The Annals of Probability
1982, Vol. 10, No. 3, 877–878

A NOTE ON CONDITIONAL DISTRIBUTIONS AND ORTHOGONAL MEASURES

By R. J. Gardner

University of Petroleum and Minerals

We solve a problem posed by J. P. Burgess and R. D. Mauldin, and thus show that a result of theirs on conditional distributions and orthogonal measures cannot be improved.

1. Introduction. In [2], D. Maharam proved the following theorem, assuming the continuum hypothesis (CH).

Theorem A. (CH). *There exists an uncountable family $\mathscr{M}$ of pairwise orthogonal Borel probability measures on the unit square I^2, such that no uncountable subset of $\mathscr{M}$ is uniformly orthogonal.*

We refer the reader to Section 2 for definitions. It is easy to show that any *countable* family of pairwise orthogonal probability measures is actually uniformly orthogonal. Maharam's example, which was suggested by work of R. L. Tweedie on Markov chains ([4]), shows (at least assuming CH) that no more can be said in general. (In fact, an easy modification shows that Theorem A remains true, without special axiomatic assumptions, if "no uncountable subset of $\mathscr{M}$" is replaced by "no subset of $\mathscr{M}$ of cardinality c".) In [1], J. P. Burgess and R. D. Mauldin demonstrated, however, that no such example is possible when the family $\mathscr{M}$ forms a conditional probability distribution. More precisely, they proved the following.

Theorem B. *Let X and Y be complete separable metric spaces, and let $\mathscr{B}(Y)$ denote the Borel sets of Y. Suppose $\{\mu(x, \cdot): x \in X\}$ is a conditional probability distribution on $\mathscr{B}(Y)$, such that $\{\mu(x, \cdot): x \in X\}$ is a pairwise orthogonal family. Then there is a nonempty compact perfect subset K of X such that the family $\{\mu(x, \cdot): x \in K\}$ is uniformly orthogonal.*

The question was then posed (see [1]): if ν is an atomless probability measure on $\mathscr{B}(X)$, can the set K of Theorem B be found to satisfy $\nu(K) > 0$? That is, is there a "fat" set K so that the family $\{\mu(x, \cdot): x \in K\}$ is uniformly orthogonal?

The purpose of this note is to give an example which answers this question negatively. Some of the ideas of Maharam's example are used.

The author would like to thank R. D. Mauldin for suggesting the problem and for several helpful discussions.

2. Definitions. Suppose X and Y are complete separable metric spaces, and $\mathscr{B}(X)$, $\mathscr{B}(Y)$ the corresponding classes of Borel subsets. A *conditional probability distribution* μ on $X \times \mathscr{B}(Y)$ is a family $\{\mu(x, \cdot): x \in X\}$ of probability measures on $\mathscr{B}(Y)$ such that for each $E \in \mathscr{B}(Y)$, $\mu(\cdot, E)$ is Borel measurable with respect to $\mathscr{B}(X)$.

A family $\mathscr{M}$ of measures defined on $\mathscr{B}(Y)$ is called *pairwise orthogonal* if given $\mu, \nu \in \mathscr{M}$ with $\mu \neq \nu$, there exist $B_{\mu\nu} \in \mathscr{B}(Y)$ such that

$$\mu(B_{\mu\nu}) = 0 = \nu(Y - B_{\mu\nu}).$$

The family $\mathscr{M}$ is *uniformly orthogonal* if for each $\mu \in \mathscr{M}$ there is a set $B_\mu \in \mathscr{B}(Y)$ such

Received February 1981.
AMS 1970 subject classifications. Primary 60B05; secondary 28A05, 28A10.
Key words and phrases. Orthogonal measures, conditional probability distribution.

that if $\nu \in \mathscr{M}$ and $\nu \neq \mu$,

$$\nu(B_\mu) = 0 = \mu(Y - B_\mu).$$

In the example which follows, we shall use I to denote the unit interval with Lebesgue measure λ, and N for the natural numbers. If $E \subset X \times Y$, we use E_x for the set $\{y: (x, y) \in E\}$.

3. The example.

THEOREM. *There is an atomless conditional probability distribution* μ *on* $I \times \mathscr{B}(N \times I^3)$, such that (i) *the family* $\{\mu(x, \cdot): x \in I\}$ *is pairwise orthogonal*; (ii) *if* $E \subset I$ *and* $\lambda(E) > 0$, *the family* $\{\mu(x, \cdot): x \in E\}$ *is not uniformly orthogonal.*

PROOF. Let ϕ be a 1–1, Borel measurable, measure-preserving map from (I, λ) into (I^2, λ^2), with $\phi(x) = (\phi^{(1)}(x), \phi^{(2)}(x))$, say. (See, for example, [3].)

Let (r_n) be an enumeration of the rationals in I, and define ϕ_n, for $n = 1, 2, \cdots$, by $\phi_n(x) = ([\phi^{(1)}(x) + r_n](\text{mod } 1), \phi^{(2)}(x))$, for each $x \in I$. Then, for each n, ϕ_n is also a 1–1, Borel measurable, measure-preserving map from (I, λ) into (I^2, λ^2).

Define $\mu(x, \cdot)$, for $x \in I$, on $\mathscr{B}(N \times I^3)$ by:

$$\mu(x, A) = \textstyle\sum_{n=1}^{\infty} \{\lambda[A \cap (\{n\} \times \{\phi_n(x)\} \times I)] + \lambda^2[A \cap (\{n\} \times \{x\} \times I^2)]\}/2^{n+1},$$

for $A \in \mathscr{B}(N \times I^3)$.

Then μ is an atomless conditional probability distribution on $I \times \mathscr{B}(N \times I^3)$, and it is easy to check that the family $\{\mu(x, \cdot): x \in I\}$ is pairwise orthogonal.

Now suppose $E \subset I$ with $\lambda(E) > 0$. Then $\lambda^2[\phi(E)] > 0$, so setting $D = \{x: \lambda[(\phi(E))_x] > 0\}$, we have $\lambda(D) > 0$. (We use Fubini's theorem and the easily proved fact that D is a measurable set.) By Steinhaus's theorem, the difference set $(E - D)$ contains an interval. So, there is a rational r_m say such that $(D + r_m) \cap E \neq \phi$. Choose $x^* \in (D + r_m) \cap E$; then

$$\lambda[(\phi_m(E))_{x^*}] > 0. \tag{1}$$

Next, suppose $\{B_x\}$ are sets in $N \times I^3$ such that $\mu(x, B_x) = 1$ for each $x \in I$. Then, in particular, $\mu(x^*, B_{x^*}) = 1$, so by the definition of $\mu(x^*, \cdot)$ we have

$$\lambda^2[B_{x^*} \cap (\{m\} \times \{x^*\} \times I^2)] = 1. \tag{2}$$

By (1), (2), and Fubini's theorem, there is some $x \in E$ with $x \neq x^*$, $\phi_m(x) \in \{x^*\} \times I$, and

$$\lambda[B_{x^*} \cap (\{m\} \times \{\phi_m(x)\} \times I)] \neq 0.$$

But this means $\mu(x, B_{x^*}) \neq 0$, by the definition of $\mu(x, \cdot)$. Thus the family $\{\mu(x, \cdot): x \in E\}$ cannot be uniformly orthogonal.

REFERENCES

[1] BURGESS, J. P. and MAULDIN, R. D. (1981) Conditional distributions and orthogonal measures. *Ann. Probability* **9** 902–906.

[2] MAHARAM, D. (1977). Orthogonal measures: an example. *Seminaire de Probabilités* II, Rennes.

[3] OXTOBY, J. C. (1970). Homeomorphic measures in metric spaces. *Proc. Amer. Math. Soc.* **24** 419–423.

[4] TWEEDIE, R. L. (1979). Topological aspects of Doeblin decompositions for Markov chains. *Z. Wahrsch. verw. Gebiete* **46** 299–305.

DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF PETROLEUM AND MINERALS
DHAHRAN, SAUDI ARABIA