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CONTINUITY OF GAUSSIAN LOCAL TIMES

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Continuity in the time parameter is considered for a natural version of the local time for stationary Gaussian processes. Bounds are given for the local and uniform modulus of continuity which are applicable in cases not covered by Kôno, notably when the incremental variance is a regularly varying function of index two.

Introduction. In this note we consider continuity in the time variable for a particular version of the local time of a stationary Gaussian process $X(t)$. Assume that X has mean zero, is separable and jointly measurable and is defined on the probability space $(\Omega, \mathscr{F}, P)$. By a local time for X we shall mean any jointly measurable function $\alpha^0(x, t, \omega)$ which is non-decreasing and right continuous in t for all x and all $\omega \in \Omega$ and which satisfies

$$\int_B \alpha^0(x, t, \omega)\, dx = \int_0^t I_{\{X(s,\omega)\in B\}}\, ds$$

for every Borel set B and a.e. ω. This definition allows α^0 to be any right continuous non-decreasing function for x in a set of Lebesgue measure zero. Thus if we wish to study the local time as a function of t for any fixed x we must be more careful about specifying a version of α^0. If we let $\sigma^2(t) = E(X(t) - X(0))^2$, then Geman (1976) has shown that when $\int_0^1 ds/\sigma(s) < \infty$, there exists a version which is a.s. continuous in t for a.e. x and, as all versions must agree for a.e. x, it follows that every version has this property. It is then possible to produce a modified version (by taking $\alpha(x, t, \omega) = 0$ for x in the exceptional set) which is a.s. continuous for every x. However, it is not at all clear how to construct this version from the sample paths of X. In this note we study a natural version of the local time which is directly computed from the sample paths of X, viz. we shall denote by $\alpha(x, t)$ the right continuous modification of

$$\liminf_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon} \int_0^t I_{\{|X(s)-x|<\varepsilon\}}\, ds. \tag{1}$$

It is well-known that the limit actually exists for a.e. x and that α is a version of the local time when one exists. We show below (Lemma 1) that the limit exists a.s. for every fixed (x, t). For this particular version (1) of the local time we are able to establish that for each fixed x, $\alpha(x, t)$ is a continuous function of t a.s. when

$$\int_0^1 ds/\sigma_*(s) < \infty. \tag{2}$$

Here $\sigma_*^2(s)$ is the conditional variance of $X(t)$ given the past, i.e. if $\mathscr{F}_T$ is the P-completion of the σ-algebra generated by $X(s)$, $-\infty < s \le T$, then $\sigma_*^2(t) = \operatorname{Var}(X(t) \mid \mathscr{F}_0)$. Although not strictly comparable (see [4, Section 3]), under regularity conditions this condition is weaker than the conditions under which Berman (1973) established the joint continuity of $\alpha(x, t)$. In particular, by framing our assumptions in terms of σ_* instead of σ, we avoid the explicit assumption of local nondeterminism used by other authors. In the final section we give some examples of processes which are not locally nondeterministic but to which our results can be applied.

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Upper bounds for the local and uniform modulus of continuity of the local time are also obtained. These bounds are applicable in certain cases not covered by Kôno (1977), notably in certain cases when $\sigma(t)$ is a regularly varying function of index *one*. Our bounds for the local modulus, as well as Kôno's when his additional assumptions are granted, are also shown to hold at stopping times and we use this fact in combination with a result of Taylor and Wendel (1966) to obtain a lower bound for the Hausdorff measure function of a level set. This extends a result of Davies (1976, 1977) who considered processes with spectral density proportional to $(a^2 + \lambda^2)^{-(\alpha+1/2)}$, $0 < \alpha < \frac{1}{2}$. A comprehensive review of local time theory has been published recently by Geman and Horowitz (1980).

1. Mainstream. We begin by showing that with the exception of a set of measure zero the limit in (1) exists for any fixed t.

LEMMA 1. *If $X(t, \omega)$ is a jointly measurable stationary Gaussian process, then*

$$\tilde{\alpha}(x, T) \equiv \lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon} \int_0^T I_{\{|X(s)-x|\leq\varepsilon\}}\, ds$$

exists for any fixed (x, T) a.s. *and*

$$\alpha(x, t) = \lim_{T\downarrow t, T\in Q} \tilde{\alpha}(x, T)$$

exists for all t a.s. *where Q is the set of all rational numbers, and $T \downarrow t$ means that T is* strictly *decreasing to t.*

PROOF. A result of Klein (1976) allows us to write $X(s) = X_0\phi(s) + Y(s)$, where $\phi(s)$ is analytic, $\phi(0) > 0$, X_0 is a standard normal variable, and $Y(s)$ is a centred (non-stationary) Gaussian process independent of X_0. Choose $\delta > 0$, rational, so that $\inf_{s\leq\delta} \phi(s) > 0$. Then for any fixed (x, T) with $T \leq \delta$

$$(2\varepsilon)^{-1} \int_0^T I_{\{|X(s)-x|\leq\varepsilon\}}\, ds = (2\varepsilon)^{-1} \int_0^T I_{\{|(Y(s)-x)/\phi(s)+X_0|\leq\varepsilon/\phi(s)\}}\, ds. \tag{3}$$

Now for fixed x, it follows from the theory of differentiation (cf. Geman and Horowitz (1980, page 13)) that

$$\lim_{\varepsilon\downarrow 0}(2\varepsilon)^{-1} \int_0^T I_{\{|(Y(s)-x)/\phi(s)+y|\leq\varepsilon\psi(s)\}}\, ds$$

exists for a.e. y when $\psi(s)$ is constant. This result is easily extended to functions ψ which are piecewise constant and then to bounded continuous functions by the usual approximation arguments. In particular we may take $\psi(s) = 1/\phi(s)$ and conclude that for fixed x, the limit exists for a.e. y. As X_0 and $Y(t)$ are independent, it follows that for fixed x, the limit as $\varepsilon \downarrow 0$ of (3) exists a.s. When $T > \delta$, let $n = [T/\delta]$ and split the integral over $[0, T]$ into the sum of integrals over $[(k-1)\delta, k\delta]$, $k = 1, \cdots, n$, $[n\delta, T]$. Because X is stationary, we also have the representations $X(k\delta + s) = X_0^k\phi(s) + Y^k(s)$, $k = 0, \cdots, n$, and the preceding argument can be applied separately to each term. Thus the limit in (1) holds for any fixed t a.s., and so also for all rational t a.s. As (1) is nondecreasing in T, the final statement of the lemma is also seen to hold.

Define $\mathscr{F}_{t^+} = \cap_{s>t} \mathscr{F}_t$. If τ is a stopping time relative to $\mathscr{F}_{t^+}$, i.e. $\{\tau \leq t\} \in \mathscr{F}_{t^+}$, define the σ-algebra $\mathscr{F}_{\tau^+}$ as those sets B such that $B \cap \{\tau \leq t\} \in \mathscr{F}_{t^+}$. We shall need to work with a version of the conditional local time. In keeping with the version (1) chosen for the unconditional local time, we shall assume that when $E\alpha(x, t)^n < \infty$ for all t the process

$$g(t, h, \omega) = E(\{\alpha(x, t+h) - \alpha(x, t)\}^n \mid \mathscr{F}_{t^+}) \tag{4}$$

is right continuous in t and h and non-decreasing in h for a.e. ω and that for any stopping time τ

$$E(\{\alpha(x, \tau + h) - \alpha(x, \tau)\}^n \mid \mathscr{F}_{\tau^+})$$

is defined as $g(\tau(\omega), h, \omega)$. The existence of these versions follows from the general theory of processes (see e.g. Dellacherie (1972, page 101)). The following lemma extends Davies' (1976) Lemma 14 and the proof is more straightforward.

LEMMA 2. *Let τ be any* a.s. *finite stopping time relative to $\{\mathscr{F}_{t^+}\}$ and assume $E(\alpha(x, t)^n) < \infty$ for all t. Then for all x, all $h > 0$, and almost every ω*

$$E(\{\alpha(x, \tau + h) - \alpha(x, \tau)\}^n \mid \mathscr{F}_{\tau^+}) \tag{5}$$

$$\leq n!(2\pi)^{-n/2} \int_{0\leq t_1<\cdots} \cdots \int_{\cdots<t_n\leq h} \det{}^{-1/2}\operatorname{Cov}(X(t_i), i = 1, \cdots, n; \mathscr{F}_0)\, dt_1 \cdots dt_n \tag{6}$$

where $\det \operatorname{Cov}(X(t_i), i = 1, \cdots, n; \mathscr{F}_0)$ *is the determinant of the conditional covariance matrix of* $(X(t_1), \cdots, X(t_n))$ *given* $\mathscr{F}_0$.

PROOF. On a set of unit measure, it follows from Lemma 1 that for all t

$$(\alpha(x, t + h) - \alpha(x, t))^n$$

$$= \lim_{T\downarrow t, T\in Q}\lim_{\varepsilon\downarrow 0} n!(2\varepsilon)^{-n} \int_{0<t_1\cdots<t_n\leq h} I_{\{|X(T+t_i)-x|\leq\varepsilon, i=1,\cdots,n\}}\, dt_1 \cdots dt_n.$$

It then follows by Fatou's lemma that $g(t, h, \omega)$ is less than or equal to

$$\lim_{T\downarrow t, T\in Q}\lim_{\varepsilon\downarrow 0} n!(2\varepsilon)^{-n} \int_{0<t_1\cdots<t_n\leq h} P(|X(T + t_i) - x| \leq \varepsilon, i = 1, \cdots, n \mid \mathscr{F}_{T^+})\, dt_1 \cdots dt_n. \tag{7}$$

The density of $(X(T + t_1), \cdots X(T + t_n))$ given $\mathscr{F}_{T^+}$ can be written as

$$((2\pi)^n \det \Lambda)^{-1/2} \exp\{-\tfrac{1}{2}(\mathbf{y} - \mathbf{u})'\Lambda^{-1}(\mathbf{y} - \mathbf{u})\} \leq (2\pi)^{-n/2}(\det \Lambda)^{-1/2} \tag{8}$$

where $\mathbf{y} = (y_1, \cdots, y_n)$, $\mathbf{u} = (u_1, \cdots, u_n)$, $u_i = E(X(T + t_i) \mid \mathscr{F}_{T^+})$ and Λ is the conditional covariance of $(X(T + t_1), \cdots, X(T + t_n))$ given $\mathscr{F}_{T^+}$. As Λ does not depend on T, the lemma is seen to hold for non-random t by applying (8) to (7). If τ is a stopping time with a countable set of values $\{s_i\}$, the result can be established from the relation $g(\tau(\omega), x, \omega) = \sum_{i=1}^{\infty} g(s_i, x, \omega)I_{\{\tau=s_i\}}$. For general stopping times τ let $\tau_n(\omega)$ = smallest number of the form $k2^{-n}$ which is greater than or equal to $\tau(\omega)$. Then the $\{\tau_n\}$ are stopping times and the lemma follows from the assumed right continuity of g.

LEMMA 3. *Assume that $\psi(t) = \int_0^t ds/\sigma_*(s)$ exists for all t. Then*

$$\begin{aligned} &\int_{0<t_1<\cdots<t_n\leq h} \det \operatorname{Cov}^{-1/2}(X(t_i), i = 1, \cdots, n; \mathscr{F}_0)\, dt_1 \cdots dt_n \\ &\quad\leq \int_{0=t_0<t_1<\cdots<t_n\leq h} \left[\prod_{i=1}^n \sigma_*(t_i - t_{i-1})\right]^{-1} dt_1 \cdots dt_n \leq (\psi(h))^n. \end{aligned} \tag{9}$$

PROOF. The first inequality follows from the fact that for $0 < t_1 < \cdots < t_n$

$$\det \operatorname{Cov}(X(t_n), \cdots, X(t_1); \mathscr{F}_0) = \prod_{j=1}^n \operatorname{Var}(X(t_j) \mid X(t_k), k < j, X(s), s \leq 0) \tag{10}$$

and that each factor of (10) is greater than or equal to

$$\operatorname{Var}(X(t_j) \mid \mathscr{F}_{t_{j-1}}) = \sigma_*^2(t_j - t_{j-1}).$$

The second inequality follows from the change of variables $s_j = t_j - t_{j-1}$ $(t_0 = 0)$ and integration over the larger domain $(s_1, \cdots, s_n) \in [0, h]^n$.

THEOREM 1. *Let τ be any* a.s. *finite stopping time relative to the σ-algebras $\{\mathscr{F}_{t^+}\}$ generated by $X(t)$. If* (2) *holds then for any fixed x*

$$\lim_{h\downarrow 0}\sup \frac{\alpha(x, \tau + h) - \alpha(x, \tau)}{\psi(h)\log\log\left(\frac{1}{\psi(h)}\right)} \le \frac{1}{\sqrt{2\pi}} \quad \text{a.s.} \tag{11}$$

where ψ is defined as in Lemma 3. *Following Kôno* (1977) *if we assume*

(12a) *$\sigma_*(t)$ is a nearly regularly function of index α, $0 < \alpha < 1$, i.e. there exist constants $0 < C_1 \le C_2 < \infty$ and a regularly varying function $r(t)$ of index α such that $C_1 r(t) \le \sigma_*(t) \le C_2 r(t)$ for t in some positive neighbourhood of the origin,*
(12b) *$t/\sigma_*(t)$ is non-decresing, and*
(12c) *$\sigma_*(t)$ is differentiable for $t > 0$ and there exists an $\varepsilon > 0$ such that $t\sigma_*'(t) \le \beta\sigma_*(t)$ for some $\beta < 1$ and all $t \in (0, \varepsilon]$,*

then

$$\lim_{h\downarrow 0}\sup \frac{\alpha(x, \tau + h) - \alpha(x, \tau)}{h/\sigma_*(h/\log\log 1/h)} < \infty \quad \text{a.s.} \tag{13}$$

PROOF. Let $h_n = \psi^{-1}((1 + \delta)^{-n})$, $\delta > 0$ and $Y_n = \alpha(x, \tau + h_n) - \alpha(x, \tau)$. By Chebychev's inequality, Lemma 2 and Lemma 3 we have that for any $\varepsilon > 0$ and $n > \max(0, -\log\log(1 + \delta))$

$$P\left(Y_n > \frac{(1 + \varepsilon)}{\sqrt{2\pi}}\psi(h_n)\log n\right) \le \inf_m m!\{(1 + \varepsilon)\log n\}^{-m}. \tag{14}$$

It follows from Stirling's formula [1, (6.1.38)] that (14) is less than

$$\sqrt{2\pi m}\, \exp\left\{-m\left(1 - \frac{1}{12m^2}\right)\right\}[m/\{(1 + \varepsilon)\log n\}^m]. \tag{15}$$

If we let m be the integer part of $(1 + \varepsilon)\log n$ then, for large n, (15) is less than a constant times $n^{-(1+\varepsilon/2)}$. It then follows by the Borel-Cantelli lemma that for all sufficiently large n

$$Y_n < (1 + \varepsilon)(2\pi)^{-1/2}\psi(h_n)\log n.$$

Now $\log n = \log\log(1/\psi(h_n)) - \log\log(1 + \delta)$ and $\psi(h_n)/\psi(h_{n-1}) = (1 + \delta)^{-1}$ and, as $\alpha(x, t)$ is non-decreasing in t, (11) follows by considering h in the interval $[h_n, h_{n-1}]$. The final statement can be established by using Kôno's (1977) estimates starting at (9).

As Theorem 1 has been established for stopping times, we can use the argument of Taylor and Wendel (1966) to obtain a lower bound for the exact Hausdorff measure function for the level sets. See also Davies (1977).

COROLLARY. *Under the conditions of Theorem* 1, *$\psi(t)\log\log 1/\psi(t)$ is a lower bound for the Hausdorff measure function of the level set $X^{-1}(X(s))$ with probability one for each fixed s. If in addition X is ergodic (iff its spectral measure has no atoms) then the result holds for $X^{-1}(u)$* a.s. *for any fixed u. If we also assume that* (12a–c) *hold, then a sharper lower bound for the Hausdorff measure function of this set is $h/\sigma_*(h/\log\log 1/h)$.*

THEOREM 2. *If* (2) *holds, then for every fixed x, $\alpha(x, t)$ is continuous in t* a.s. *Also*

$$\lim_{\delta\downarrow 0}\sup_{0<t-s\le\delta} \frac{\alpha(x, t) - \alpha(x, s)}{\psi(\delta)\log(1/\delta)} < \infty.$$

PROOF. It is convenient to work with a left continuous modification $\alpha^*(x, t) =$

$\lim_{s\uparrow t} \alpha(x, s)$. Of course this is continuous exactly when α is. Define a sequence of stopping times $\{\tau_n\}$ with respect to $\{\mathscr{F}_{t^+}\}$ via the recursion

$$\tau_0 = 0$$

$$\tau_n = \inf\{\tau_{n-1} + 1, \text{ first } t \geq \tau_{n-1}, \text{ such that } \alpha^*(x, t) - \alpha^*(x, \tau_{n-1}) \geq \varepsilon\}, \quad n \geq 1.$$

Since $E(\alpha^*(x, t)) \leq t\phi(x/\sigma)/\sigma$ where ϕ is the standard normal density and $\sigma^2 = EX^2(0)$, $\alpha^*(x, t) < \infty$ a.s. so that $\tau_n \to \infty$ a.s. It is easily checked that Theorem 1 applies to α^* so that $\tau_n - \tau_{n-1} > 0$ a.s. and since α^* is left continuous $\alpha^*(x, \tau_n) - \alpha^*(x, \tau_{n-1}) \leq \varepsilon$ so that this function can have no jumps of size greater than ε. Since ε was arbitrary it follows that α^* (and α) is a.s. continuous.

To bound the uniform modulus of continuity we note that, as α is non-decreasing,

$$\sup_{0<t-s\leq\delta, 0\leq t,s\leq 1}(\alpha(x, t) - \alpha(x, s)) \leq \sup_{2\leq k\leq[t_n^{-1}]+1}(\alpha(x, kt_n) - \alpha(x, (k-2)t_n))$$

for $\delta < t_n \equiv \psi^{-1}(2^{-n})$ so that when $t_{n-1} \leq \delta \leq t_n$

$$P[\sup_{t-s\leq\delta, 0\leq t,s\leq 1}(\alpha(x, t) - \alpha(x, s)) > 2\log(t_n^{-1})\psi(t_n)]$$
$$\leq ([t_n^{-1}] + 1)P(\alpha(x, 2t_n) > 2\log(t_n^{-1})\psi(t_n))$$

which, using the methods of Theorem 1, can be bounded by a constant times t_n. The proof is now completed by an application of the Borel-Cantelli lemma.

REMARK. The method used to establish the modulus of continuity provides an alternative proof of continuity without the use of stopping times or conditional expectation under the stronger hypothesis that $\psi(t)\log(1/t) \to 0$ as $t \downarrow 0$.

2. Lower bounds for $\sigma_*^2(t)$. Crucial to the application of Theorems 1 and 2 is the need to find lower bounds for $\sigma_*(t)$ at least in some neighbourhood of the origin. This has been taken up in Cuzick (1977). We say that X is strongly ϕ-nondeterministic if there exists a non-decreasing function ϕ, with $\phi(0) = 0$ such that $\liminf_{t\downarrow 0} \sigma_*^2(t)/\phi(t) > 0$. The main result is that if the process $X(t)$ has a spectral measure whose absolutely continuous component has a density $f(\lambda)$ which satisfies

$$f(\lambda/t) \geq t\phi(t)h(\lambda) \quad \text{with} \quad \int_0^\infty \frac{\log h(\lambda)\, d\lambda}{1 + \lambda^2} > -\infty \tag{16}$$

then X is strongly ϕ-nondeterministic. For example if

$$f(\lambda) \geq K(1 + \lambda)^{-\alpha} \quad \text{for some} \quad 1 < \alpha < 3, \quad K > 0 \tag{17}$$

then (16) holds with $\phi(t) = t^{\alpha-1}$. In this case when the inequality is replaced by equality in (17) then X is locally nondeterministic and our results for local time are essentially contained in the work of Berman (1973) and Kôno (1977). By contrast if

$$f(\lambda) = (1 + \lambda)^{-3}\log^\beta(e + \lambda), \quad \beta \geq 0$$

then it can be shown that

$$\sigma^2(t) = Ct^2\log^{\beta+1}\left(\frac{1}{t}\right)\left(1 + O\left(\log^{-1}\left(\frac{1}{t}\right)\right)\right)$$

for some $C > 0$ and the lemma given below shows that the process is not locally nondeterministic. However (16) does hold with $\phi(t) = t^2\log^\beta(1/t)$ and the lemma shows that this lower bound for $\sigma_*^2(t)$ is of the correct order of magnitude. For this example, Theorem 2 establishes that $\alpha(x, t)$ is a.s. continuous in t if $\beta > 2$ and Theorem 1 shows that an upper bound for the local modulus of continuity is $\log^{-(\beta/2-1)}(1/t)\log\log\log(1/t)$. These results are new.

LEMMA 4. *Assume that* $\sigma^2(t) = t^2 \log^\alpha(1/t)(1 + O(\log^{-1}(1/t)))$, $\alpha > 0$. *Then* $\sigma_*^2(t) \le Kt^2 \log^{\alpha-1}(1/t)$ *for some* $K < \infty$.

PROOF. For all $t > 0$,

$$\sigma_*^2(t) \le \operatorname{Var}(X(t) \mid X(0), X(-t)). \tag{18}$$

As $\sigma^2(t) = 2(1 - EX(0)X(t))$, it follows from standard multinormal expressions that the right hand side of (18) is equal to

$$\frac{1}{2}\left(\frac{\sigma^2(2t)}{\sigma^2(t)}\right)(4\sigma^2(t) - \sigma^2(2t)) - \frac{1}{2}\sigma^2(t)\sigma^2(2t).$$

The last term of this expression is seen to be negligible and, as

$$\begin{aligned} 4\sigma^2(t) - \sigma^2(2t) &= 4t^2\left(\log^\alpha\left(\frac{1}{t}\right) - \log^\alpha\left(\frac{1}{2t}\right)\right) + O\left(t^2 \log^{\alpha-1}\left(\frac{1}{t}\right)\right) \\ &= O\left(t^2 \log^{\alpha-1}\left(\frac{1}{t}\right)\right), \end{aligned}$$

the result follows.

REMARK. We have used the term *strongly* ϕ-nondeterministic here to distinguish the fact that the variance of $X(t)$ is conditional on *all* of the past ($X(s)$, $s \le 0$) as opposed to the local ϕ-nondeterminism of Cuzick (1978) and local nondeterminism of Berman (1973) (when $\phi = \sigma^2$) in which $X(t)$ is conditional on only a *finite* number of prior observations confined to an arbitrarily small interval in the immediate past. The example given in Cuzick (1977) shows this distinction is important. The results of Theorems 1 and 2 can be shown to hold under the weaker assumption of *strong local* ϕ-nondeterminism if we redefine $\sigma_*^2(t)$ to be the variance of $X(t)$ given $X(s)$, $s \in [-\varepsilon, 0]$, for some (fixed) $\varepsilon > 0$.

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