigma) = \int_{[r_0,b)} f(x)\phi(x, \sigma)m(dx), \qquad f \in L^2([r_0, b), m(dx))$$

$$\Phi^{-1}g(x) = \int_0^\infty g(\sigma)\phi(x, \sigma)\Omega_b(d\sigma), \qquad g \in L^2((0, \infty), \Omega_b(d\sigma)).$$

*All functions expressed as improper integrals are to be interpreted as limits in* $L^2$ *of proper integrals.*

PROOF. See, for example, Titchmarsh (1962, page 51 and also Chapter 6), whose arguments can be adapted to the present setting with little change. In particular, the canonical measure $\Omega_b(\cdot)$ in (4.4) can be identified with the spectral measure $\pi^{-1}k(\cdot)$ in Titchmarsh (1962, page 54, Equation (3.3.1)).

REMARKS. (1) Use of this theorem gives the eigenfunction expansion

$$f(x) = \Phi^{-1}(\Phi(f)), \qquad f \in L^2([r_0, b), m(dx)),$$

of $f(x)$ in terms of the initial value solutions $\phi(x, \sigma)$ as $\sigma$ varies through the support of $\Omega_b(d\sigma)$ (which is called the *spectrum*).

(2) If the spectrum is discrete, then the eigenfunction expansion for $f(x)$ becomes a *series* in terms of orthogonal functions. In particular, if $\lambda_0$ is an atom of the spectrum,

$$\int_{[r_0,b)} \phi^2(x, \lambda_0)m(dx) = [\Omega_b\{\lambda_0\}]^{-1}. \tag{5.3}$$

(3) The classification of $r_0$ as a *limit-point* or *limit-circle* type of boundary can be made (Elliott, 1955) but is irrelevant for our purposes.

(4) Eigenfunction expansions based on Theorem 5.1 can be used to express the probability transition density of the diffusion on $[r_0, r_1]$ (see Ito and McKean, 1965, pages 149–161).

(5) A change in the initial condition (5.2) would change the normalization of the eigenfunctions and hence would lead to a different definition of the spectral measure. The present normalization has been chosen in order to identify the spectral measure with the canonical measure $\Omega_b(\cdot)$ for the infinitesimal hitting times. However, note that if the natural scale is rescaled by a constant factor $c$ in (3.1), then both the canonical measure and spectral measure are altered by the same constant factor so that the identification between the two remains unchanged.

(6) If $b$ is given a reflecting regular boundary condition, then an expansion analogous to Theorem 5.1 can be constructed in terms of the appropriate initial value solution. However, it is not possible to identify the spectral measure with the canonical measure $\Omega_b(d\sigma)$ in this case.

**6. Discrete spectrum.** Suppose that $\psi(x, \lambda)$ can be chosen so that

(a) for all $x \in (r_0, r_1)$ $\psi(x, \lambda)$ is an *entire* function of $\lambda$, real-valued for $\lambda < 0$, and
(b) if $\psi(x_0, \lambda_0) = 0$ then $\psi^-(x_0, \lambda_0) \neq 0$.

This setting includes *all* diffusion hitting times for which $r_0$ is *not* a natural boundary, in which case we can take $\psi(x, \lambda) = u(x, \lambda)$, the *initial value solution at* $r_0$ given in Theorem 4.1 of Kent (1980).

In general note that (b) prohibits removable zeros because $\psi(x_0, \lambda_0) = 0$, $\psi^-(x_0, \lambda_0) = 0$ is equivalent to $\psi(x, \lambda_0) = 0$ for all $x$. Hence $\psi$ can have no zeros for $\lambda \in C\backslash[0, \infty)$ because for all $r_0 < a < b < r_1$, $\psi(a, \lambda)/\psi(b, \lambda)$ has no zeros or poles for these values of $\lambda$. Further, there cannot be a zero at $\lambda = 0$ because $\lim_{\lambda\uparrow 0}\{\psi(a, \lambda)/\psi(b, \lambda)\} \in (0, 1]$ for all $r_0 < a < b < r_1$.

Then the zeros of $\psi(x, \cdot)$ are simple and positive, and can be arranged in an infinite sequence increasing to $\infty$, $0 < \lambda_{1,x} < \lambda_{2,x} < \cdots$. (Adapt the argument in Titchmarsh, 1962, pages 53–54 to show the zeros are *simple*.) Clearly, $\lim_{\eta\downarrow 0} \arg \psi(x, \sigma + i\eta)$ is constant for $\sigma$ real, except at the points $\sigma = \lambda_{k,x}$, where there is a jump of $-\pi$. Thus the canonical measure has a density $Q_{ab}(\sigma) = q_{ab}(\sigma)\, d\sigma$ given by

$$(6.1) \qquad q_{ab}(\sigma) = \textstyle\sum_{k=1}^{\infty} I[\sigma > \lambda_{k,b}] - \sum_{k=1}^{\infty} I[\sigma > \lambda_{k,a}]$$

where $I[\cdot]$ is an indicator function. As in Theorem 5.1 of Kent (1980) we see that $\tau_{ab}$ can be expressed as an *infinite convolution of elementary mixtures of exponential distributions.*

If $r_0$ is *not* natural, then $\sum \lambda_{k,b}^{-1} < \infty$, but this condition need not be satisfied if $r_0$ *is* natural. (For an example, see Titchmarsh, 1962, pages 90–91, 144–145.)

Integrating (6.1) with respect to $\sigma$, and comparing with (4.5), (4.4) and (5.3) shows that $\Omega_b(d\sigma)$ is *discrete*, with mass at $\lambda_{k,b}$ given by

$$\begin{aligned} -\partial\lambda_{k,b}/\partial s(b) &= \psi^-(b, \lambda_{k,b})/\psi'(b, \lambda_{k,b}) \\ &= \left\{\int_{[r_0,b)} \phi^2(x, \lambda_{k,b}) m(dx)\right\}^{-1}. \end{aligned}$$

Here the prime denotes differentiation with respect to $\lambda$ and $\psi^+(x, \lambda_{k,b}) = \psi^-(x, \lambda_{k,b})$ at $x = b$.

**7. Continuous spectrum.** Suppose $\psi(x, \lambda)$, satisfying (3.8)–(3.10), can be chosen in such a way that it can be extended as a *continuous nonzero* function of $x \in (r_0, r_1)$ and $\lambda$ to the upper side of the cut $\lambda \geq 0$ as $\operatorname{Im} \lambda \downarrow 0$. Let $\lambda \geq 0$ denote a point on the upper side of the cut.

Then $\arg \psi(x, \lambda)$ is defined for $\lambda \geq 0$ and (3.11) takes the simple form

$$(7.1) \qquad q_{ab}(\sigma) = \arg \psi(a, \sigma) - \arg \psi(b, \sigma), \qquad 0 < \sigma < \infty,$$

where $Q_{ab}(\sigma) = q_{ab}(\sigma)\, d\sigma$. Further $q_{ab}(\sigma)$ is jointly continuous in $(a, b, \sigma)$ and $q_{ab}(0) = 0$.

It is easily checked that $\psi(x, \lambda)$ satisfies (3.5) throughout the region $\operatorname{Im} \lambda \geq 0$. In particular $\psi^+(x, \lambda)$ ($\psi^-(x, \lambda)$) exists and is jointly right-(left-) continuous in $x$ and continuous in $\lambda$ in the region $x \in (r_0, r_1)$, $\operatorname{Im} \lambda \geq 0$. Thus, $\Omega_b(d\sigma)$ has a density ($\Omega_b(d\sigma) = \omega_b(\sigma)\, d\sigma$) and (1.1) can be written in density form

$$\partial q_{ab}(\sigma)/\partial s(b) = \omega_b(\sigma).$$

Further, $\omega_b(\sigma)$ is jointly continuous in $b$ and $\sigma$, and $\omega_b(0) = 0$.

**8. Properties of diffusion hitting times near $t = 0$.** It is important to note that the class of diffusion hitting times does *not* consist of the *whole* Bondesson class because diffusion hitting times have an exponentially small probability of being small. Ray's estimate (Ito and McKean, 1965, page 134) tells us that for all $n \geq 1$,

$$(8.1) \qquad \lim_{t\downarrow 0} t^{-n} P(\tau_{ab} < t) = 0.$$

In the case of a diffusion generated by a classical differential operator, this estimate can be strengthened. Suppose

$$(8.2) \qquad A = \alpha(x)\, d^2/dx^2 + \beta(x)\, d/dx + \gamma(x)$$

where $\alpha(x) > 0$ is continuously differentiable, and where $\beta(x)$ and $\gamma(x) \leq 0$ are continuous on $(r_0, r_1)$. Transform the $x$-axis so that $\alpha(x) \equiv 1/2$, and suppose we wish to find the m.g.f. $G_{ab}(\lambda)$ of $\tau_{ab}$ where $r_0 < a < b < r_1$. Suppose $|\beta(x)| \leq M$ and $-\gamma(x) \leq N$, for $a^* \leq x \leq b$ where $r_0 < a^* < a$. Then by comparing this diffusion with the two Brownian motions governed by

$$A_1 = (1/2)\ d^2/dx^2 + M\ d/dx$$

reflected at $a^*$, and by

$$A_2 = (1/2)\ d^2/dx^2 - M\ d/dx - N$$

absorbed at $a^*$, respectively, it is not difficult to check that

$$\lim_{\lambda\downarrow-\infty}\{\log G_{ab}(\lambda)\}/\{-(-2\lambda)^{1/2}\} = b - a, \tag{8.3}$$

which implies

$$\lim_{t\downarrow 0}\{\exp(d/t)P(\tau_{ab} < t)\} = 0, \qquad 0 < d < 1/2(b-a)^2. \tag{8.4}$$

However, *limits* of diffusion hitting times need not satisfy these bounds. For example, the exponential distribution, which is too large near 0 for (8.1) to hold, can arise as a limit (Mandl, 1968, pages 102–106). On the other hand, the m.g.f. $G(\lambda) = I_{-i\sqrt{\lambda}}(a)/I_0(a)$ of Section 13 has asymptotic behaviour $\log G(\lambda) \sim -1/2(-\lambda)^{1/2}\log(-\lambda)$, so $G(\lambda)$ tends to 0 *more* quickly than (8.3) as $\lambda \downarrow -\infty$.

A concrete description of the *closure* of the class of diffusion hitting times remains an open question.

**9. Shift of spectrum.** Suppose the killing measure $k(dx)$ is augmented by a constant multiple of the speed measure, $k(dx) \to k(dx) + c\,m(dx)$. In (8.2) this transformation takes the form $\gamma(x) \to \gamma(x) - c$.

The effect of this transformation is clear from (3.2). If $G_{ab}^{(c)}(\lambda)$ denotes the hitting time m.g.f. for the transformed diffusion then $G_{ab}^{(c)}(\lambda) = G_{ab}(\lambda - c)$. Thus, the canonical and spectral measures are shifted to the right by an amount $c$,

$$Q_{ab}^{(c)}(\sigma) = Q_{ab}(\sigma - c), \qquad \Omega_b^{(c)}(\sigma) = \Omega_b(\sigma - c).$$

The deficiency $\alpha$ of course increases with $c$, but the infimum remains unchanged.

**10. The Thorin class of distributions.** The class of *Thorin distributions* or *generalized gamma convolutions* was introduced by Thorin (1977). This class consists of all distributions in the Bondesson class for which the canonical measure has a density, $Q(d\sigma) = q(\sigma)\,d\sigma$, such that $q(\sigma)$ is a *nondecreasing* function of $\sigma$. Differentiating the logarithm of (2.1) leads to the representation

$$G'(\lambda)/G(\lambda) = \delta + \int_{(0,\infty)} (\sigma - \lambda)^{-1} q(d\sigma) \tag{10.1}$$

(see Bondesson, 1981, pages 48–49).

The Thorin class is closed under convolutions, raising the m.g.f. to positive powers, and weak limits. Thus, we see from Section 4 that $\tau_{ab}$ *lies in the Thorin class for all* $r_0 < a < b < r_1$ *if and only if the infinitesimal left-hitting times at* $b$ *lie in the Thorin class for all* $r_0 < b < r_1$.

In particular, in the case of a *discrete* spectrum (Section 6), $\tau_{ab}$ will *not* lie in the Thorin class except possibly at special values of $a$ and $b$, or perhaps in limiting cases.

**11. Conditional diffusions.** The diffusion models of Section 3 allow a diffusion hitting time to equal $\infty$ with positive probability. If this feature is undesirable it can be removed by conditioning the hitting times $\tau_{ab}$, $r_0 < a < b < r_1$, to be finite with probability one. The resulting process is again a diffusion and its hitting times $\tau_{ab}^*$ satisfy $P(\tau_{ab}^* < \infty) = 1$, $r_0 < a < b < r_1$. The m.g.f. of $\tau_{ab}^*$ is given by

$$G_{ab}^*(\lambda) = G_{ab}(\lambda)/G_{ab}(0-), \qquad r_0 < a < b < r_1,$$

so that the distribution of $\tau_{ab}^*$ equals that of $\tau_{ab} \mid \tau_{ab} < \infty$.

Let $\psi(x, \lambda)$ denote the solution of (3.2)–(3.3). Then the speed measure, natural scale and killing measure of the new process are given by

$$m^*(dx) = \psi(x, 0-)^2 m(dx), \qquad s^*(dx) = \psi(x, 0-)^{-2} s(dx), \qquad k^*(dx) \equiv 0.$$

In terms of the classical differential operator (8.2), the new coefficients are given by

$$\alpha^*(x) = \alpha(x), \quad \beta^*(x) = \beta(x) + 2\,\alpha(x)\, d/dx \log \psi(x, 0-), \quad \gamma^*(x) \equiv 0.$$

The type of boundary behaviour can change at both $r_0$ and $r_1$ but no ambiguity can arise (except possibly what to do if the diffusion ever reaches $r_1$).

**12. Time changes.** Let $X(t)$, $t \geq 0$, denote a sample path of the diffusion in Section 3. Consider the additive functional up to the hitting time $\tau_{ab}$ defined by

$$T = \int_0^{\tau_{ab}} g(X(t))\, dt, \tag{12.1}$$

where $X(0) = a$ and where $g(x)$ is a given function of $x$. The hitting time itself can be included in this framework by choosing $g(x) \equiv 1$, so that $\tau_{ab} = T$. More general choices for $g(x)$ can be studied by introducing a random time change into the diffusion (Ito and McKean, 1965, Chapter 5).

To keep the discussion simple, suppose that the diffusion satisfies

(a) $P(\tau_{ab} < \infty) = 1$ for all $r_0 < a < b < r_1$.

Note (a) implies that (but is not implied by) the vanishing of the killing measure, $k(dx) \equiv 0$. Also suppose that $g(x)$ satisfies

(b) $0 < \int_a^b g(x)\, m(dx) < \infty$ for all $r_0 < a < b < r_1$ (including $a = r_0$ if $r_0$ is reflecting regular).

Under conditions (a) and (b) the following result holds.

THEOREM 12.1. *For all $r_0 < a < b < r_1$ the additive functional $T$ in* (12.1) *has the same distribution as the hitting time $\hat{\tau}_{ab}$ for a new diffusion defined by*

$$\hat{m}(dx) = g(x)m(dx), \qquad \hat{s}(dx) = s(dx), \qquad \hat{k}(dx) = k(dx) \equiv 0. \tag{12.2}$$

*No ambiguity about the boundary behaviour at $r_0$ can arise for this new diffusion. For the classical differential operator* (8.2), *the new coefficients are defined by*

$$\hat{\alpha}(x) = g(x)^{-1}\alpha(x), \qquad \hat{\beta}(x) = g(x)^{-1}\beta(x), \qquad \hat{\gamma}(x) = \gamma(x) \equiv 0.$$

*In particular, the distribution of $T$ lies in the Bondesson class.*

**13. Example 1. The Hartman-Watson distribution.** Consider the diffusion on $(0, \infty)$ generated by

$$A_\nu = 1/2\{x^2 d^2/dx^2 + x d/dx - x^2 - \nu\} \tag{13.1}$$

where $\nu \geq 0$ is a parameter. It is easily checked that both boundaries are natural and that the appropriate solutions of (3.2) (which is just Bessel's modified differential equation) lead to the following m.g.f.'s for the first hitting times:

$$G_{ab}(\lambda) = I_\mu(a)/I_\mu(b), \qquad 0 < a < b < \infty, \tag{13.2}$$

$$G_{ab}(\lambda) = K_\mu(a)/K_\mu(b), \qquad \infty > a > b > 0, \tag{13.3}$$

where

$$\mu = (\nu - 2\lambda)^{1/2}.$$

Note that since $\arg \lambda \in (0, 2\pi)$, we can take $\arg(\nu - 2\lambda) \in (-\pi, \pi)$, so $\arg \mu \in (-\pi/2, \pi/2)$. Here and elsewhere facts about the Bessel functions $I_\rho(z)$, $K_\rho(z)$, $J_\rho(z)$, $Y_\rho(z)$ are taken from Abramowitz and Stegun (1972). Normalizing these distributions to be finite with probability one and letting the right-hand point tend to $\infty$ yields the distributions in the Bondesson class with m.g.f.'s.

$$I_\mu(a)/I_\nu(a), \tag{13.4}$$

$$K_\nu(b)/K_\mu(b). \tag{13.5}$$

We wish to determine the canonical measures for these distributions. Note that by Remark (1) after Theorem 2.1, (13.4) and (13.5) have infima equal to 0.

We start with (13.4) with $\nu = 0$ so that $\mu = -i(2\lambda)^{1/2}$. First, since (13.4) is the m.g.f. of a Bondesson distribution, $I_\mu(x)$ is *nonzero* for arg $\lambda \in (0, 2\pi)$. Secondly, note that $I_{\bar\rho}(x) = \overline{I_\rho(x)}$ since for $x > 0$, $I_\rho(x)$ is an entire function of $\rho$, real-valued for $\rho$ real. Hence for arg $\lambda = 0$ (that is, $\mu$ imaginary), we have the relationship

$$\begin{aligned} K_\mu(x) &= (1/2)\pi[I_\mu(x) - I_{-\mu}(x)]/[i \sinh \pi(2\lambda)^{1/2}] \\ &= \pi \operatorname{Im}[I_\mu(x)]/[\sinh \pi(2\lambda)^{1/2}] \end{aligned}$$

(Abramowitz and Stegun, 1972, page 375).

Now $I_\rho(x)$ and $K_\rho(x)$ cannot both vanish together because they have nonvanishing Wronskian; thus the above formula shows that $I_\mu(x)$ is in fact also nonzero for arg $\lambda = 0$. Therefore, $\psi(x, \lambda) = I_\mu(x)$ satisfies the assumption of Section 7, and the methods described there can be used to find the canonical measures for (13.2) and (13.4).

On the other hand, $K_\rho(x)$ is an *even* function of $\rho$ so that (for general $\nu \geq 0$) $K_\mu(x)$ is an entire function of $\lambda$. It is not difficult to check that $\psi(x, \lambda) = K_\mu(x)$ satisfies the assumptions of Section 6, and hence the methods described there can be used to find the canonical measures of (13.3) and (13.5). In particular, (13.5) can be written in the form $\prod_{k=1}^{\infty} (1 - \lambda/\lambda_{k,b})^{-1}$. This is the m.g.f. of an *infinite convolution of exponential densities.*

Note that the canonical measures for the hitting times of $A_\nu$ are related to those of $A_0$ by a shift, as described in Section 9.

The infinite divisibility of (13.2)–(13.5) was first proved in Hartman (1976) using arguments similar to those involving diffusion hitting times. The distribution with m.g.f. (13.4) is known as the *Hartman-Watson mixing distribution.*

The importance of the Hartman-Watson distribution is due to the following result (Hartman and Watson, 1974). Let $\Omega_q$ denote the unit sphere in $R^q$, $q \geq 2$, and let $B_0(t)$, $t \geq 0$, be a standard Brownian motion on $\Omega_q$ started at some $e \in \Omega_q$. If $T$ is a random variable with distribution (13.4) independent of $B_0(t)$, then

$$B_0(T) \sim M(a, e) \tag{13.6}$$

where $M(a, e)$ denotes the *von Mises-Fisher* distribution of concentration parameter $a$ and mean direction $e$. If for fixed $t$ we call $B_0(t)$ the *spherical normal* distribution of "variance" $t$ and mean direction $e$, then (11.6) shows that the von Mises-Fisher distribution can be obtained as a "variance" mixture of spherical normals. The infinite divisibility of the Hartman-Watson distribution is important because it implies the infinite divisibility of the von Mises-Fisher distribution (see Kent, 1977, 1981).

An appealing model in which the representation (13.6) arises naturally has been given in Pitman and Yor (1981). Their model is closely related to the diffusion (13.1) and is probabilistically satisfying because the infinite divisibility of (13.4) follows directly from their construction. Here is their model.

Let $B(t)$, $t \geq 0$, be a standard Brownian motion $R^q$, $q \geq 2$, started at the origin with constant drift vector $e(\in \Omega_q)$. Split $B(t)$ into its radial component $R(t) = \|B(t)\|$ and angular component $\Theta(t) = B(t)/R(t)$. Then the following properties hold.

(a) If $\sigma_{0a}$ denotes the first time $B(t)$ hits the circle of radius a, then $\Theta(\sigma_{0a})$ has a von Mises-Fisher distribution (see, for example, Kent (1978) Theorem 4.1).

(b) $R(t)$, $t \geq 0$, is itself a diffusion (a surprising result!) called the *Bessel process of parameter* $\nu = (q - 2)/2$ *and drift* 1, with generator

$$A^* = (1/2)\, d^2/dr^2 + \{(2\nu + 1)/2r + I_{\nu+1}(r)/I_\nu(r)\}\, d/dr. \tag{13.7}$$

(c) The reversed process $uB(1/u)$, $u \geq 0$, is also a Brownian motion (started at $e$ without any drift). Hence the distribution of $\Theta(t)$ *given* $R(s)$, $t \leq s < \infty$, is spherical normal with mean direction $e$ and "variance"

$$(13.8) \qquad \int_0^{1/t} \{u\, R(1/u)\}^{-2}\, du = \int_t^{\infty} R(s)^{-2}\, ds = T, \qquad \text{say.}$$

The fixed time $t$ can be replaced by any random time which depends only on the radial process.

If we insert $\sigma_{0a}$ for $t$ in (13.8), we see from (a) and (c) that $\Theta(\sigma_{0a})$ can be given the representation (13.6). Using (b) note that $\sigma_{0a}$ is the same as the hitting time $\tau^*_{0a}$ for the process (13.7), so that the distribution of $T$ can also be described by

$$(13.9) \qquad T = \int_0^{\infty} X(s)^{-2} ds = \lim_{b\uparrow\infty} \int_0^{\tau^*_{ab}} X(s)^{-2}\, ds$$

where $X(s)$ follows the diffusion (13.7) started at $a$. To find the distribution of $T$ we can use a time change and a conditioning argument. (Pitman and Yor use a different argument.) Since $P(\tau^*_{ab} < \infty) = 1$, $a < b < \infty$, and since $P(\tau^*_{a\infty} = \infty) = 1$ for the diffusion (13.7), we can make the time substitution given in Section 12 with $g(x) = x^{-2}$, and describe $T$ as the hitting time $T = \tau^{**}_{a\infty}$ for the diffusion generated by

$$(13.10) \qquad A^{**} = (1/2)x^2 d^2/dx^2 + \{(1/2)x(2\nu + 1) + x^2 I_{\nu+1}(x)/I_\nu(x)\}\, d/dx.$$

Now by Section 11, (13.10) is obtained from (13.1) by conditioning the left-hitting times of (13.1) to be finite with probability 1. Hence, as we have already calculated, $T = \tau^{**}_{a\infty}$ has m.g.f. given by (13.4). Since $T$ has been expressed as a diffusion hitting time its infinite divisibility follows immediately.

Finally, we remark that the Pitman-Yor construction can be carried out for non-integer values of $q \geq 2$ by using the Brownian motion-Bessel process described in Kent (1978, Section 4). However, for $1 < q < 2$ this approach breaks down, in part because the integral in (13.9) now diverges with positive probability. It is not clear whether any aspects of the representation (13.6) can be salvaged in this case.

**14. Example 2. The Bessel process.** Consider now the Bessel process on $[0, \infty)$ (without drift) with generator

$$(14.1) \qquad A_\nu = (1/2)[d^2/dx^2 + \{(2\nu + 1)/x\}\, d/dx]$$

where $\nu$ is a real-valued parameter. The *left*-hitting times were studied in Kent (1978, 1980), and here we shall look at *right*-hitting times $\tau^{(\nu)}_{ab}$, where we let $0 < b < a < \infty$ if $\nu \geq 0$, and $0 \leq b < a < \infty$ if $\nu < 0$. Note that $r_1 = \infty$ is *natural* for all $\nu$.

The m.g.f. $G^{(\nu)}_{ab}(\lambda)$ was given (as a Laplace transform) in Kent (1978). Note that for $\nu > 0$, $\tau^{(\nu)}_{ab}$ and $\tau^{(-\nu)}_{ab}$ have the *same* canonical measure but different deficiencies ($e^{-\alpha} = (b/a)^{2\nu}$ and $e^{-\alpha} = 1$, respectively).

Starting first with $\tau^{(-\nu)}_{ab}$, $\nu > 0$, it is easily checked that the canonical measure has a continuous density

$$(14.2) \qquad q^{(-\nu)}_{a0}(\sigma) = g_\nu(a\{2\sigma\}^{1/2}), \qquad 0 < \sigma < \infty,$$

where

$$(14.3) \qquad g_\nu(z) = -(1/\pi) \arctan\{J_\nu(z)/Y_\nu(z)\},$$

and the branch of arctan is chosen so that $g_\nu(z) \to 0$ as $z \downarrow 0$.

Thus, for general $\tau^{(\nu)}_{ab}$ we have

$$(14.4) \qquad q^{(\nu)}_{ab}(\sigma) = g_{|\nu|}(a\{2\sigma\}^{1/2}) - g_{|\nu|}(b\{2\sigma\}^{1/2}), \qquad 0 < \sigma < \infty$$

(including $\nu = 0$ by Theorem 2.1).

Differentiating (14.3) gives $g'_\nu(z) = 2/\{\pi^2 z^2 M^2_\nu(z)\}$, where $M^2_\nu(z) = M^2_{-\nu}(z) = J^2_\nu(z) + Y^2_\nu(z) > 0$ for $z > 0$, $\nu$ real. Then the canonical measure for the infinitesimal right-hitting times at $a$, obtained by differentiating (14.4) with respect to the natural scale $s(da) = ((1/2)a)^{-(2\nu+1)}$ is given by

$$\omega_a^{(\nu)}(\sigma) = 2((1/2)a)^{2\nu+1}/\pi^2 a\, M^2_\nu(a\{2\sigma\}^{1/2}), \qquad 0 < \sigma < \infty, \tag{14.5}$$

for all $\nu$ and $0 < a < \infty$.

Let $\nu > 0$. Since $g'_\nu(z) > 0$, (14.2) is an *increasing* function of $\sigma$; *hence* $\tau_{a0}^{(-\nu)}$ *is a Thorin distribution*. In fact, $1/\tau_{a0}^{(-\nu)}$ has a gamma distribution, and in this situation the representation formula (10.1) is known as *Grosswald's formula* (Grosswald, 1976; Ismail, 1977). A limiting argument shows that Grosswald's formula is also valid for $\nu = 0$, but there is no direct probabilistic interpretation in this case.

The distribution of $\tau_{a0}^{(-\nu)}$ for $\nu > 0$ was earlier obtained by Hammersley (1961, page 18) as an additive functional of Brownian motion up to a first hitting time. By the time change argument of Section 12, Hammersley's result is equivalent to looking the first hitting time to 0 for the Bessel process.

To obtain further results, note that Nicholson's formula (Watson, 1944, page 446) implies that $M^2_\nu(z)$ is a *decreasing* function of $z > 0$, for $\nu$ real. Hence (14.5) is an *increasing* function of $\sigma$. *Therefore, all of the infinitesimal right-hitting times* (*and by Section 10, all of the right-hitting times*) *are Thorin distributions.* This last result was derived in conversation with Lennart Bondesson (see Bondesson, 1981), and extends the earlier work of Ismail and Kelker (1979) and Kent (1978) where just infinite divisibility is shown.

Finally, consider the diffusion modified as in Section 9, generated by $A_\nu - c$ $(c \geq 0)$, with hitting times $\tau_{ab}^{(\nu;c)}$, and in particular consider $\tau_{a0}^{(-\nu;c)}$, where $\nu > 0$. This distribution, normalized to be finite with probability one, is a Thorin distribution lying in a subclass of the *generalized inverse Gaussian distributions.* Its infinite divisibility was established in Barndorff-Neilsen et al. (1978) using the diffusion obtained from $A_\nu - c$ by the conditioning arguments of Section 11. It was first shown to lie in the Thorin class by Bondesson (1979) and Halgreen (1979).

**15. Connection with other work.** Since this paper was completed J. Pitman (personal communication) has pointed out that the infinitesimal hitting time distributions of Section 4 also appear in Ito and McKean (1965, pages 214–217) in a context where the sample paths of the diffusion *do have a natural meaning*. For simplicity suppose the diffusion is *persistent* with sample paths $X(t)$, $0 \leq t < \infty$, and let $L(t, x)$ denote the *local time* of the diffusion. For $\gamma > 0$ and $r_0 < b < r_1$ let

$$S_\gamma = \sup\{t > 0 : L(t, b) = \gamma\}$$

denote the inverse local time process at $b$, and set

$$V_\gamma = \int_0^{S_\gamma} I[X(t) < b]\, dt, \qquad W_\gamma = \int_0^{S_\gamma} I[X(t) \leq b]\, dt.$$

Then Ito and McKean show that

$$\{H_b^L(\lambda)\}^\gamma = E_b\{\exp(\lambda V_\gamma)\}, \qquad \{H_b^R(\lambda)\}^\gamma = E_b\{\exp(\lambda W_\gamma)\}.$$

Pitman has also found a nice description of the Bondesson class in terms of Levy measures. Recall that a general infinitely divisible distribuiton on $[0, \infty]$ has an m.g.f. of the form

$$G(\lambda) = \exp\{-\alpha + \delta\lambda + \int_{(0,\infty)} (e^{\lambda u} - 1)\Theta(du)\},$$

where $\Theta(du)$ is called the *Levy measure.* Then $\Theta(du)$ represents a member of the

Bondesson class if and only if $\Theta(du)$ *has a density* $\Theta(du) = \theta(u)du$ *and* $\theta(u)$ *is a completely monotone function of* $u$. Further it is easily checked that $\theta(u)$ is the Laplace transform of the canonical measure $Q(d\sigma)$. This description is also implicit in Bondesson (1981).

In a similar analysis Thorin (1977) showed that $\theta(u)$ represents a member of the Thorin class if and only if $u\theta(u)$ is a completely monotone function of $u$.

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