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RATES OF ESCAPE OF RANDOM WALKS

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The rates at which a transient random walk can escape to infinity are discussed.

1. Introduction. The purpose of this note is to give a one parameter family of examples of one dimensional random walks exhibiting different rates of escape. More specifically, for each $\delta \in (0, 1)$ we give a random walk such that

$$(*) \qquad \liminf_{n\to\infty} \frac{|S_n|}{n^\alpha} = \begin{cases} \infty \text{ a.s.} & \text{if } \alpha < \delta \\ 0 \text{ a.s.} & \text{if } \alpha > \delta. \end{cases}$$

We achieve this by using a result of Kesten [2]. In the same paper Kesten gives an example of a transient random walk such that $\liminf n^{-\alpha}|S_n| = 0$ a.s. for all $\alpha > 0$. This would correspond to $\delta = 0$ in (*). If $\delta > 1$, then the stable random walk of index δ^{-1} satisfies (*), while the Strong Law of Large Numbers provides examples when $\delta = 1$. Thus in one dimension, for every $\delta \geq 0$ there is a random walk satisfying (*). One can also show, as we point out later, that the same situation holds in two dimensions. This behaviour is in contrast to the d-dimensional case for $d \geq 3$, where $n^{-\alpha}|S_n| \to \infty$ a.s. for every $\alpha < \frac{1}{2}$; see Kesten [3].

2. The examples. We begin by introducing some notation:

$a_n \sim b_n$ iff $\dfrac{a_n}{b_n} \to 1$ as $n \to \infty$

$a_n \approx b_n$ iff there is a constant $M > 1$ such that $1/M \leq a_n/b_n \leq M$ as $n \to \infty$.
We write ℓx for $\log x$, $\ell_2 x$ for $\log \log x$, etc. Define

$$H(x) = \frac{\exp\{\gamma(\ell_2 x)(\ell_3 x)\}}{\gamma(\ell x)^{\gamma-1}(\ell_3 x)} \quad \text{for} \quad \gamma \in (0, \infty).$$

Observe that

$$\frac{d}{dx}\left[\frac{-(\ell x)^\gamma}{\exp\{\gamma(\ell_2 x)(\ell_3 x)\}}\right] = \frac{1}{xH(x)}.$$

Also one can easily check that $H(x)$ is slowly varying.

The examples follow along the lines of Kesten's example mentioned in the introduction. Let $X_1, X_2, \cdots$ be independent, identically distributed, symmetric random variables with non-lattice distribution satisfying

$$P\{|X_1| > x\} \sim \frac{H(x)}{x} \quad \text{as} \quad x \to \infty.$$

Then

$$\liminf_{n\to\infty} \frac{|S_n|}{n^\alpha} = \begin{cases} \infty \text{ a.s.} & \text{if } \alpha < e^{-1/\gamma} \\ 0 \text{ a.s.} & \text{if } \alpha > e^{-1/\gamma}. \end{cases}$$

PROOF. We will make use of the following criterion of Kesten [2];

$$\liminf_{n\to\infty} \frac{|S_n|}{n^\alpha} < \infty \quad \text{a.s. iff} \quad \sum_n \frac{P\{|S_n| < Kn^\alpha\}}{\sum_{i=0}^{n-1} P\{|S_i| < n^\alpha\}} = \infty \text{ for some } K > 0.$$

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First note that X_1 is in the domain of attraction of the Cauchy distribution and so by a Local Limit Theorem of Stone [4],

$$P\{|S_n| < k_n\} \approx \frac{k_n}{nH(n)} \quad \text{as} \quad n \to \infty \quad \text{if} \quad \frac{k_n}{nH(n)} \to 0.$$

Since $e^{-1/\gamma} \in (0, 1)$, it suffices to consider $\alpha \in (0, 1)$, in which case

$$P\{|S_n| < Kn^\alpha\} \approx \frac{n^\alpha}{nH(n)}.$$

Next we must estimate $\Delta_n = \sum_{i=0}^{n-1} P\{|S_i| < n^\alpha\}$. Let

$$\varphi(n) = \frac{\gamma n^\alpha (\ell n^\alpha)^{\gamma-1}(\ell_3 n^\alpha)(\ell_2 n)}{\exp\{\gamma(\ell_2 n^\alpha)(\ell_3 n^\alpha)\}}.$$

One easily checks by using the Mean Value Theorem, that $(\ell n^\alpha)^{\gamma-1} \sim (\ell\varphi(n))^{\gamma-1}$, $(\ell_3 n^\alpha) \sim (\ell_3\varphi(n))$ and $\exp\{\gamma(\ell_2 n^\alpha)(\ell_3 n^\alpha)\} \sim \exp\{\gamma(\ell_2\varphi(n))(\ell_3\varphi(n))\}$. Thus

$$\frac{n^\alpha}{\varphi(n)H(\varphi(n))} \sim \frac{1}{\ell_2 n} \to 0.$$

Now

$$\sum_{\varphi(n)}^{n-1} P\{|S_i| < n^\alpha\} \le \Delta_n \le \varphi(n) + \sum_{\varphi(n)}^{n-1} P\{|S_i| < n^\alpha\}$$

and since $n^\alpha/\varphi(n)H(\varphi(n)) \to 0$,

$$\begin{aligned}\sum_{\varphi(n)}^{n-1} P\{|S_i| < n^\alpha\} &\approx \sum_{\varphi(n)}^{n-1} \frac{n^\alpha}{kH(k)} \\ &\approx \frac{n^\alpha(\ell\varphi(n))^\gamma}{\exp\{\gamma(\ell_2\varphi(n))(\ell_3\varphi(n))\}} - \frac{n^\alpha(\ell n)^\gamma}{\exp\{\gamma(\ell_2 n)(\ell_3 n)\}} \\ &\approx \frac{n^\alpha(\ell n)^\gamma}{\exp\{\gamma(\ell_2 n^\alpha)(\ell_3 n^\alpha)\}}.\end{aligned}$$

Thus

$$\Delta_n \approx \frac{n^\alpha(\ell n)^\gamma}{\exp\{\gamma(\ell_2 n^\alpha)(\ell_3 n^\alpha)\}}.$$

So we obtain

$$\Delta_n^{-1} P\{|S_n| < Kn^\alpha\} \approx \frac{(\ell_3 n)}{n(\ell n)\exp\{\gamma(\ell_2 n)(\ell_3 n) - \gamma(\ell_2 n^\alpha)(\ell_3 n^\alpha)\}} \approx \frac{(\ell_3 n)}{n(\ell n)(\ell_2 n)^{\gamma(\ell/\alpha)}}$$

again by the Mean Value Theorem. Thus

$$\sum_n \Delta_n^{-1} P\{|S_n| < Kn^\alpha\} \approx \sum_n \frac{(\ell_3 n)}{n(\ell n)(\ell_2 n)^{\gamma(\ell/\alpha)}}$$

and the result now follows.

REMARK 1. In the case $\delta = 1$, the Strong Law of Large Numbers provides non-symmetric examples. We indicate here how to obtain symmetric examples. Let X be a symmetric, non-lattice random variable with distribution satisfying $P\{|X| > x\} \sim L(x)/x$ where

$$L(x) = \exp\{(\ell_2 x)(\ell_3 x)^\beta\} \text{ with } \beta > 1.$$

Now proceeding as above one shows that for $\alpha < 1$, $\liminf n^{-\alpha}|S_n| = \infty$ a.s. while for $\alpha > 1$, $\liminf n^{-\alpha}|S_n| = \limsup n^{-\alpha}|S_n| = 0$ a.s. since X is in the domain of attraction of the Cauchy distribution.

In passing we mention that taking $\beta \in (0, 1)$ gives rise to a transient random walk of index $\delta = 0$.

REMARK 2. As mentioned in the introduction, (*) can also be satisfied for every $\delta \geq 0$ in two dimensions. For $\delta > \frac{1}{2}$ the symmetric stable random walk of index δ^{-1} satisfies (*). For $0 < \delta < \frac{1}{2}$, let Y be a symmetric, one dimensional, non-lattice random variable with distribution satisfying $P\{|Y| > x\} \sim H'(x^2)$ where $H(x)$ is as before with $\gamma = -(\ell 2\delta)^{-1}$. One easily verifies that Y is in the domain of attraction of the Normal distribution and a correct normalizing sequence is $\sqrt{nH(n)}$. Now let Z be an independent copy of Y and let $X = (Y, Z)$. Then X is in the domain of attraction of the two dimensional Normal distribution with the same normalizing sequence, thus by Stone's Local Limit Theorem,

$$P\{|S_n| < k_n\} \approx \frac{k_n^2}{nH(n)} \quad \text{as} \quad n \to \infty \quad \text{if} \quad \frac{k_n^2}{nH(n)} \to 0.$$

Everything now goes through as before using Kesten's condition in two dimensions; see Erickson [1]. For $\delta = \frac{1}{2}$ take Y satisfying $P\{|Y| > x\} \sim L'(x^2)$ where L is as in Remark 1 with $\beta > 1$. Proceeding as above one obtains for $\alpha < \frac{1}{2}$, $\liminf n^{-\alpha}|S_n| = \infty$ a.s., while for $\alpha > \frac{1}{2}$, $\liminf n^{-\alpha}|S_n| = \limsup n^{-\alpha}|S_n| = 0$ a.s. for every random variable with zero mean in the domain of attraction of the Normal. Finally for $\delta = 0$, proceed as above using the example from Remark 1 with $\beta \in (0, 1)$.

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