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ORTHOGONAL MEASURES: AN EXAMPLE

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A family $\{\mu_\alpha\}$ of measures on a σ-field $\mathscr{B}$ on a space X is "uniformly orthogonal" means that for each α, $\exists H_\alpha \in \mathscr{B}$ such that $\mu_\alpha(X - H_\alpha) = \mu_\beta(H_\alpha) = 0$ if $\beta \neq \alpha$. Assuming CH, an example is given of an orthogonal family of measures on the Borel sets of I^2 such that no uncountable subfamily is uniformly orthogonal. Assuming $\sim CH + MA$, such an uncountable family obviously exists.

A family $\mathscr{M}$ of measures, defined on a Borel field $\mathscr{B}$ of subsets of a space X, is said to be *pairwise orthogonal* if, given $\lambda, \mu \in \mathscr{M}$ with $\lambda \neq \mu$, there exists $H_{\lambda\mu} \in \mathscr{B}$ such that $\lambda(H_{\lambda\mu}) = 0 = \mu(X - H_{\lambda\mu})$. $\mathscr{M}$ will be called *uniformly orthogonal* provided there is, for each $\lambda \in \mathscr{M}$, a set $H_\lambda \in \mathscr{B}$ such that, for each $\mu \in \mathscr{M} - \{\lambda\}$, $\lambda(H_\mu) = 0 = \lambda(X - H_\lambda)$. Clearly every uniformly orthogonal family is pairwise orthogonal, and every *countable* pairwise orthogonal family is uniformly orthogonal. One simple example of an uncountable pairwise orthogonal family $\mathscr{M}$ that is not uniformly orthogonal is provided by taking X to be the unit interval I, $\mathscr{B}$ the Borel sets of X, and $\mathscr{M}$ to consist of Lebesgue measure, together with all 1-point measures. Here, however, the family does have an uncountable subfamily consisting of uniformly orthogonal measures; we have only to omit Lebesgue measure. The following example shows that in general we cannot obtain an uncountable uniformly orthogonal family from a pairwise orthogonal family by discarding measures - provided the continuum hypothesis is assumed.

THEOREM. *Assuming the continuum hypothesis, there exists an uncountable family $\mathscr{M}$ of pairwise orthogonal Borel probability measures on the unit square I^2, such that no uncountable subset of $\mathscr{M}$ is uniformly orthogonal.*

We need a well-known lemma (see, for example, Oxtoby (1970), page 76), as follows.

LEMMA. *Assuming the continuum hypothesis, there exists a partition of the unit interval I into a family $\mathscr{N}$ of c pairwise disjoint non-empty Borel null sets such that each null set in I is covered by a countable subfamily of $\mathscr{N}$.*

PROOF. Well-order the null G_δ sets as $\{G_\alpha : \alpha < \omega_1\}$, define $N_\alpha = G_\alpha - \cup\{G_\beta : \beta < \alpha\}$, and omit empty N_α's.

Construction. Let $\mathscr{N} = \{N_\alpha : \alpha < \omega_1\}$ be a partition as in the Lemma, and let $\{y_\alpha : \alpha < \omega_1\}$ well-order I without repetition. For each $\alpha < \omega_1$ let μ_α denote the (linear) Lebesgue measure on $I \times \{y_\alpha\} \subset I^2$. For each $\alpha > 0$, take a sequence $\{u_{\alpha\beta} : \beta < \alpha\}$ of positive real numbers such that $\sum \{u_{\alpha\beta} : \beta < \alpha\} = \frac{1}{2}$. Take a Borel measure $m_{\alpha\beta}$ on $N_\alpha \times \{y_\beta\}$ $(\beta < \alpha < \omega_1)$ such that $m_{\alpha\beta}(N_\alpha \times \{y_\beta\}) = u_{\alpha\beta}$. Now, for each Borel set $H \subset I^2$ and $\alpha < \omega_1$, define

$$m_\alpha(H) = \tfrac{1}{2}\, \mu_\alpha(H \cap (I \times \{y_\alpha\})) + \sum \{m_{\alpha\beta}(H \cap (N_\alpha \times \{y_\beta\})) : \beta < \alpha\}$$

if $\alpha \geq 1$, and define $m_0(H) = \mu_0(H \cap (I \times \{y_0\}))$. Then put $\mathscr{M} = \{m_\alpha : \alpha < \omega_1\}$, an uncountable family of Borel probability measures on I^2. It is easy to see that they are pairwise orthogonal. On the other hand, fixing $\gamma < \omega_1$, suppose H_γ is a Borel subset of I^2

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such that $m_\gamma(H_\gamma) = 1$; then also $\mu_\gamma(H_\gamma \cap (I \times \{y_\gamma\})) = 1$. That is, $\mu(H^\gamma) = 1$ where μ is Lebesgue measure and $H^\gamma = \{x \in I: (x, y_\gamma) \in H_\gamma\}$. By construction of the sets N_α, H^γ must contain all but a countable subfamily of the sets N_α, and hence H_γ can be null with respect to only countably many measures m_β with $\beta > \gamma$. It follows at once that every uniformly orthogonal subfamily of $\mathscr{M}$ is countable, as required.

REMARKS. 1. By taking a little more trouble, we could ensure that the measures m_α were all non-atomic (in addition to their other properties).

2. The continuum hypothesis is essential for the theorem. It is relatively consistent (with usual set theory) that the union of fewer than c null sets in I (with respect to any finite Borel measure) is always null. (See, for example, Shoenfield (1975) for the case of Lebesgue measure, the same argument works for the more general measures considered here.) From this assumption it follows easily that, if $\aleph_1 < c$, each family of $\aleph_1$ pairwise orthogonal finite Borel measures on I (or, what comes to the same thing, on I^2) is uniformly orthogonal.

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